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The Asymptotic Stability of $x_{n+1} - a^2 x_{n-1} + b x_{n-k} = 0$

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ABSTRACT. We give the necessary and sufficient conditions for the asymptotic stability of the linear delay difference equation $x_{n+1} - a^2 x_{n-1} + b x_{n-k} = 0, \quad n = 0, 1, \cdots$, where a and b are arbitrary real numbers and k is a positive integer greater than 1. The obtained conditions are given in terms of parameters a and b of difference equations. The method of proof is based on arithematic of complex numbers as well as properties of analytic functions.

1. Introduction

In [2] S.A. Kuruklis gave the necessary and sufficient conditions for the asymptotic stability of the difference equation

$$\text{(A)} \qquad x_{n+1} - a x_n + b x_{n-k} = 0, \quad n = 0, 1, \cdots,$$

where a and b are arbitrary real and k is a positive integer. The following is the main result obtained in [2]:

Theorem A. *Let a be a nonzero real, b an arbitrary real, and k a positive integer greater than 1. Equation (A) is asymptotically stable if and only if $|a| < \frac{k+1}{k}$, and*

$$|a| - 1 < b < \left\{a^2 - 2|a|\cos\phi + 1\right\}^{\frac{1}{2}} \qquad \textit{for } k : \textit{odd},$$
$$|b - a| < 1 \quad \textit{and} \quad |b| < \left\{a^2 - 2|a|\cos\phi + 1\right\}^{\frac{1}{2}} \qquad \textit{for } k : \textit{even},$$

where ϕ is the solution in $\left(0, \frac{\pi}{k+1}\right)$ of $\frac{\sin k\theta}{\sin(k+1)\theta} = \frac{1}{|a|}$.

Theorem A generalizes results in [1] and [3]. The proof of Theorem A uses the fact that a linear difference equation is asymptotically stable if and only if all

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roots of its characteristic equation lie inside the unit disk. The term x_{n+1} in the difference equation (A) depends on the terms x_n and x_{n-k}. We are interested in the situation when the term x_{n+1} depends on the terms x_{n-1} and x_{n-k}. From this motivation, we propose to study the following linear difference equation

$$x_{n+1} - a^2 x_{n-1} + b x_{n-k} = 0, \quad n = 0, 1, \cdots, \tag{1}$$

where a and b are arbitrary real and k is a positive integer greater than 1. The method of proof is based on arithematic of complex numbers as well as properties of analytic functions. The characteristic equation of (1) is

$$\lambda^{k+1} - a^2 \lambda^{k-1} + b = 0. \tag{2}$$

In the case that $a \neq 0$, if we let $\mu = \frac{\lambda}{a}$ then (2) becomes

$$F(\mu) \equiv \mu^{k+1} - \mu^{k-1} + c = 0, \tag{3}$$

where $c = \frac{b}{a^{k+1}}$. Thus roots of (2) are inside the unit disk if and only if roots of (3) are inside the disk $|\mu| < \frac{1}{|a|}$, $a \neq 0$. For $c = 0$, (3) becomes

$$F(\mu) \equiv \mu^{k-1}(\mu^2 - 1) = 0,$$

which has a root at 0 of multiplicity $k-1$ and two simple roots at -1 and 1. As c varies the roots of (3) move in a continuous fashion.

2. Main results

To prove our results we will need several lemmas which deal with the behavior of the roots of (3) as c varies. The first three lemmas deal with the locations of real roots of (3) where the proofs are similar to Lemma 1 and Lemma 2 in [2] and will be omitted.

Lemma 2.1. *Let k be a positive integer greater than 1 and let c be a nonzero real number. Let $\beta = \left(\frac{k-1}{k+1}\right)^{\frac{k-1}{2}} \left(\frac{2}{k+1}\right)$ and $\alpha = \sqrt{\frac{k-1}{k+1}}$.*

(a) *$c > 0$ and k is even. In this case (3) has one negative real root in $(-\infty, -1)$. Furthermore, if $0 < c < \beta$, then (3) has two positive real roots, one in $(0, \alpha)$ and the other in $(\alpha, 1)$. If $c = \beta$, then α is a double root. If $c > \beta$, then (3) does not have positive real roots.*

(b) *$c > 0$ and k is odd. In this case $F(\mu)$ is an even function. If $0 < c < \beta$, then (3) has two negative real roots, one in $(-1, -\alpha)$ and the other in $(-\alpha, 0)$, and two positive real roots, one in $(0, \alpha)$ and the other in $(\alpha, 1)$. If $c = \beta$, then $-\alpha$ and α are double roots of (3). If $c > \beta$, then (3) does not have real roots.*

(c) *$c < 0$ and k is even. In this case (3) has one positive real root in $(1, +\infty)$. Furthermore, if $-\beta < c < 0$, then (3) has two negative real roots, one in $(-1, -\alpha)$ and the other in $(-\alpha, 0)$. If $c = -\beta$ then $-\alpha$ is a double root of (3). If $c < -\beta$ then (3) does not have real roots.*

(d) $c < 0$ *and* k *is odd. In this case* $F(\mu)$*is an even function and* (3) *has one positive root in* $(1, +\infty)$ *and one negative root in* $(-\infty, -1)$.

Lemma 2.2. *The absolute values of the roots of* (3) *increase as* $|c|$ *increases except in the following cases:*

(a) k *even and* $0 < c < \beta$. *In this case the largest positive root, which lies in* $(\alpha, 1)$, *decreases as* c *increases.*

(b) k *odd and* $0 < c < \beta$. *In this case the largest positive root, which lies in* $(\alpha, 1)$, *decreases as* c *increases and the absolute value of the smallest negative root, which lies in* $(-1, -\alpha)$, *decreases as* c *increases.*

(c) k *even and* $-\beta < c < 0$. *In this case the absolute value of the largest negative root, which lies in* $(-\alpha, 0)$, *decreases as* $|c|$ *increases. Let* $\mu = r(\cos\theta + i\sin\theta)$ *be a root of* (3). *If we substitute this into* (3), *then equate real and imaginary parts we obtain*

$$r^{k-1}\left\{\cos(k-1)\theta - r^2\cos(k+1)\theta\right\} = c, \tag{4}$$

$$r^{k-1}\left\{\sin(k-1)\theta - r^2\sin(k+1)\theta\right\} = 0. \tag{5}$$

It follows from (5) that

$$r^2 = \frac{\sin(k-1)\theta}{\sin(k+1)\theta}. \tag{6}$$

From (4) and (6) we get

$$\begin{aligned} c &= r^{k-1}\left\{\cos(k-1)\theta - \frac{\sin(k-1)\theta}{\sin(k+1)\theta}\cos(k+1)\theta\right\} \\ &= r^{k-1}\frac{\sin 2\theta}{\sin(k+1)\theta}. \end{aligned}$$

Hence,

$$c = r^{k-1}\frac{\sin 2\theta}{\sin(k+1)\theta}. \tag{7}$$

Moreover, as in [2] we can show that for $0 \le \theta < 2\pi$,

$$\frac{dr}{d\theta} > 0. \tag{8}$$

The next lemma gives us the relationship between the absolute value of c and the arguments of the roots of (3) which lie on $|\mu| = \frac{1}{|a|}$ with $|a| < \frac{1}{\alpha}$.

Lemma 2.3. *Let* $\mu = r(\cos\theta + i\sin\theta)$*, with* $0 \le \theta < 2\pi$*, be a root of* (3) *such that* $r = \frac{1}{|a|}$ *where* a *is a real number satisfying* $|a| < \frac{1}{\alpha}$*. Then*

$$|c| = \frac{1}{|a|^{k+1}} \left\{a^4 - 2a^2\cos 2\theta + 1\right\}^{\frac{1}{2}}. \tag{9}$$

Remark 2.1. Since $a^4 - 2a^2\cos 2\theta + 1 \ge \left(a^2 - 1\right)^2$, the square root in (9) is legitimate. If $\theta = 0$ or π, then

$$|c| = \frac{\left|a^2 - 1\right|}{|a|^{k+1}} \tag{10}$$

and if $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, then

$$|c| = \frac{a^2 + 1}{|a|^{k+1}}. \tag{11}$$

We will now find the minimum value of $|c|$ for which (3) has a root on the circle $|\mu| = \frac{1}{|a|}$ with $|a| < \frac{1}{\alpha}$. Note that the arguments of the roots of (3) certainly satisfy (6) which also has 0 and π as solutions, although (3) might not have a positive or negative root on $|\mu| = \frac{1}{|a|}$. Also, when k is odd then $\frac{\pi}{2}$ is a solution of (6) but when k is even $\frac{\pi}{2}$ is not a solution of (6). However, for $\theta \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$ we may write (6) as

$$S(\theta) = \frac{\sin(k-1)\theta}{\sin(k+1)\theta} = \frac{1}{a^2}. \tag{12}$$

Remark 2.2. Note that from (12) we have

$$S(\pi - \theta) = S(\theta). \tag{13}$$

It follows that if $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfies (12), then $\pi - \theta$ also satisfies (12).

Note that the arguments of all complex roots of (3) on $|\mu| = \frac{1}{|a|}$ which are in the first quadrant are solutions of (12). The following two lemmas state that if k is an integer greater than 1 then the converse also holds.

Lemma 2.4. *Assume that* $c \ne 0$ *and* $|a| < \frac{1}{\alpha}$*. Let* k *be a positive integer greater than 3. Then the number of solutions of* (12) *in* $\left(0, \frac{\pi}{2}\right)$ *is* $\frac{k-1}{2}$ *if* k *is odd and* $\frac{k}{2}$ *if* k *is even.*

Proof. **Case 1.** k is odd.
Assume that $c > 0$. Then by (6) and (7), we have $\sin(k-1)\theta > 0$ and $\sin(k+1)\theta > 0$, from which it follows that

$$\frac{2m\pi}{k-1} < \theta < \frac{(2m+1)\pi}{k-1} \quad , \quad m = 0, 1, \cdots, \left[\frac{k-3}{4}\right]$$

and

$$\frac{2n\pi}{k+1} < \theta < \frac{(2n+1)\pi}{k+1} \quad , \quad n = 0, 1, \cdots, \left[\frac{k-1}{4}\right].$$

Thus

$$\frac{2p\pi}{k-1} < \theta < \frac{(2p+1)\pi}{k+1} \quad , \quad p = 0, 1, \cdots, \left[\frac{k-3}{4}\right],$$

where $[x]$ is the integer part of x.By (6), (8), and (12), $S(\theta)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. Since $S(0) = \frac{k-1}{k+1}$, $S\left(\frac{p\pi}{k-1}\right) = 0$ and $S\left(\frac{(2p+1)\pi}{k+1}\right) = +\infty$ for $p = 1, 2, \cdots, \left[\frac{k-3}{4}\right]$, the number of solutions of (12) when $c > 0$ is $\left[\frac{k-3}{4}\right] + 1$.

Now assume that $c < 0$, then as in the case $c > 0$ we obtain

$$\frac{(2p+1)\pi}{k-1} < \theta < \frac{(2p+2)\pi}{k+1} \quad , \quad p = 0, 1, \cdots, \left[\frac{k-5}{4}\right].$$

Since $S\left(\frac{p\pi}{k-1}\right) = 0$ and $S\left(\frac{(2p+2)\pi}{k+1}\right) = +\infty$ for $p = 1, 2, \cdots, \left[\frac{k-5}{4}\right]$, the number of solutions of (12) when $c < 0$ is $\left[\frac{k-5}{4}\right] + 1$. Therefore, the total number of solutions of (12) in $\left(0, \frac{\pi}{2}\right)$ is $\left[\frac{k-3}{4}\right] + \left[\frac{k-5}{4}\right] + 2$ and it is easy to show that this number is equal to $(k-1)/2$.

Case 2. k is even.

With the same argument as in Case 1 we obtain that the number of solutions of (12) in $\left(0, \frac{\pi}{2}\right)$ is $\left[\frac{k-1}{4}\right] + \left[\frac{k-3}{4}\right] + 2$ and it is easy to show that this number is equal to $k/2$. $\square$

Lemma 2.5. *Let k be an integer greater than 1. Then the number of solutions of (3) which lie on $|\mu| = \frac{1}{|a|}$ and in the first quadrant $(0 < \theta < \frac{\pi}{2})$ where $|a| < \frac{1}{\alpha}$ is $\frac{k-1}{2}$ when k is odd, and $\frac{k}{2}$ if k is even.*

Proof. Let μ be a root of (3) which lies on $|\mu| = \frac{1}{|a|}$. From (3) we get

$$\mu^{k-1} - \mu^{k+1} = c. \tag{14}$$

Since c is real, we also have

$$\overline{\mu}^{k-1} - \overline{\mu}^{k+1} = c, \tag{15}$$

where $\overline{\mu}$ denotes the complex conjugate of μ. From (14) and (15) we obtain

$$\begin{aligned} \mu^{k-1} - \mu^{k+1} &= \overline{\mu}^{k-1} - \overline{\mu}^{k+1} \\ &= \frac{1}{a^{2(k-1)}\mu^{k-1}} - \frac{1}{a^{2(k+1)}\mu^{k+1}} \\ &= \frac{1}{a^{2(k-1)}\mu^{k-1}}\left(1 - \frac{1}{a^4\mu^2}\right). \end{aligned}$$

It follows that

$$a^{2(k+1)}\mu^{2k}(1-\mu^2) - a^4\mu^2 + 1 = 0. \tag{16}$$

Since the left hand side of (16) is a polynomial of degree $2k+2$, there are $2k+2$ roots of (16). It is straightforward to check that $\frac{1}{|a|}$ and $-\frac{1}{|a|}$ are simple real roots of (16). We will show that all other roots of (16) lie on $|\mu| = \frac{1}{|a|}$. Suppose that $\mu = \frac{1}{|a|}e^{i\theta}$ is a root of (16) on $|\mu| = \frac{1}{|a|}$, then

$$a^{2(k+1)}\frac{1}{a^{2k}}e^{i2k\theta}(1-\frac{1}{a^2}e^{i2\theta}) - a^4\frac{1}{a^2}e^{i2\theta}+1=0$$

or

$$e^{i2k\theta} = \frac{a^2e^{i2\theta}-1}{a^2-e^{i2\theta}}. \tag{17}$$

Let $z = e^{i2\theta}$. Then (17) becomes

$$z^k = \frac{a^2z-1}{a^2-z}. \tag{18}$$

We claim that complex roots of (18) are distinct if $|a| < \frac{1}{\alpha}$. We rewrite (18) as

$$z^{k+1} - a^2z^k + a^2z - 1 = 0. \tag{19}$$

Letting $F(z) = z^{k+1} - a^2z^k + a^2z - 1$, then $F'(z) = (k+1)\,z^k - a^2kz^{k-1} + a^2$. Assume that z_0 is a root of (19). Then by (18) we obtain

$$\begin{aligned} F'(z_0) &= (k+1)\,z_0^k - a^2kz_0^{k-1} + a^2 \\ &= (k+1)\left(\frac{a^2z_0-1}{a^2-z_0}\right) - \frac{a^2k}{z_0}\left(\frac{a^2z_0-1}{a^2-z_0}\right) + a^2. \end{aligned}$$

Note also that $(k+1)\left(\frac{a^2z-1}{a^2-z}\right) - \frac{a^2k}{z}\left(\frac{a^2z-1}{a^2-z}\right) + a^2 = 0$ if and only if

$$z = \frac{\left(1+k+a^4(k-1)\right) \pm \sqrt{(a^4-1)\left(a^4(k-1)^2-(1+k)^2\right)}}{2a^2k}. \tag{20}$$

Note that

$$\begin{aligned} &\left(1+k+a^4(k-1)\right)^2 + (a^4-1)\left(a^4(k-1)^2-(1+k)^2\right) = \\ &2\left((k-1)^2a^8 - 2a^4 + (k+1)^2\right) \end{aligned} \tag{21}$$

and it is straightforward to show that $\sqrt{2\left((k-1)^2a^8-2a^4+(k+1)^2\right)}$ is not equal to $2a^2k$ for any positive integer k and $0 < |a| < \frac{1}{\alpha}$. Thus z in (20) are two complex numbers which do not lie on the unit circle when $0 < |a| < \frac{1}{\alpha}$. Thus for $0 < |a| < \frac{1}{\alpha}$, (19) has $k+1$ distinct roots on the unit circle. It follows that if $0 < |a| < \frac{1}{\alpha}$ then

(16) has $2k+2$ distinct roots on $|\mu| = \frac{1}{|a|}$. Now since (16) has two simple real roots at $\frac{1}{|a|}$ and $-\frac{1}{|a|}$, there are $2k$ distinct complex roots. Thus there are k complex roots of (16) on the upper half plane (that is for $0 < \theta < \pi$). Now if $\frac{1}{|a|}e^{i\theta}$ satisfies (16) then we can show that $\frac{1}{|a|}e^{i(\pi-\theta)}$ also satisfies (16). Note that $\frac{\pi}{2}$ is a solution of (16) when k is odd. Therefore, the number of solutions of (3) which lie on $|\mu| = \frac{1}{|a|}$ in the first quadrant ($0 < \theta < \frac{\pi}{2}$) is $\frac{k-1}{2}$ if k is odd and $\frac{k}{2}$ if k is even. This completes the proof of Lemma 2.5. □

From Remark 2.2, Lemma 2.4, and Lemma 2.5, we have the following result.

Proposition 2.6. *Assume that $c \neq 0$ and $|a| < \frac{1}{\alpha}$. Let k be an integer greater than 3. Then the arguments of the complex roots of* (3) *which lie on $|\mu| = \frac{1}{|a|}$ are the only solutions of* (12) *except when k is even where the arguments $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are not solutions of* (12).

Remark 2.3. In [2] the author proved a result similar to Proposition 2.6 by showing that the number of solutions of $S(\theta) = \frac{\sin k\theta}{\sin(k+1)\theta} = \frac{1}{|a|}$ with $0 < \theta < \pi$ and $|a| < \frac{k+1}{k}$ is equal to the number of complex roots of (3) which lie on the upper half plane for a fixed value of c which is not correct (although the conclusion of his result is correct). In fact we must show that the number of solutions of $S(\theta) = \frac{\sin k\theta}{\sin(k+1)\theta} = \frac{1}{|a|}$ with $0 < \theta < \pi$ and $|a| < \frac{k+1}{k}$ is equal to the number of complex roots of (3) which lie on $|\mu| = \frac{1}{|a|}$ for different values of c (which can be shown as in Lemma 2.5 above).

Remark 2.4. Let k be an integer greater than 3. The minimum value of $|c|$ given in (18) for which (3) has a complex root on $|\mu| = \frac{1}{|a|}$ with $|a| < \frac{1}{\alpha}$ occurs when θ is the solution of (12) and $\theta \in \left(0, \frac{\pi}{k+1}\right)$.

Remark 2.5. For $k = 2$ or $k = 3$, it is easy to show that (3) has exactly one complex root $\mu = \frac{1}{|a|}e^{i\theta_0}$, where $\theta_0 \in \left(0, \frac{\pi}{2}\right)$ and that θ_0 is the only solution of (12) in $\left(0, \frac{\pi}{2}\right)$. Thus, Proposition 1.6 holds true for $k = 2$ and $k = 3$ also.

The next four lemmas provide us with the necessary and sufficient conditions for the roots of (3) when $c \neq 0$ to be inside the disk $|\mu| < \frac{1}{|a|}$.

Lemma 2.7. *Let k be an odd integer greater than 1 and $c > 0$. Then all the roots of* (3) *lie inside the disk $|\mu| < \frac{1}{|a|}$ if and only if $|a| < \frac{1}{\alpha}$ and*

$$\frac{a^2-1}{|a|^{k+1}} < c < \frac{1}{|a|^{k+1}}\left\{a^4 - 2a^2\cos 2\phi + 1\right\}^{\frac{1}{2}}, \tag{22}$$

where ϕ is the solution of $\frac{\sin(k-1)\theta}{\sin(k+1)\theta} = \frac{1}{a^2}$ in $\left(0, \frac{\pi}{k+1}\right)$.

Proof. Lemma 2.1 implies that if $\frac{1}{|a|} \leq \alpha$ and $0 < c \leq \beta$, then (3) has real roots which lie outside the disk $|\mu| < \frac{1}{|a|}$. Hence, for all roots of (3) to be inside the disk

$|\mu| < \frac{1}{|a|}$ it is necessary that $|a| < \frac{1}{\alpha}$. We now consider the following two cases.
Case 1. $1 < |a| \le \frac{1}{\alpha}$.
As c increases from 0 to β the largest positive root and the smallest negative root of (3) move from 1 and -1 toward α and $-\alpha$ respectively, while all other roots move outward away from 0. Let c_1 and c_2 be the values of c for which the largest positive and the smallest negative roots, and a complex root reach the circle $|\mu| = \frac{1}{|a|}$. Then

$$c_1 = \frac{a^2-1}{|a|^{k+1}} \text{ and } c_2 = \frac{1}{|a|^{k+1}}\left\{a^4 - 2a^2\cos 2\phi + 1\right\}^{\frac{1}{2}}.$$

Note that $c_1 < c_2$. From Remark 2.4, the smallest value of c_2 for which a complex root of (3) lies on the circle $|\mu| = \frac{1}{|a|}$ is obtained when ϕ is the solution of $\frac{\sin(k-1)\theta}{\sin(k+1)\theta} = \frac{1}{a^2}$ in $\left(0, \frac{\pi}{k+1}\right)$. Thus, as c increases, first the largest positive and the smallest negative root reach the circle $|\mu| = \frac{1}{|a|}$ (simultaneously) and then a complex root reaches the circle $|\mu| = \frac{1}{|a|}$. Therefore, all the roots of (3) lie inside the disk $|\mu| < \frac{1}{|a|}$ if and only if $c_1 < c < c_2$ and (22) holds.
Case 2. $|a| \le 1$.
By Lemma 2.1, (3) has only complex roots which reach the circle $|\mu| = \frac{1}{|a|}$. Let c_2 be the values of c for which a complex root reaches the circle $|\mu| = \frac{1}{|a|}$. As in Case 1, $c_2 = \frac{1}{|a|^{k+1}}\left\{a^4 - 2a^2\cos 2\phi + 1\right\}^{\frac{1}{2}}$ and all the roots of (3) lie inside the disk $|\mu| < \frac{1}{|a|}$ if and only if $0 < c < c_2$ and (22) holds. □

The next three lemmas can be proved similarly to Lemma 2.7 and will be omitted.

Lemma 2.8. *Let k be an odd integer greater than 1 and $c < 0$. Then all the roots of (3) lie inside the disk $|\mu| < \frac{1}{|a|}$ if and only if $|a| < 1$ and $c > \frac{a^2-1}{|a|^{k+1}}$.*

Lemma 2.9. *Let k be an even integer greater than 1 and $c > 0$. Then all the roots of (3) lie inside the disk $|\mu| < \frac{1}{|a|}$ if and only if $|a| < 1$ and $c < \frac{1-a^2}{|a|^{k+1}}$.*

Lemma 2.10. *Let k be an even integer greater than 1 and $c < 0$. Then all the roots of (3) lie inside the disk $|\mu| < \frac{1}{|a|}$ if and only if $|a| < 1$ and $c > \frac{a^2-1}{|a|^{k+1}}$.*

We now give the necessary and sufficient conditions for the roots of (2) to be inside the unit disk.

Lemma 2.11. *Let k be an odd integer greater than 1, a nonzero real number, and b arbitrary real number. The roots of (2) are inside the unit disk if and only if $|a| < \frac{1}{\alpha}$ and $a^2 - 1 < b < \left\{a^4 - 2a^2\cos 2\phi + 1\right\}^{\frac{1}{2}}$, where ϕ is the solution of $\frac{\sin(k-1)\theta}{\sin(k+1)\theta} = \frac{1}{a^2}$ in $\left(0, \frac{\pi}{k+1}\right)$.*

Proof. **Case 1.** $b > 0$.

Then, $c > 0$ and Lemma 2.7 implies that all roots of (2) are inside the unit disk if and only if $|a| < \frac{1}{\alpha}$ and $a^2 - 1 < b < \left\{a^4 - 2a^2 \cos 2\phi + 1\right\}^{\frac{1}{2}}$.
Case 2. $b < 0$.
Then, $c < 0$ and Lemma 2.8 implies that all roots of (2) are inside the unit disk if and only if $|a| < 1$ and $a^2 - 1 < b$. The Lemma follows from Case 1 and Case2. □

Lemma 2.12. *Let k be an even integer greater than* 1, *a nonzero real number and b arbitrary real number. The roots of* (2) *are inside the unit disk if and only if $|a| < 1$ and $a^2 + |b| < 1$.*

Proof. Similar to Lemma 2.11 and will be omitted. □

From Lemma 2.11 and Lemma 2.12, we obtain the following necessary and sufficient conditions for (1) to be asymptotically stable.

Theorem 2.13. *Let a be a nonzero real number, b an arbitrary real number, and k an integer greater than* 1. *Then for k odd,* (1) *is asymptotically stable if and only if $|a| < \frac{1}{\alpha}$ and $a^2 - 1 < b < \left\{a^4 - 2a^2 \cos 2\phi + 1\right\}^{\frac{1}{2}}$, where ϕ is the solution of $\frac{\sin(k-1)\theta}{\sin(k+1)\theta} = \frac{1}{a^2}$ in $\left(0, \frac{\pi}{k+1}\right)$. For k even,* (1) *is asymptotically stable if and only if $|a| < 1$ and $a^2 + |b| < 1$.*

Remark 2.6. When $a = 0$ or $k = 1$ it is easy to show that the necessary and sufficient conditions for (1) to be asymptotically stable is that $\left|a^2 - 1\right| < 1$.

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