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Some Finite Integrals Involving The Product of Srivastava's Polynomials and A Certain $\bar{H}$-Function with Applications

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Abstract. The aim of this paper is to evaluate four finite integrals involving the product of Srivastava's polynomials, a generalized hypergeometric function and $\bar{H}$ -function proposed by Inayat Hussian which contains a certain class of Feynman integrals. At the end, we give an application of our main findings by connecting them with the Riemann-Liouville type of fractional integral operator. The results obtained by us are basic in nature and are likely to find useful applications in several fields notably electric networks, probability theory and statistical mechanics.

1. Introduction

The $\bar{H}$-function will be defined and represented as follows [1]:

$$\bar{H}^{M,N}_{P,Q}[Z] = \bar{H}^{M,N}_{P,Q}[z|^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \bar{\phi}(\xi) z^{\xi} d\xi, \tag{1.1}$$

where

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}}. \tag{1.2}$$

Buschman and Srivastava [1] has proved that the integral on the right hand side

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of (1.1) is absolutely convergent when $\Omega > 0$ and $|argz| < \frac{1}{2}\pi\Omega$, where

$$\Omega = \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} A_j\alpha_j - \sum_{j=M+1}^{Q} B_j\beta_j - \sum_{j=N+1}^{P} \alpha_j > 0$$

here, and throughout the paper $a_j(j = 1, 2, . \cdots, P)$ and $b_j(j = 1, 2, \cdots, Q)$ are complex parameters, $\alpha_j \geq 0(j = 1, \cdots, P), \beta_j \geq 0(j = 1, \cdots, Q)$ (not all zero simultaneously) and the exponents $A_j(j = 1, \cdots, N)$ and $B_j(j = M + 1, \cdots, Q)$ can take on non-negative values. For further details of $\bar{H}$-function one can refer the original paper of Buschman and Srivastava [1]. Srivastava [7] introduced the general class of polynomials (see also Srivastava and Singh [9])

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, n = 0, 1, 2, \cdots , \tag{1.3}$$

where m and n are arbitrary integers and the coefficents $A_{n,k}(n, k \geq 0)$ are arbitrary constants real or complex. Generalized hypergeometric function is defined as follows:

$${}_PF_Q[(a_P); (b_Q); z] ={}_P F_Q[\frac{(a_P)}{(b_Q)}; z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{P} (a_j)_n}{\prod_{j=1}^{Q} (b_j)_n} \frac{z^n}{n!}, \tag{1.4}$$

where for brevity, (a_P denotes the array of parameters $a_1, \cdots, a_P$ with similar interpretation for (b_Q)etc.. For further details one can refer Rainville[6].

2. Main Results

$$\int_0^t x^{\rho-1}(t-x)_R^{\sigma-1}F_S[(g_R); (h_S); ax^u(t-x)^{\eta}]S_n^m[yx^{\mu}(t-x)^{\nu}]S_{n'}^{m'}[y'x^{\mu'}(t-x)^{\nu'}]\cdot$$

$$\bar{H}_{P,Q}^{M,N}[zx^{\mu}(t-x)^{\nu}|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}]dx = t^{\rho+\sigma+1}\cdot$$

$$\sum_{k=0}^{[n/m]} \sum_{k'=0}^{[n'/m']} \sum_{r=0}^{\infty} \frac{(-n)_{mk}(-n')_{m'k'}}{k!k'!} A_{n,k}A'_{n'k'}y^k y'^{k'} t^{(\mu+\nu)k+(\mu'+\nu')k'} f(r)\cdot$$

$$\bar{H}_{P+2,Q+1}^{M,N+2}[zt^{\mu+\nu}|_{B^\star,(1-\rho-\sigma-ur-\eta r-\mu k-\nu k-\mu' k'-\nu' k',\nu+\mu;1)}^{(1-\rho-ur-\mu k-\mu' k',\mu;1),(1-\sigma-\eta r-\nu k-\nu' k',\nu;1),A^\star}]t^{(u+\eta)r}, \tag{2.1}$$

where $f(r) = \frac{\prod_{j=1}^{R} (g_j)_r}{\prod_{j=1}^{S} (h_j)_r} \frac{a^r}{r!}$. The conditions of validity of (2.1) are

(i) $\mu \geq 0, \nu \geq 0$ (not both zero simultaneously) $|argz| < \frac{1}{2}\pi\Omega, \Omega > 0$,

(ii) $R \leq S$ or $R = S+1$ and $|at^{u+\eta}| < 1$ [none of $h_j (j = 1, 2, \cdots, S)$ is a negative integer or zero],

(iii) u and μ are non-negative integers such that $u + \eta \geq 1$,

(iv) m and m' are arbitrary positive integers and the coefficents $A_{n,k}, A'_{n'k'} (n, k, n'k' \geq 0)$ are arbitrary real or complex,

(v) $Re(\rho)+\mu(\min)_{1\leq j\leq M}[Re(b_j/\beta_j)] > 0$ and $Re(\sigma)+\nu(\min)_{1\leq j\leq M}[Re(b_j/\beta_j)] > 0$.

$$\int_0^t x^{\rho-1}(t-x)_R^{\sigma-1} F_S[(g_R);(h_S);ax^u(t-x)^\eta] S_n^m[yx^{-\mu}(t-x)^{-\nu}]\cdot$$

$$S_{n'}^{m'}[y'x^{-\mu'}(t-x)^{-\nu'}]\bar{H}_{P,Q}^{M,N}[zx^{-\mu}(t-x)^{-\nu}|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}]dx$$

$$= t^{\rho+\sigma-1}\sum_{k=0}^{[n/m]}\sum_{k'=0}^{[n'/m']}\sum_{r=0}^{\infty}\frac{(-n)_{mk}(-n')_{m'k'}}{k!k'!}A_{n,k}A'_{n'k'}y^k y'^{k'} t^{(\mu+\nu)k+(\mu'+\nu')k'} f(r)\cdot$$

$$\bar{H}_{P+1,Q+2}^{M+2,N}[zt^{-\mu-\nu}|_{(\rho+ur-\mu k-\mu'k',\mu),(\sigma-\eta r-\nu k-\nu'k';\nu),B^\star}^{A^\star,(\rho+\sigma-\mu k-\nu k-\mu'k'-\nu'k',\mu+\nu)}]t^{(u+\eta)r}. \tag{2.2}$$

Provided that $Re(\rho)+\mu(\max)_{1\leq j\leq N}[Re((a_j-1)/\alpha_j)] > 0$ and $Re(\sigma)-\nu(\max)_{1\leq j\leq N}[Re((a_j-1)/\alpha_j)] > 0$ and the sets of conditions (i) to (iv) given with (2.1) are satisfied.

$$\int_0^t x^{\rho-1}(t-x)_R^{\sigma-1} F_S[(g_R);(h_S);ax^u(t-x)^\eta] S_n^m[yx^{\mu}(t-x)^{-\nu}]\cdot$$

$$S_{n'}^{m'}[y'x^{\mu'}(t-x)^{-\nu'}]\bar{H}_{P,Q}^{M,N}[zx^{\mu}(t-x)^{-\nu}|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}]dx$$

$$= t^{\rho+\sigma-1}\sum_{k=0}^{[n/m]}\sum_{k'=0}^{[n'/m']}\sum_{r=0}^{\infty}\frac{(-n)_{mk}(-n')_{m'k'}}{k!k'!}A_{n,k}A'_{n'k'}y^k y'^{k'} t^{(\mu-\nu)k+(\mu'-\nu')k'} f(r)\cdot$$

(2.3)

$$\bar{H}_{P+1,Q+2}^{M+1,N+1}[zt^{\mu-\nu}|_{(\sigma+\eta r-\nu k-\nu'k',\nu),B^\star,(1-\rho-\sigma-ur-\eta r-\mu k+\nu k-\nu'k'+\nu'k';\mu-\nu;1)}^{(1-\rho-ur-\mu k-\mu'k',\mu;1)A^\star}]t^{(u+\eta)r}.$$

Provided that $\mu \geq 0, \nu \geq 0$ such that $\mu-\nu \geq 0$ and $Re(\rho)+\mu(\min)_{1\leq j\leq M}[Re(b_j/\beta_j)] > 0$ and $Re(\sigma) - \nu(\max 0_{1\leq j\leq N}[Re((a_j-1)/\alpha_j)] > 0$. It is being assumed that the conditions (i) to (ii) given with (2.1) are satisfied.

$$\int_0^t x^{\rho-1}(t-x)_R^{\sigma-1} F_S[(g_R);(h_S);ax^u(t-x)^\eta] S_n^m[yx^{-\mu}(t-x)^{\nu}]\cdot$$

$$S_{n'}^{m'}[y'x^{-\mu'}(t-x)^{\nu'}]\bar{H}_{P,Q}^{M,N}[zx^{-\mu}(t-x)^{\nu}|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}]dx$$

$$= t^{\rho+\sigma-1}\sum_{k=0}^{[n/m]}\sum_{k'=0}^{[n'/m']}\sum_{r=0}^{\infty}\frac{(-n)_{mk}(-n')_{m'k'}}{k!k'!}A_{n,k}A'_{n'k'}y^k y'^{k'}t^{(\mu-\nu)k+(\mu'-\nu')k'}f(r)\cdot$$

$$\bar{H}_{P+2,Q+1}^{M+1,N+1}[zt^{-\mu+\nu}|_{(\sigma+\eta r-\nu k-\nu'k',\nu),B^\star}^{(1-\rho-ur-\mu k-\mu'k',\mu;1)A^\star,(\rho+\sigma+ur+\eta r+\mu k-\nu k+\nu'k'-\nu'k';\nu-\mu)}].$$

$$t^{(u+\eta)r}. \tag{2.4}$$

Provided that $\mu \geq 0, \nu \geq 0$ such that $\nu-\mu \geq 0$ and $Re(\rho)-\mu(\max)_{1\leq j\leq N}[Re((a_j-1)/\alpha_j)] > 0$ and $Re(\sigma)+\nu(\min)_{1\leq j\leq M}[Re(b_j/\alpha_j)] > 0$. It is being assumed that the conditions (i) to (ii) given with (2.1) are satisfied.

Proof. To derive (2.1), we use series representation for the generalized hypergeometric function, change the order of integration and summation (which is permissible under the conditions stated), evaluate the resulting x-integral by applying the following integral due to [2]

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}S_n^m[yx^{\mu}(t-x)^{\nu}]S_{n'}^{m'}[y'x^{\mu'}(t-x)^{\nu'}]\cdot$$

$$\bar{H}_{P,Q}^{M,N}[zx^{\mu}(t-x)^{\nu}|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}]dx$$

$$= t^{\rho+\sigma+1}\sum_{k=0}^{[n/m]}\sum_{k'=0}^{[n'/m']}\frac{(-n)_{mk}(-n')_{m'k'}}{k!k'!}A_{n,k}A'_{n'k'}y^k y'^{k'}t^{(\mu+\nu)k+(\mu'+\nu')k'}.$$

$$\bar{H}_{P+2,Q+1}^{M,N+2}[zt^{\mu+\nu}|_{B^\star,(1-\rho-\sigma-\mu k-\nu k-\mu'k'-\nu'k',\nu+\mu;1)}^{(1-\rho-\mu k-\mu'k',\mu;1),(1-\sigma-\nu k-\nu'k',\nu;1),A^\star}],$$

where the sets (i) to (v) of the conditions mentioned with (2.1) are satisfied. The integrals (2.2) to (2.4) can be evaluated on lines similar to those of first integral.

3. Special Cases

If we take $n = 0$ and $n' = 0$ in (2.1) to (2.4), we arrive at the following integrals which are also new and sufficently general in nature:

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}_R F_S[(g_R);(h_S);ax^u(t-x)^{\eta}]\cdot$$

$$\bar{H}_{P,Q}^{M,N}[zx^{\mu}(t-x)^{\nu}|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}]dx = t^{\rho+\sigma+1}.$$

$$\sum_{r=0}^{\infty} f(r)\bar{H}^{M,N+2}_{P+2,Q+1}[zt^{\mu+\nu}|^{(1-\rho-ur,\mu;1),(1-\sigma-\eta r,\nu;1),A^\star}_{B^\star,(1-\rho-\sigma-ur-\eta r,\nu+\mu;1)}]t^{(u+\eta)r}. \tag{3.1}$$

The conditions of validity of (3.1) can be easily obtained from those of (2.1).

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}{}_RF_S[(g_R);(h_S);ax^u(t-x)^\eta]\cdot$$

$$\bar{H}^{M,N}_{P,Q}[zx^{-\mu}(t-x)^{-\nu}|^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}]dx = t^{\rho+\sigma-1}.$$

$$\sum_{r=0}^{\infty} f(r)\bar{H}^{M+2,N}_{P+1,Q+2}[zt^{-\mu-\nu}|^{A^\star,(\rho+\sigma,\mu+\nu)}_{(\rho+ur,\mu),(\sigma-\eta r;\nu),B^\star}]t^{(u+\eta)r}. \tag{3.2}$$

The conditions of validity of (3.2) can be easily obtained from those of (2.2).

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}{}_RF_S[(g_R);(h_S);ax^u(t-x)^\eta]\cdot$$

$$\bar{H}^{M,N}_{P,Q}[zx^{\mu}(t-x)^{-\nu}|^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}]dx = t^{\rho+\sigma-1}.$$

$$\sum_{r=0}^{\infty} f(r)\bar{H}^{M+1,N+1}_{P+1,Q+2}[zt^{\mu-\nu}|^{(1-\rho-ur,\mu;1),A^\star}_{(\sigma+\eta r,\nu),B^\star,(1-\rho-\sigma-ur-\eta r;\mu-\nu;1)}]t^{(u+\eta)r}. \tag{3.3}$$

The conditions of validity of (3.3) can be easily obtained from those of (2.3).

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}{}_RF_S[(g_R);(h_S);ax^u(t-x)^\eta]\cdot$$

$$\bar{H}^{M,N}_{P,Q}[zx^{-\mu}(t-x)^{\nu}|^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}]dx = t^{\rho+\sigma-1}.$$

$$\sum_{r=0}^{\infty} f(r)\bar{H}^{M+1,N+1}_{P+2,Q+1}[zt^{-\mu+\nu}|^{(1-\rho-ur,\mu;1),A^\star,(\rho+\sigma+ur+\eta r;\nu-\mu)}_{(\sigma+\eta r,\nu),B^\star}]t^{(u+\eta)r}. \tag{3.4}$$

The conditions of validity of (3.4) can be easily obtained from those of (2.4). On sutabily specializing the parameters of the $\bar{H}$-function and the general class of polynomials in our main integrals , we can obtain a large number of new integrals as their special cases but we do not record them here on account of lack of space.

4. Application

We shall define the Rieman-Liouville fractional derivative of function $f(x)$ of order σ (or alternatively, σ^{th} order fractional integral) $(2, p.181; (11), p.49)$ by

$$D_x^{-\sigma}\{f(x)\} = \{ {}^{\frac{1}{\Gamma(-\sigma)}\int_a^x (x-t)^{-\sigma-1} f(t)dt, Re(\sigma)<0}_{\frac{dq}{dx^q}\,{}_aD_x^{\sigma-q}\{f(x)\}, q-1\leq Re(\sigma)<q} , \tag{4.1}$$

where q is a positive integer and the integral exists. For simplicity the special case of the fractional derivative operator ${}_aD_x^{\sigma}$, when $a = 0$ will be written as D_x^{σ}. Thus we have

$$D_x^{\sigma} =_0 D_x^{\sigma}. \tag{4.2}$$

Now by setting $t = b$ in the main integral (2.1), it can be rewritten as the following fractional integral formula :

$$D_t^{-\sigma}\{b_R^{\rho-1}F_S[(g_R); (h_S); at^u(b-t)^\eta]S_n^m[yt^\mu(b-t)^\nu]\cdot$$

$$S_{n'}^{m'}[y^{'}t^{\mu^{'}}(b-t)^{\nu^{'}}]\bar{H}_{P,Q}^{M,N}[zt^\mu(b-t)^\nu|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}}]\} = \frac{b^{\rho+\sigma+1}}{\Gamma(\sigma)}\cdot$$

$$\sum_{k=0}^{[n/m]}\sum_{k^{'}=0}^{[n^{'}/m^{'}]}\sum_{r=0}^{\infty}\frac{(-n)_{mk}(-n^{'})_{m'k'}}{k!k'!}A_{n,k}A^{'}_{n'k'}y^k y^{'k^{'}} b^{(\mu+\nu)k+(\mu^{'}+\nu^{'})k^{'}} f(r)\cdot$$

$$\bar{H}_{P+2,Q+1}^{M,N+2}[zb^{\mu+\nu}|_{B^\star,(1-\rho-\sigma-ur-\eta r-\mu k-\nu k-\mu' k'-\nu' k',\nu+\mu;1)}^{(1-\rho-ur-\mu k-\mu^{'}k^{'},\mu;1),(1-\sigma-\eta r-\nu k-\nu^{'}k^{'},\nu;1),A^\star}]b^{(u+\eta)r}, \tag{4.3}$$

where $Re(\sigma) > 0$ and all the conditions of validity mentioned with (2.1) are satisfied. The fractional integral formula given by (4.3) is also quite general in nature and can easily yield Riemann-Liouville fractional integrals of a large number of simpler functions and polynomials merely by specializing the parameters of $\bar{H}, S_n^m, S_{n'}^{m'}$ occuring in it which may find applications in electromagnetic theory and probability.

References

[1] R.G. Buschman and H.M. Srivastava, *The $\bar{H}$-function associated with a certain class of Feynman integrals*, J.Phys. A: Math. Gen., **23** (1990), 4707-4710.

[2] A. et.al. Erdelyi, Tables of Integral Transforms, vol.II, McGraw-Hill, New-York (1954).

[3] K.C. Gupta and R.C. Soni, New properties of a generalization of hypergeometric series associated with feynman integrals,Kyungpook Mathematical Journal, vol.4(1), (2001), 97-104.

[4] A.A. Inayat-Hussian, *New properties of hypergeometric series deriable from feynman integrals:I; Transformation and reduction formulae*, J.Phys. A:Math.Gen. **20** (1987), 4109-4117.

[5] A.A. Inayat-Hussian, *New properties of hypergeometric series deriable from feynman integrals:II; A generalization of the H-function*, J.Phys. A:Math.Gen. **20** (1987), 4119-4128.

[6] E.D. Rainville, Special Functions, The Macmillan Company Inc., NewYork (1963).

[7] H.M. Srivastava, *A contour integral involving Fox's H-function*, Indian J.Math., **14** (1972), 1-6.

[8] H.M. Srivastava, K.C. Gupta and S.P. Goyal, The H-function of One and Two Variables with Application; South Asian Publisher, New Delhi (1982).

[9] H.M. Srivastava and N.P. Singh, *The integration of certain products of the multivariable H-function with a general class of polynomials*, Rend. Circ. Math.Palermo, (Ser2), **32** (1983), 157-187.