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# COMMON FIXED POINT THEOREMS FOR A GENERALIZED CONTRACTIVE TYPE MAPPINGS IN METRIC SPACES

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ABSTRACT. In this paper, we give a generalized contractive type condition for a pair of self maps of a metric space and analyze the existence of common fixed points for these maps of a metric space.

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## 1. Introduction

Recently, in [1,3,4], the authors obtained some fixed point theorems for self maps of metric space satisfying integral type contractive condition. Since then, in [5], the author gave a generalized contractive type condition for two maps in metric space and proved some common fixed point theorems for these maps. That is, the author extend well-known results in [1,2,4].

In this paper, we give a generalized contractive type condition for two self maps of a metric space and then we prove common fixed point theorems for these maps. That is, we generalize the results of [3].

For a self map $f$ of a nonempty set $X$ and $x \in X$, we denote $O(x,f)$ by the orbit of $f$ at $x$ and $O(x,f,n)$ by the $n$-th orbit of $f$ at $x$. That is,

$$O(x,f) = \Big\{f^k x : k = 0,1,2,\cdots,\Big\} \text{ and } O(x,f,n) = \Big\{f^k x : k = 0,1,2,\cdots,n\Big\}.$$

For $A \subset X$, we denote the diameter of $A$ by $\delta(A)$.

Let $\{x_n\}$ be a bounded sequence in a metric space $(X,d)$ and $r_n = \delta\Big(\{x_n, x_{n+1}, x_{n+2}, \cdots\}\Big)$ for $n = 1,2,3,\cdots$. Then $r_n$ is a finite number for all $n \in \mathbb{N}$ and $\{r_n\}$ is nonincreasing and $r_n \geq 0$ for all $n \in \mathbb{N}$. Thus there exists a $r \geq 0$ such that $\lim_{n\to\infty} r_n = r$.

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From now on, we denote

$$M(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,gy), d(x,gy), d(y,fx)\Big\}$$

for self maps $f$ and $g$ of a metric space $(X, d)$ and $x, y \in X$. Also, we denote $\Lambda$ by the class of nondecreasing continuous function $\alpha : [0,\infty) \to [0,\infty)$ such that

($\alpha$1) $\alpha(0) = 0$,

($\alpha$2) $\alpha(s) > 0$ for all $s > 0$.

Note that if $\alpha(s) = \int_0^s \varphi(t)dt$, then $\alpha \in \Lambda$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable, nonnegative and $\int_0^u \varphi(t)dt > 0$ for each $u > 0$.

And we denote $\Phi$ by the class of nondecreasing right upper semi-continuous function $\phi : [0,\infty) \to [0,\infty)$ satisfying:

($\phi$1) $\phi(t) < t$ for all $t > 0$,

($\phi$2) for each $t > 0$, $\lim_{n\to\infty} \phi^n(t) = 0$.

Note that $\phi(0) = 0$.

## 2. Common fixed point theorems

In this section, we give some generalized contractive type common fixed point theorems for a pair of self maps of a metric space.

**Theorem 2.1.** *Let $f$ and $g$ be self maps of a complete metric space $(X, d)$ and $\phi \in \Phi$ and $\alpha \in \Lambda$ satisfying: for each $x, y \in X$,*

$$\alpha\Big(d(fx, gy)\Big) \le \phi\Big(\alpha(M(x,y))\Big) \tag{2.1.1}$$

*If there exists a point $x_0 \in X$ such that $O(x_0, gf) \cup O(fx_0, fg)$ is bounded, then $f$ and $g$ have a unique fixed point $z$ in $X$.*

*Moreover, the iteration sequence $\{x_n\}$ with $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ converges to $z$.*

*Proof.* Suppose $O(x_0, gf) \cup O(fx_0, fg)$ is bounded for some $x_0 \in X$ and let $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n = 0, 1, 2, \cdots$. Then for each $n = 0, 1, 2, \cdots$, $x_n \in O(x_0, gf) \cup O(fx_0, fg)$. Hence $\{x_n\}$ is a bounded sequence in $(X, d)$. Let $r_n = \delta\Big(\{x_n, x_{n+1}, x_{n+2}, \cdots\}\Big)$ for $n \in \mathbb{N}$. Then there exists a $r \ge 0$ such that $\lim_{n\to\infty} r_n = r$.

We now show that $r = 0$. Let $l \in \mathbb{N}$ be fixed. From (2.1.1) we have for $n \ge l$

$$\begin{aligned}
&\alpha\Big(d(x_{2n+1}, x_{2n})\Big) \\
&= \alpha\Big(d(fx_{2n}, gx_{2n-1})\Big) \\
&\le \phi\Big(\alpha(M(x_{2n}, x_{2n-1}))\Big) \\
&= \phi\Big(\alpha(max\{d(x_{2n}, x_{2n-1}), d(x_{2n+1}, x_{2n-1}), \\
&\qquad d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n}), d(x_{2n-1}, x_{2n+1})\})\Big) \\
&\le \phi\Big(\alpha(max\{r_{2n-1}, r_{2n-1}, r_{2n-1}, 0, r_{2n-1}\})\Big) \\
&= \phi\Big(\alpha(r_{2l-1})\Big).
\end{aligned} \tag{2.1.2}$$

Taking sup over $n \ge l$ in (2.1.2) we have $\alpha(r_{2l}) \le \phi(\alpha(r_{2l-1}))$. Letting $l \to \infty$, we have $\alpha(r) \le \phi(\alpha(r)) < \alpha(r)$, which is a contradiction if $r \neq 0$. Hence $r = 0$ and so $\lim_{n\to\infty} r_n = 0$. From (2.1.2), we have

$$\alpha\Big(\lim_{n\to\infty} d(x_{2n}, x_{2n+1})\Big) \le \phi\Big(\alpha\Big(\lim_{n\to\infty} r_{2l-1}\Big)\Big) = \phi(\alpha(0)) = 0.$$

Thus $lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$.

Similary, we can show that $\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = 0$. Therefore, we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. We now show that $\{x_n\}$ is Cauchy.
If not, then there exist $\epsilon > 0$ and subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ of $\{x_n\}$ with $m_i > n_i$ such that $d(x_{n_i}, x_{m_i}) \ge 2\epsilon$ for each $i$.

From (2.1.2), we have $d(x_{n_i+1}, x_{n_i}) < \frac{\epsilon}{2}$ and $d(x_{m_i}, x_{m_i-1}) < \frac{\epsilon}{2}$. Then

$$d(x_{n_i+1}, x_{m_i}) \ge d(x_{n_i}, x_{m_i}) - d(x_{n_i+1}, x_{n_i}) > \epsilon$$

and

$$d(x_{n_i}, x_{m_i-1}) \ge d(x_{n_i}, x_{m_i}) - d(x_{m_i-1}, x_{m_i}) > \epsilon$$

which imply

$$d(x_{n_i+1}, x_{m_i-1}) \ge d(x_{n_i}, x_{m_i}) - d(x_{m_i-1}, x_{m_i}) - d(x_{n_i+1}, x_{n_i}) > \epsilon.$$

We can assume that $n_i$ are even numbers and $m_i$ are odd numbers and $d(x_{n_i}, x_{m_i}) \ge \epsilon$ for each $i$.

Let $k_i = \min\Big\{m_i : d(x_{n_i}, x_{m_i}) > \epsilon,\ m_i \text{ are odd number}\Big\}$. Then we have

$$\begin{aligned}
&\epsilon < d(x_{n_i}, x_{k_i}) \\
&\le d(x_{n_i}, x_{k_i-2}) + d(x_{k_i-2}, x_{k_i-1}) + d(x_{k_i-1}, x_{k_i}) \\
&\le \epsilon + d(x_{k_i-2}, x_{k_i-1}) + d(x_{k_i-1}, x_{k_i})
\end{aligned}$$

which implies $\lim_{i\to\infty} d(x_{n_i}, x_{k_i}) = \epsilon$.

We have

$$\begin{aligned} & d(x_{n_i}, x_{k_i}) - d(x_{n_i}, x_{n_i+1}) - d(x_{k_i}, x_{k_i+1}) \\ & \leq d(x_{n_i+1}, x_{k_i+1}) \\ & \leq d(x_{n_i}, x_{x_i}) + d(x_{n_i}, x_{n_i+1}) + d(x_{k_i}, x_{k_i+1}), \end{aligned}$$

which implies $lim_{i\to\infty} d(x_{n_i+1}, x_{k_i+1}) = \epsilon$. From (2.1.1) we have

$$\begin{aligned} & \alpha\Big(d(x_{n_i+1}, x_{k_i+1})\Big) \\ & \leq \phi\Big(\alpha(max\{d(x_{n_i}, x_{k_i}), d(x_{n_i+1}, x_{n_i}), d(x_{k_i+1}, x_{k_i}), d(x_{n_i+1}, x_{k_i}), \\ & \quad d(x_{k_i+1}, x_{n_i})\})\Big) \\ & \leq \phi\Big(\alpha(\max\{d(x_{n_i}, x_{k_i}), d(x_{n_i+1}, x_{n_i}), d(x_{k_i+1}, x_{k_i}), \\ & \quad d(x_{n_i+1}, x_{n_i}) + d(x_{n_i}, x_{k_i}), d(x_{k_i+1}, x_{k_i}) + d(x_{k_i}, x_{n_i})\})\Big). \end{aligned}$$

Taking limit as $n \to \infty$, we have $\alpha(\epsilon) \leq \phi(\alpha(\epsilon)) < \alpha(\epsilon)$ which is a contradiction.

Thus $\{x_n\}$ is Cauchy. By the completeness of $(X, d)$, $\{x_n\}$ converges to some $z$ in $X$.

From (2.1.1), we have

$$\begin{aligned} \alpha\Big(d(fz, x_{2n})\Big) &= \alpha\Big(d(fz, gx_{2n-1})\Big) \\ &\leq \phi\Big(\alpha(M(z, x_{2n-1}))\Big) \\ &= \phi\Big(\alpha(max\{d(z, x_{2n-1}), d(z, fz), d(x_{2n-1}, x_{2n}), \\ &\quad d(z, x_{2n}), d(x_{2n}, fz)\})\Big). \end{aligned} \tag{2.1.3}$$

Taking limit in (2.1.3) as $n \to \infty$, we have $\alpha(d(fz, z)) \leq \phi(\alpha(d(fz, z)))$ which implies that $\alpha(d(fz, z)) = 0$. Hence $d(fz, z) = 0$ or $z = fz$.

Now, we show that $z = gz$. From (2.1.1), we have

$$\begin{aligned} \alpha\Big(d(x_{2n+1}, gz)\Big) &= \alpha\Big(d(fx_{2n}, gz)\Big) \\ &\leq \phi\Big(\alpha(M(x_{2n}, z))\Big) \\ &= \phi\Big(\alpha(max\{d(x_{2n}, z), d(x_{2n}, x_{2n+1}), d(z, gz), \\ &\quad d(x_{2n}, gz), d(z, x_{2n+1})\})\Big). \end{aligned} \tag{2.1.4}$$

Taking limit in (2.1.4) as $n \to \infty$, we have $\alpha(d(z,gz)) \leq \phi(\alpha(d(z,gz)))$ which implies that $\alpha(d(z,gz)) = 0$. Hence $d(z,gz) = 0$ or $z = gz$.

Therefore, $z = fz = gz$, that is $z$ is a common fixed point of $f$ and $g$. For the uniqueness, let $z$ and $w$ be common fixed points of $f$ and $g$.

From (2.1.1), we have

$$\begin{aligned}
\alpha\Big(d(z,w)\Big) &= \alpha\Big(d(fz,gw)\Big) \\
&\leq \phi\Big(\alpha(M(z,w))\Big) \\
&= \phi\Big(\alpha(\max\{d(z,w), d(z,fz), d(w,gw), d(z,gw), d(w,fz)\})\Big) \\
&= \phi\Big(\alpha(\max\{d(z,w), d(z,z), d(w,w), d(z,w), d(w,z)\})\Big) \\
&= \phi\Big(\alpha(d(z,w))\Big)
\end{aligned}$$

which implies that $\alpha(d(z,w)) = 0$. Hence $d(z,w) = 0$ or $z = w$. □

The following example show that if we don't have the condition of which $O(x_0, gf) \cup O(fx_0, fg)$ is bounded for some $x_0 \in X$, then Theorem 2.1 does not hold. So we have to have the above condition.

**Example.** Let $f, g : \mathbb{N} \to \mathbb{N}$ be defined by $fn = gn = n+1$ and let $\alpha(t) = 2^t - 1$ and $\phi(t) = \frac{1}{2}t$ for $t \geq 0$. Then $\alpha \in \Lambda$ and $\phi \in \Phi$.

For $m > n$ we have

$$\alpha(d(n,gm)) = 2^{m-n+1} - 1 = 2^{m-n}2 - 1 \geq 2(2^{m-n} - 1) = 2\alpha(d(fn,gm)).$$

Hence we have

$$\begin{aligned}
\alpha(d(fn,gm)) &= (2^{m-n} - 1) \\
&\leq \frac{1}{2}(2^{m-n+1} - 1) \\
&= \phi\Big(\alpha(d(n,gm))\Big) \\
&\leq \phi\Big(\alpha(max\{d(n,m), d(n,fn), d(m,gm), d(m,fn), d(n,gm)\})\Big).
\end{aligned}$$

Thus (2.1.1) is satisfied. But the orbits are not bounded and $f$ and $g$ have no common fixed points.

**Corollary 2.2.** *Let $f$ and $g$ be self maps of a complete metric space $(X,d)$ and $\phi \in \Phi$ satisfying: for each $x, y \in X$,*

$$\int_0^{d(fx,gy)} \varphi(s)ds \leq \phi\left(\int_0^{M(x,y)} \varphi(s)ds\right)$$

*where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is nonnegative, Lebesque integrable map which is summable and for each* $\epsilon > 0$, $\int_0^{\epsilon} \varphi(s)ds > 0$.

*If there exists a point* $x_0 \in X$ *such that* $O(x_0, gf) \cup O(fx_0, fg)$ *is bounded, then* $f$ *and* $g$ *have a unique fixed point* $z$ *in* $X$.

*Moreover, the iteration sequence* $x_n$ *with* $x_{2n+1} = fx_{2n}$ *and* $x_{2n+2} = gx_{2n+1}$ *converges to* $z$.

Taking $\phi(t) = kt$ for $k \in [0, 1)$ in Theorem 2.1 and Corollary 2.2, we have the next corollaries.

**Corollary 2.3.** *Let* $f$ *and* $g$ *be self maps of a complete metric space* $(X, d)$ *and* $\alpha \in \Lambda$ *satisfying: there exists* $k \in [0, 1)$ *such that for each* $x, y \in X$,

$$\alpha\Big(d(fx, gy)\Big) \leq k\alpha\Big(M(x, y)\Big)$$

*If there exists a point* $x_0 \in X$ *such that* $O(x_0, gf) \cup O(fx_0, fg)$ *is bounded, then* $f$ *and* $g$ *have a unique fixed point* $z$ *in* $X$.

*Moreover, the iteration sequence* $x_n$ *with* $x_{2n+1} = fx_{2n}$ *and* $x_{2n+2} = gx_{2n+1}$ *converges to* $z$.

**Corollary 2.4.** *Let* $f$ *and* $g$ *be self maps of a complete metric space* $(X, d)$ *satisfying: there exists* $k \in [0, 1)$ *such that for each* $x, y \in X$,

$$\int_0^{d(fx,gy)} \varphi(s)ds \leq k \int_0^{M(x,y)} \varphi(s)ds$$

*If there exists a point* $x_0 \in X$ *such that* $O(x_0, gf) \cup O(fx_0, fg)$ *is bounded, then* $f$ *and* $g$ *have a unique fixed point* $z$ *in* $X$.

*Moreover, the iteration sequence* $x_n$ *with* $x_{2n+1} = fx_{2n}$ *and* $x_{2n+2} = gx_{2n+1}$ *converges to* $z$.

**Remark 2.4.** Let $f = g$ and $\alpha(u) = \int_0^u \varphi(t)dt$ in Corollary 2.3. Then we have Theorem 4 of [3], where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable, nonnegative and $\int_0^s \varphi(t)dt > 0$ for each $s > 0$.

## References


1. A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integal type* , Int. J. Math. Math. Sci. **29(9)**(2002), 532-536.
2. B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226**(1977), 257-290.

3. B. E. Rhoades, *Two fixed point theorems for mappings satisfying a general contractive condition of integal type* , Int. J. Math. Math. Sci. **2003(63)**(2003), 4007-4013.
4. P. Vijayaraju, B. E. Rhoades and R. Mohanraj, *A fixed point theorem for a pair of maps satisfying a general contractive condition of integal type*, Int. J. Math. Math. Sci. **2005(15)**(2005), 2359-2364.
5. Xian Zhang, *Common fixed point theorems for some new generalized contractive type mappings*, J. Math. Anal. Appl. **333**(2007), 780-786.

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