

A NEW APPROACH FOR RANKING FUZZY NUMBERS BASED ON α -CUTS

HADI BASIRZADEH* AND ROOHOLLAH ABBASI

ABSTRACT. Comparison between two or more fuzzy numbers, along with their ranking, is an important subject discussed in scholarly articles. We endeavor in this paper to present a simple yet effective parametric method for comparing fuzzy numbers. This method offer significant advantages over similar methods, in comparing intersected fuzzy numbers, rendering the comparison between fuzzy numbers possible in different decision levels. In the process, each fuzzy number will be given a parametric value in terms of α , which is dependent on the related α -cuts. We have compared this method to Cheng's centroid point method [5] (The relation of calculating centroid point of a fuzzy number was corrected later on by Wang [12]). The proposed method can be utilized for all types of fuzzy numbers whether normal, abnormal or negative.

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1. Introduction

An important topic, put forward immediately after the definition of fuzzy numbers, is the order of, and comparison between, fuzzy numbers. This is particularly important in the theory of fuzzy decision making. As a matter of fact, the ranking of fuzzy numbers is not as easy as that of scalar numbers. Different methods have been presented for ranking fuzzy numbers and various articles have been written on the subject. See for example [9], [10] and [11] for the history of these methods.

A considerable number of methods presented so far for ranking fuzzy numbers do not sound flawless. Some of them for example, have limitations, are difficult in calculation, or they are non-intuitive, which makes them inefficient in practical applications, especially in the process of decision making. However, in some of these methods, as ones in which fuzzy numbers are compared according to their

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centroid point (see [5], [6], [12]), the decision maker does not play any role in the comparison between fuzzy numbers. Nevertheless, there are certain methods in which fuzzy numbers are compared in a parametric manner (see e.g. [7], [10]).

Fuzzy and the nature of uncertainty is not always attributed to the inaccurate statistical information, but these conditions mainly occur in practice when we model linguistic expressions.

For this reason, when two fuzzy numbers are compared, it is quite natural that the result of the comparison would either be fuzzy or, at least, parametric, due to its subjective and interpretive nature.

This can also be seen in the evolution of operators in the theory of fuzzy sets. It is specifically seen in the theory of fuzzy decision that the parametric operators act better than non-parametric operators in case of experimental data [13]. Two factors play significant roles in fuzzy decision systems:

- (1) Contribution of the decision-maker in the decision making process,
- (2) Simplicity of calculation.

This paper attempts to propose a method for ranking and comparing fuzzy numbers to account for the above-mentioned factors as much as possible. The proposed method has also been compared with centroid point method (taking account of Wang's correction [12]).

2. Definition of an arbitrary fuzzy number

A fuzzy number has been defined in various forms. We appropriately employ the following definition of a fuzzy number [1], [4]: We present an arbitrary fuzzy number \tilde{A} by an ordered pair of functions $(\underline{A}(r), \overline{A}(r))$, where $0 \leq r \leq \omega$ and ω is an arbitrary constant between zero and one ($0 \leq \omega \leq 1$), in a parametric form which satisfies the following requirements:

- (1) $\underline{A}(r)$ is a bounded left continuous non-decreasing function over $[0, \omega]$.
- (2) $\overline{A}(r)$ is a bounded left continuous non-increasing function over $[0, \omega]$.
- (3) $\underline{A}(r) \leq \overline{A}(r)$, $0 \leq r \leq \omega$.

A crisp number "k" is simply represented by $\underline{A}(r) = \overline{A}(r) = k$, $0 \leq r \leq \omega$.

By appropriate definitions, the fuzzy numbers space $\{\underline{A}(r), \overline{A}(r)\}$ becomes a convex cone E^1 which is embedded isomorphically and isometrically in a Banach space. If \tilde{A} be an arbitrary fuzzy number then the α -cut of \tilde{A} is $[\tilde{A}]_\alpha = [\underline{A}(\alpha), \overline{A}(\alpha)]$, $0 \leq \alpha \leq \omega$.

If $\omega = 1$, then the above-defined number is called a normal fuzzy number.

fig.1 represents an arbitrary fuzzy number.

Here \tilde{A}_ω represents a fuzzy number in which " ω " is the maximum membership value that a fuzzy number takes on. Whenever a normal fuzzy number is meant, the fuzzy number is shown by \tilde{A} , for convenience.

3. A new approach for ranking fuzzy numbers

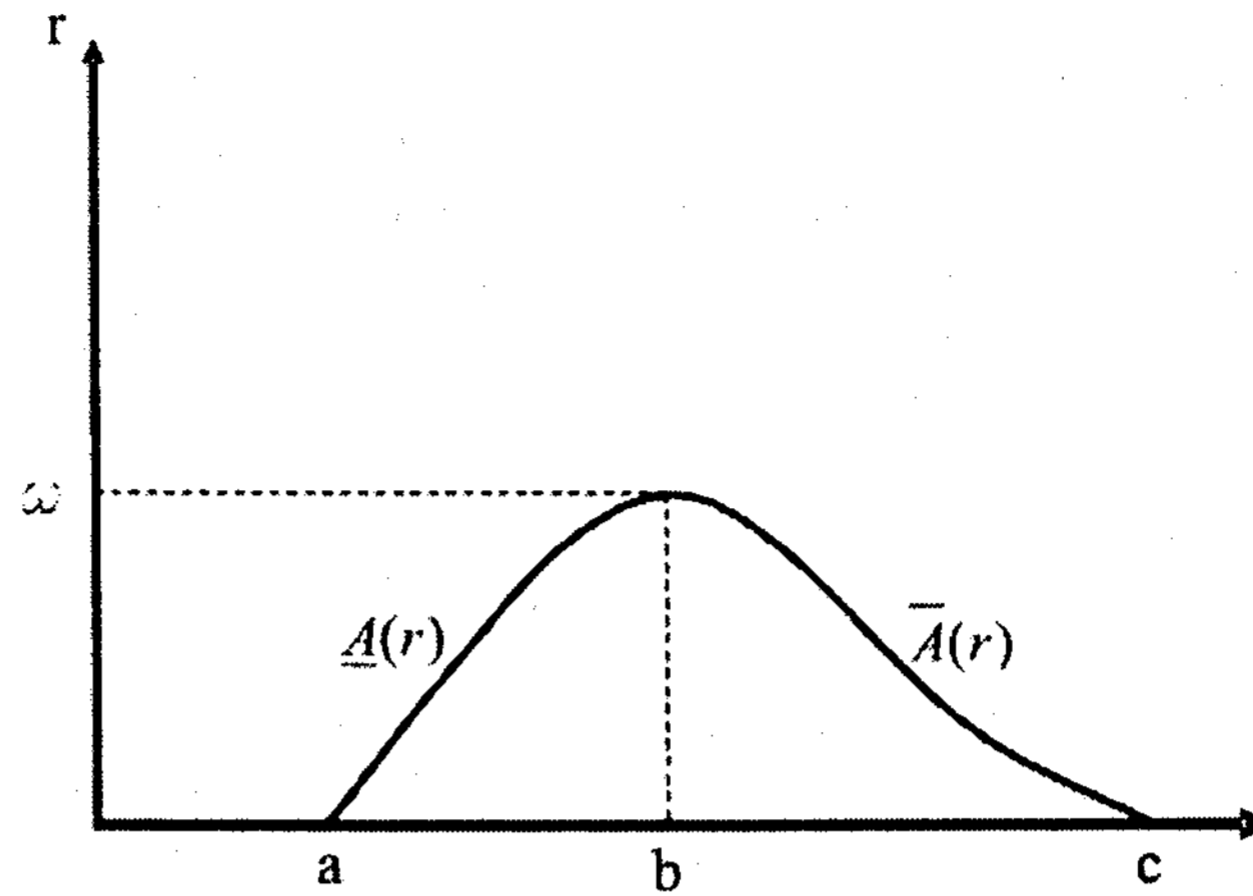


FIGURE 1. an arbitrary Fuzzy number ($0 \leq \omega \leq 1$)

As mentioned earlier it seems that parametric methods of comparing fuzzy numbers, especially in fuzzy decision making theory, are more efficient than non-parametric methods. For example, in Cheng's centroid point method [5], fuzzy numbers are compared according to their Euclidean distances from the origin (see fig.2).

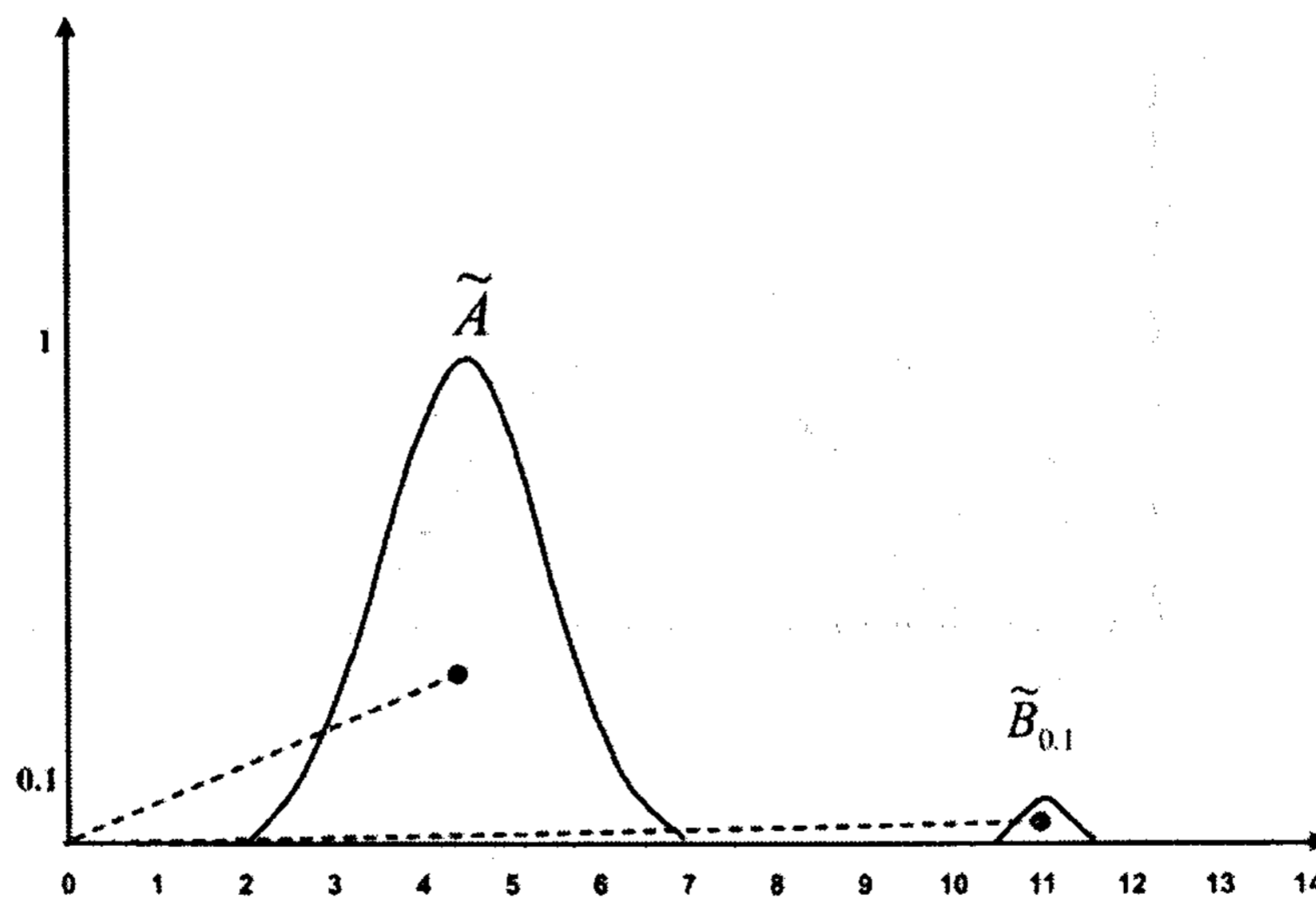


FIGURE 2. distances of centroid points of \tilde{A} and $\tilde{B}_{0.1}$ from the origin.

fig.2 represents two fuzzy numbers \tilde{A} and $\tilde{B}_{0.1}$. According to Cheng's centroid point method $\tilde{A} < \tilde{B}_{0.1}$, but is this comparison practically correct?

If in a decision making process membership values over 0.1 were significant, would $\tilde{B}_{0.1}$ have any member with this membership value?

Negative fuzzy numbers in Cheng's centroid point method were not compared. However some time later Chu and Tsao [6] tried to solve this problem using the area between the centroid point to the origin. But their study was not flawless either. Abbasbandy and Assady [4] found that Chu and Tsao's area method occasionally causes non-intuitive ranking. They presented the sign distance method. But their method was non-parametric and only was applicable for normal fuzzy numbers.

It's clearly seen that non-parametric methods for comparing fuzzy numbers have some drawbacks in practice.

According to the above-mentioned definition of a fuzzy number, let $\tilde{A}_\omega = (\underline{A}(r), \bar{A}(r))$, $(0 \leq r \leq \omega)$ be a fuzzy number, then the value $Q_\alpha(\tilde{A}_\omega)$, is assigned to \tilde{A}_ω for a decision level higher than " α " which is calculated as follows:

$$Q_\alpha(\tilde{A}_\omega) = \int_\alpha^\omega \{\underline{A}(r) + \bar{A}(r)\} dr, \text{ where } 0 \leq \alpha < 1$$

This quantity will be used as a basis for comparing fuzzy numbers in decision level higher than α .

It is clear that if $\alpha \geq \omega$, then $Q_\alpha(\tilde{A}_\omega) = 0$. In order to clarify the concept of the above-mentioned quantity, consider the following fuzzy number:

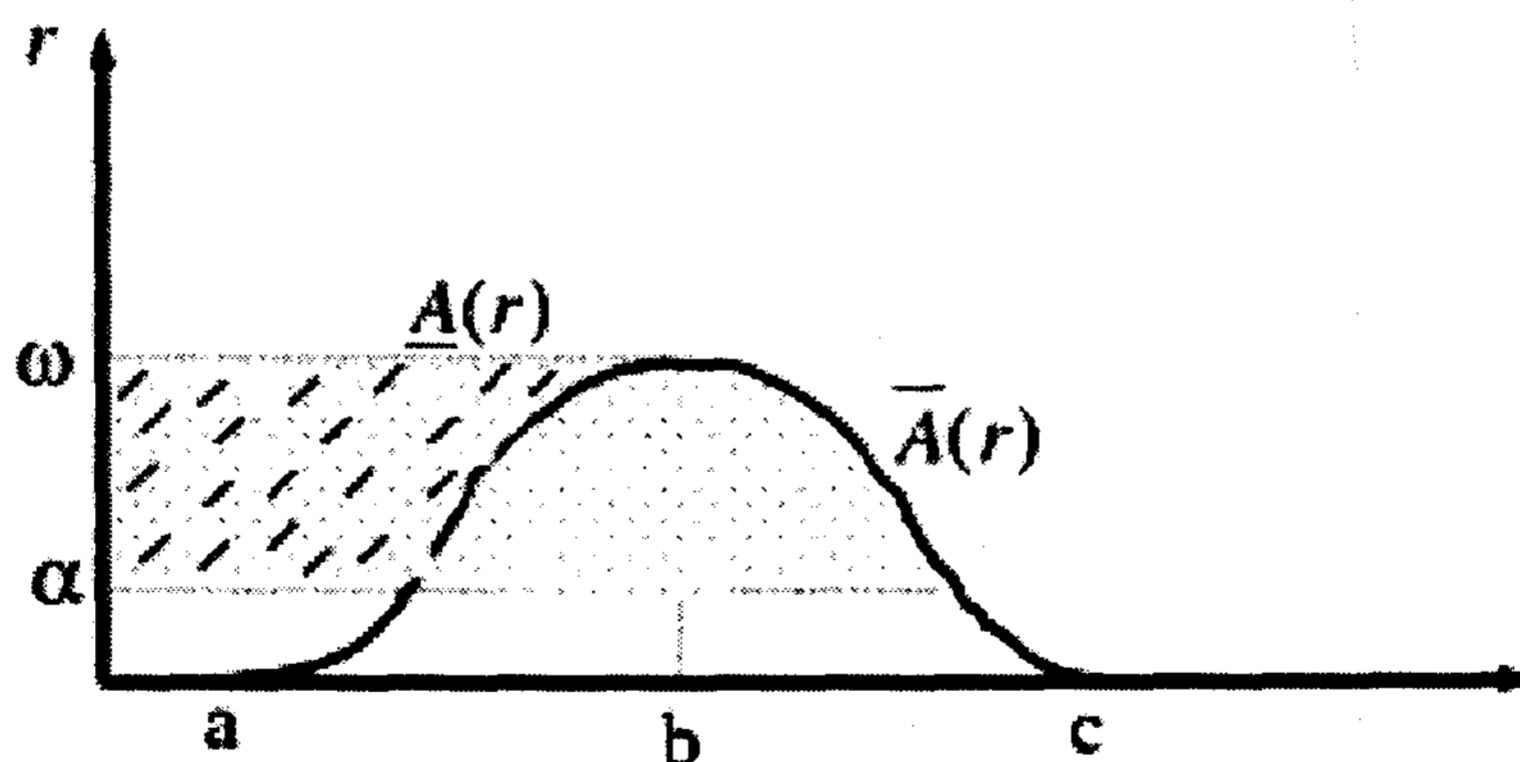


FIGURE 3. $Q_\alpha(\tilde{A}_\omega)$ Quantity

As shown in fig.3, the presented quantity is the summation of the dotted area and the cross-hatched area.

$$Q_\alpha(\tilde{A}_\omega) = \int_\alpha^\omega \{\underline{A}(r) + \bar{A}(r)\} dr = \int_\alpha^\omega \underline{A}(r) dr + \int_\alpha^\omega \bar{A}(r) dr$$

(cross-hatched area)(dotted area)

Definition 1. If \tilde{A}_ω and \tilde{B}_ω are two arbitrary fuzzy numbers and $\omega, \omega' \in [0, 1]$, then we have:

- (1) $\tilde{A}_\omega \leq \tilde{B}'_\omega \iff \forall \alpha \in [0, 1] Q_\alpha(\tilde{A}_\omega) \leq Q_\alpha(\tilde{B}'_\omega)$
- (2) $\tilde{A}_\omega = \tilde{B}'_\omega \iff \forall \alpha \in [0, 1] Q_\alpha(\tilde{A}_\omega) = Q_\alpha(\tilde{B}'_\omega)$
- (3) $\tilde{A}_\omega \geq \tilde{B}'_\omega \iff \forall \alpha \in [0, 1] Q_\alpha(\tilde{A}_\omega) \geq Q_\alpha(\tilde{B}'_\omega)$

Definition 2. If we compare two arbitrary fuzzy numbers including \tilde{A}_ω and \tilde{B}_ω at decision levels higher than " α " and $\alpha, \omega, \omega' \in [0, 1]$, then we have:

- (1) $\tilde{A}_\omega \leq_\alpha \tilde{B}_\omega \iff Q_\alpha(\tilde{A}_\omega) \leq Q_\alpha(\tilde{B}_\omega)$
- (2) $\tilde{A}_\omega =_\alpha \tilde{B}_\omega \iff Q_\alpha(\tilde{A}_\omega) = Q_\alpha(\tilde{B}_\omega)$
- (3) $\tilde{A}_\omega \geq_\alpha \tilde{B}_\omega \iff Q_\alpha(\tilde{A}_\omega) \geq_\alpha Q_\alpha(\tilde{B}_\omega)$

where $\tilde{A}_\omega \leq_\alpha \tilde{B}_\omega$, i.e., at decision levels higher than α , \tilde{B}_ω is greater than or equal to \tilde{A}_ω .

If α is close to one, the pertaining decision is called a "high level decision", in which case only parts of the two fuzzy numbers, with membership values between " α " and "1", will be compared. Likewise, if " α " is close to zero, the pertaining decision is referred to as a "low level decision", since members with membership values lower than both the fuzzy numbers are involved in the comparison. For instance, as shown in Fig.4, according to the presented quantity, the results clearly vary with different decision levels, e.g. $\tilde{A} \leq_{0.8} \tilde{B}$, $\tilde{A} \geq_{0.1} \tilde{B}$:

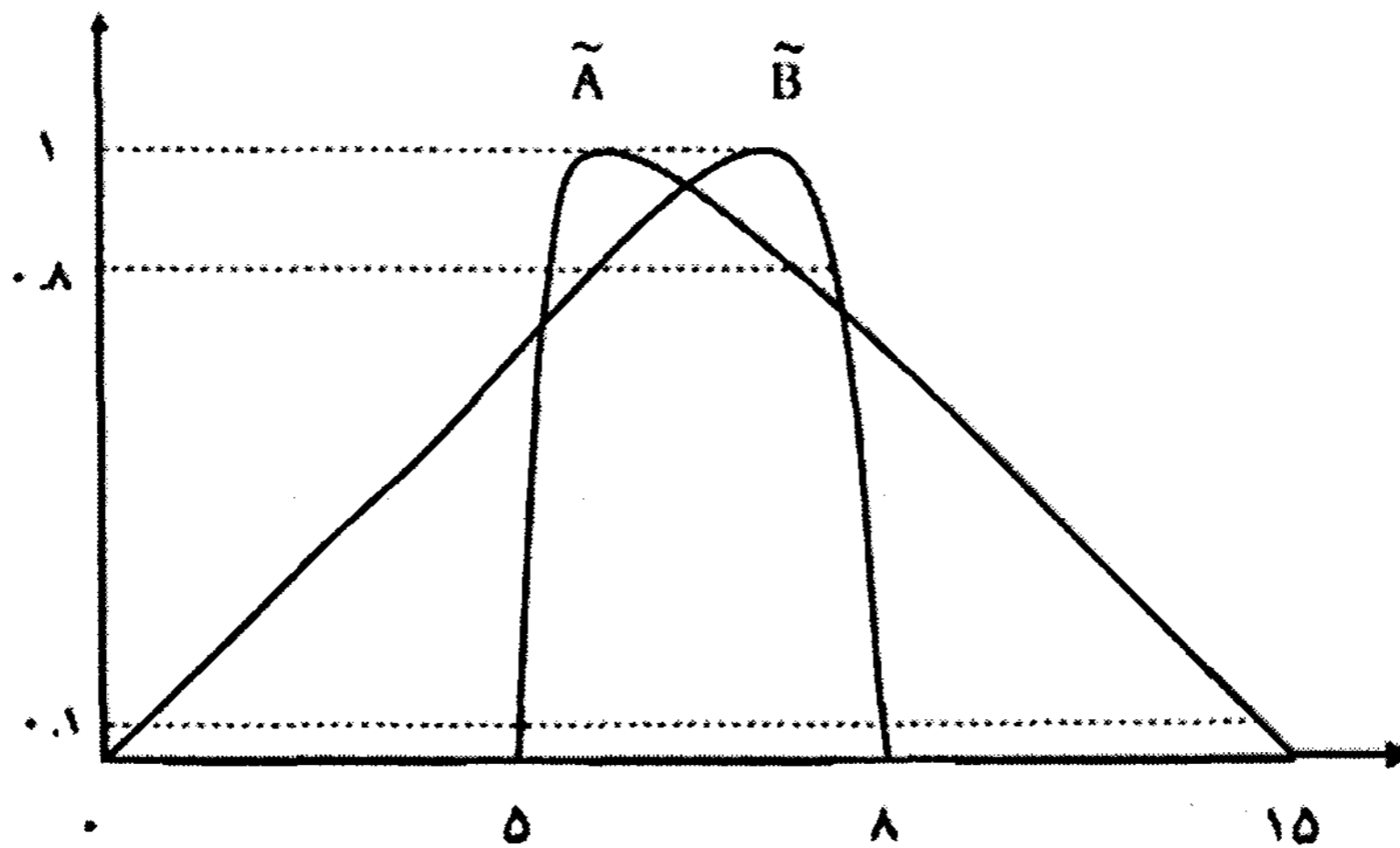


FIGURE 4. Comparison of \tilde{A} and \tilde{B} at two different decision levels

Two relevant classes of fuzzy numbers, which are frequently used in practical purposes and are rather easy to work with, are "triangular and trapezoidal fuzzy numbers", as shown in Fig.5a and Fig.5b. some methods of approximating a

fuzzy number with a trapezoidal fuzzy number, have also been presented (see e.g. [1], [8]) and hence there is no concern in this respect.

Any arbitrary fuzzy number may be approximated with a trapezoidal fuzzy number, before being used.

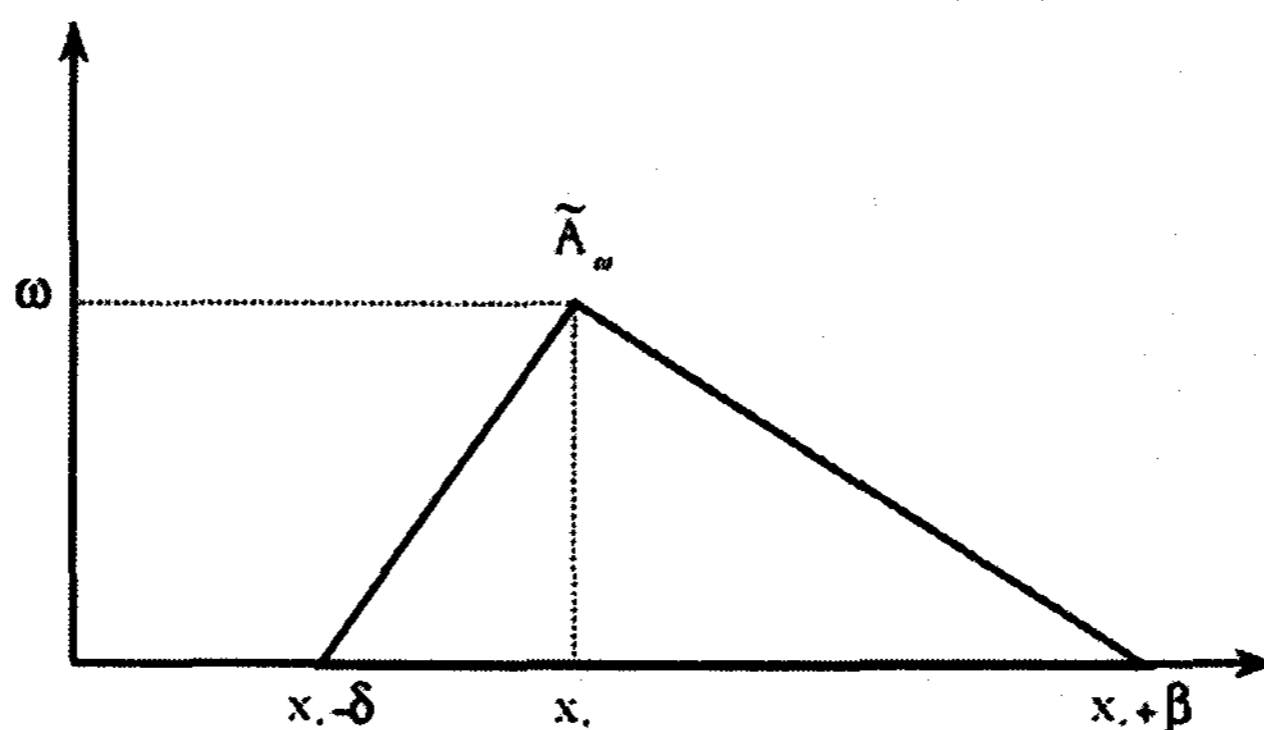


FIGURE 5. A triangular fuzzy number

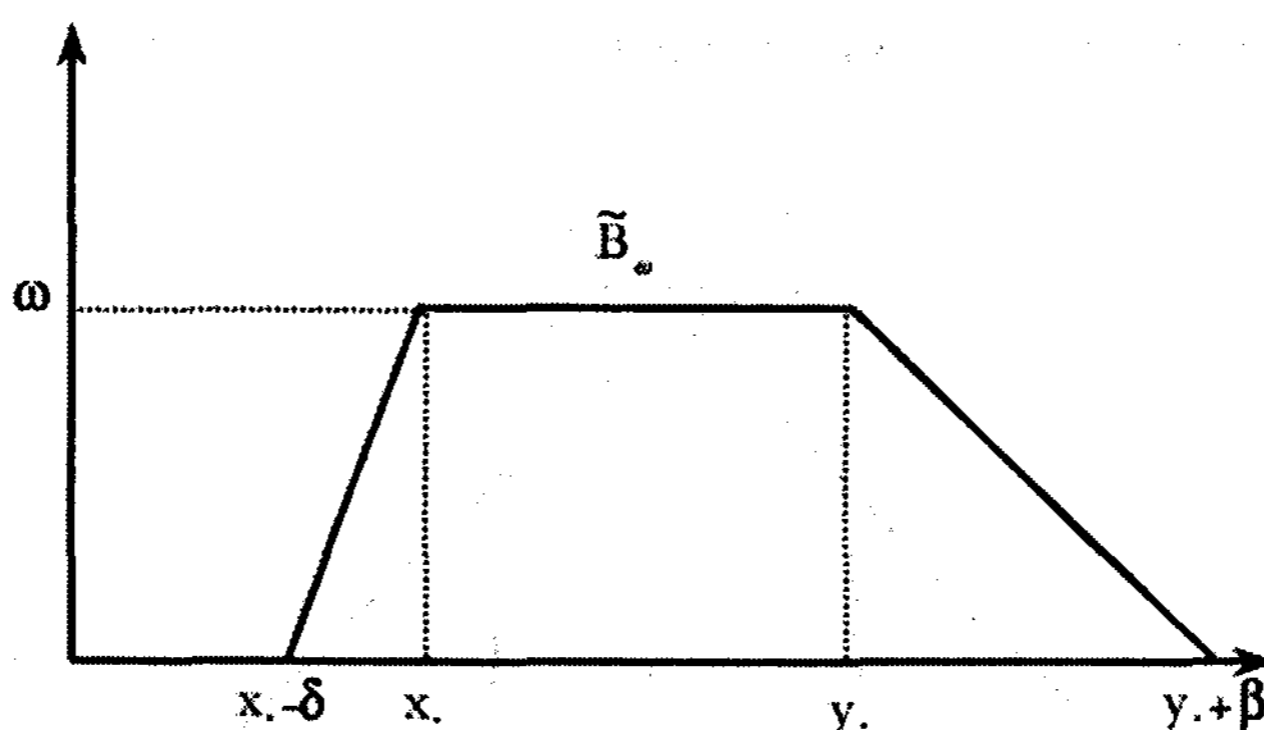


FIGURE 6. A trapezoidal fuzzy number

As shown in Fig.5a and Fig.5b, $\tilde{A}_\omega = (\underline{A}(r), \overline{A}(r)) = (x_0 - \delta + \frac{\delta}{\omega}r, x_0 + \beta - \frac{\beta}{\omega}r)$ and $\tilde{B}_\omega = (\underline{B}(r), \overline{B}(r)) = (x_0 - \delta + \frac{\delta}{\omega}r, y_0 + \beta - \frac{\beta}{\omega}r)$ are triangular and trapezoidal fuzzy numbers, respectively. The parametric values assigned to the two fuzzy numbers, represented by $Q_\alpha^{Tri}(\tilde{A}_\omega)$ and $Q_\alpha^{Tra}(\tilde{B}_\omega)$ respectively, may be calculated as follows:

If $\omega > \alpha$, then:

$$Q_\alpha^{Tri}(\tilde{A}_\omega) = \int_\alpha^\omega \{\underline{A}(r) + \overline{A}(r)\} dr =$$

$$\int_{\alpha}^{\omega} \left(2x_0 + (\beta - \delta) \left(1 - \frac{r}{\omega} \right) \right) dr = 2x_0[\omega - \alpha] + \frac{(\beta - \delta)}{2\omega} (\omega - \alpha)^2$$

where the value corresponding to the triangular fuzzy number \tilde{A}_ω pertains to a decision level higher than α .

$$Q_{\alpha}^{Tra}(\tilde{B}_\omega) = \int_{\alpha}^{\omega} \{ \underline{B}(r) + \overline{B}(r) \} dr = \int_{\alpha}^{\omega} \left\{ x_0 + y_0 + (\beta - \delta) \left(1 - \frac{r}{\omega} \right) \right\} dr = (x_0 + y_0)(\omega - \alpha) + \frac{(\beta - \delta)}{2\omega} (\omega - \alpha)^2$$

where the value corresponding to the trapezoidal fuzzy number \tilde{B}_ω pertains to a decision level higher than α .

Obviously, if $\alpha \geq \omega$, then the above quantity will be zero. It can also be seen that if \tilde{A} is a normal triangular or trapezoidal fuzzy number ($\omega = 1$) the above quantities reduce to:

$$Q_{\alpha}^{Tri}(\tilde{A}) = 2x_0[1 - \alpha] + \frac{(\beta - \delta)}{2} (1 - \alpha)^2$$

$$Q_{\alpha}^{Tra}(\tilde{B}) = (x_0 + y_0)(1 - \alpha) + \frac{(\beta - \delta)}{2} (1 - \alpha)^2, 0 \leq \alpha < 1.$$

As the above relation show, if the fuzzy number is symmetrical ($\delta = \beta$), the relations may be simplified more (the second terms on the right-hand side of the above equations are canceled out). For simplicity, based on the results obtained, hereafter the triangular fuzzy number \tilde{A} and the trapezoidal fuzzy number \tilde{B} will be represented as $\tilde{A} = (x_0, \delta, \beta, \omega)$ and $\tilde{B} = (x_0, y_0, \delta, \beta, \omega)$ respectively. The following examples may be helpful to clarify the proposed method:

Example 1. Assume that $\tilde{A} = (6, 2, 6, 1)$ and $\tilde{B} = (7, 6, 2, 1)$ are two triangular fuzzy numbers(see Fig.6). According to the proposed method, we have:

$$Q_{\alpha}^{Tri}(\tilde{A}) = 12(1 - \alpha) + 2(1 - \alpha)^2$$

$$Q_{\alpha}^{Tri}(\tilde{B}) = 14(1 - \alpha) - 2(1 - \alpha)^2$$

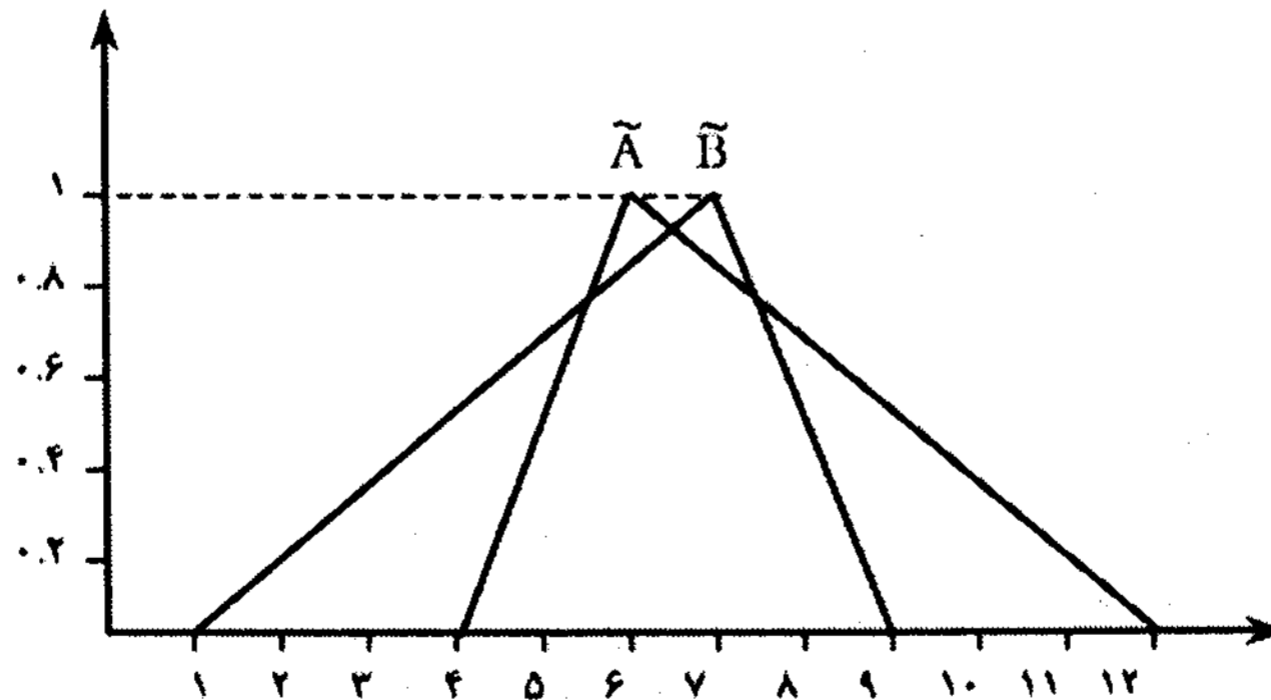


FIGURE 7. \tilde{A} and \tilde{B}

$\alpha \setminus$ F.N	\tilde{A}	\tilde{B}	Ranking
0.1	12.42	10.98	$\tilde{A} >_{0.1} \tilde{B}$
0.5	6.5	6.5	$\tilde{A} =_{0.5} \tilde{B}$
0.8	2.48	2.72	$\tilde{A} <_{0.8} \tilde{B}$

Table 1

As shown in the table, the result of comparing the two fuzzy numbers \tilde{A} and \tilde{B} varies with different values of α . The same results can also be seen in the method of ranking fuzzy numbers presented by Detyniecki and Yager [7] if proper parameters are selected.

However in the centroid point method, centroid points of \tilde{A} and \tilde{B} will be (7.33,0.33) and (5.67,0.33) respectively; therefore the Euclidean distances from the centroid point of \tilde{A} to the origin and from the centroid point of \tilde{B} to the origin, are 7.34 and 5.68 respectively, i.e. in the centroid point method, always: $\tilde{B} < \tilde{A}$

Example 2.

$$\tilde{A} = (5, 2, 2, 1), \quad \tilde{B} = (5, 2, 2, 0.8), \quad \tilde{C} = (7, 9, 2, 1, 1), \quad \tilde{D} = (8, 9, 1, 1, 0.4).$$

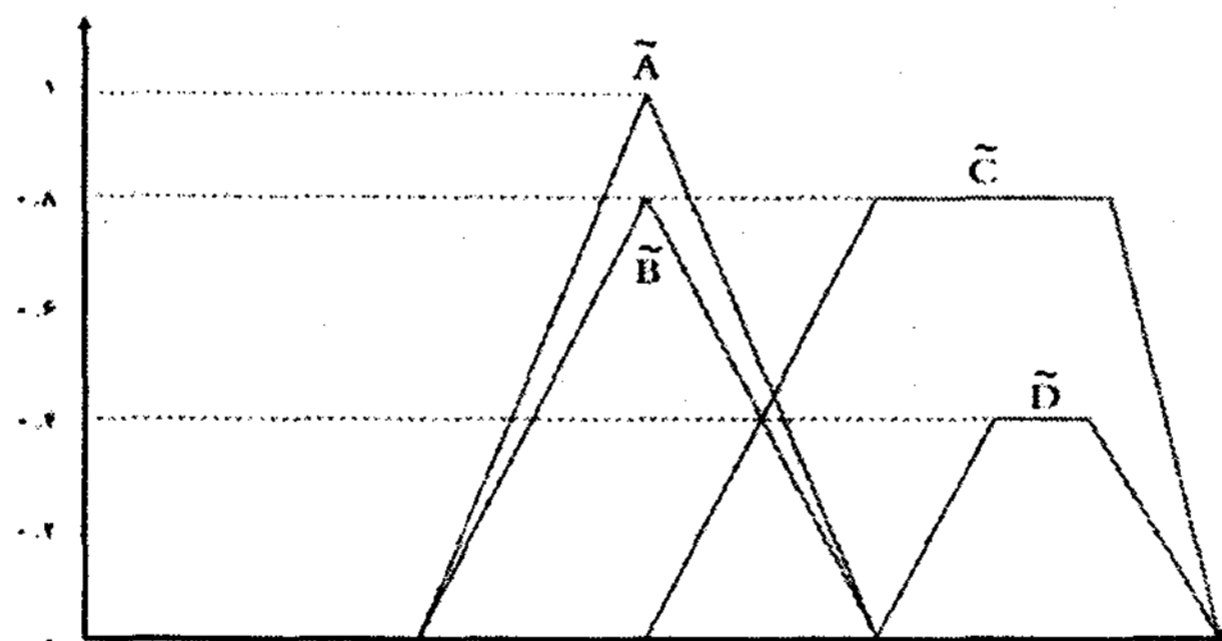


FIGURE 8

A diagram such as that shown in Fig.7, has been given in [5]. According to the presented method, we have:

$$Q_{\alpha}^{Tri}(\tilde{A}) = 10(1 - \alpha), 0 \leq \alpha < 1$$

$$Q_{\alpha}^{Tri}(\tilde{B}) = 10(0.8 - \alpha) \text{ if } 0 \leq \alpha < 0.8 \text{ and}$$

$$Q_{\alpha}^{Tri}(\tilde{B}) = 0 \text{ if } 0.8 \leq \alpha < 1.$$

It can be seen that for each value of α ($0 \leq \alpha < 1$), $Q_{\alpha}^{Tri}(\tilde{B}) < Q_{\alpha}^{Tri}(\tilde{A})$, hence always: $\tilde{B} < \tilde{A}$.

This relation is also the case for Cheng's centroid point method ($\tilde{B} < \tilde{A}$). To compare \tilde{C} and \tilde{D} , we have:

$$Q_{\alpha}^{Tra}(\tilde{C}) = 16(1 - \alpha) - 0.5(1 - \alpha)^2, \quad Q_{\alpha}^{Tra}(\tilde{D}) = 17(0.4 - \alpha).$$

It can be seen that for each value of α ($0 \leq \alpha < 1$), $Q_{\alpha}^{Tra}(\tilde{D}) < Q_{\alpha}^{Tra}(\tilde{C})$ hence always: $\tilde{D} < \tilde{C}$. However, in the centroid point method, $\tilde{C} < \tilde{D}$. consequently, in this case the result is contrary to the centroid point method. For example, as the figure implies, if \tilde{C} is compared to \tilde{D} at a decision level higher than 0.5, \tilde{D} does not have a member with a membership value higher than 0.5.

Example 3.

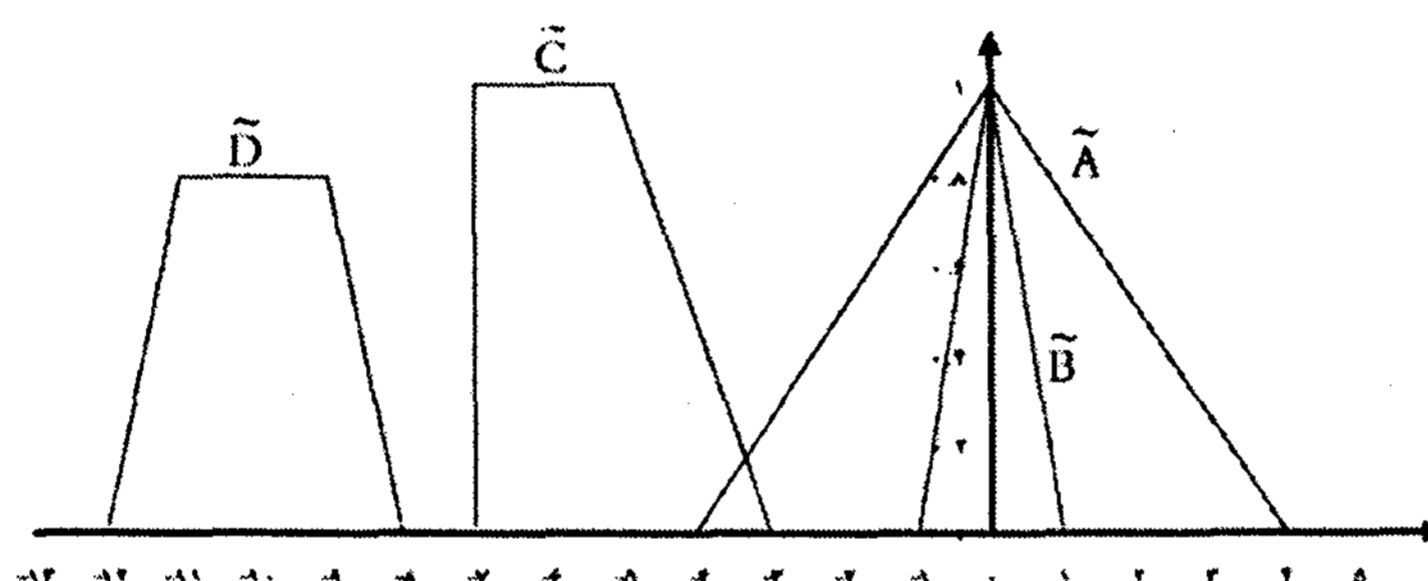


FIGURE 9

$$\tilde{A} = (0, 4, 4, 1), \tilde{B} = (0, 1, 1, 1), \tilde{C} = (-7, -5, 0, 2, 1), \tilde{D} = (-11, -9, 1, 1, 0.8)$$

According to the proposed method, it can be concluded that always $\tilde{A} = \tilde{B}$ ($Q_{\alpha}(\tilde{A}) = Q_{\alpha}(\tilde{B}) = 0$), which is similar to the result obtained using the centroid point method. For fuzzy numbers \tilde{C} and \tilde{D} we have:

$$Q_{\alpha}(\tilde{C}) = -12(1 - \alpha) + (1 - \alpha)^2, \quad Q_{\alpha}(\tilde{D}) = -20(0.8 - \alpha)$$

It can also be seen that for decision levels higher than 0.7, \tilde{D} is greater than \tilde{C} ($\tilde{C} <_{0.7} \tilde{D}$). However, for a decision level higher than 0.5, \tilde{C} is greater than \tilde{D} ($\tilde{C} >_{0.5} \tilde{D}$). In the centroid point method we have: $\tilde{C} < \tilde{D}$! and the method which that presented by Abbasbandy and Asady is for normal fuzzy numbers only, and is not applicable for this example.

Example 4. The following fuzzy numbers are to be ranked for different decision levels:

$$\tilde{A} = (1, 0.5, 0.5, 0.6), \tilde{B} = (3, 1, 1, 0.6), \tilde{C} = (6, 2, 6, 0.4), \tilde{D} = (1.5, 2, 0, 1, 1)$$

$$\tilde{E} = (3, 4, 3, 1, 0.8), \tilde{F} = (-2, 1, 1, 1), \tilde{G} = (-4, -2, 0, 1, 0.6), \tilde{H} = (-8, -4, 1, 1, 0.5)$$

The parametric values corresponding to each fuzzy number in Fig.9 are as follows:

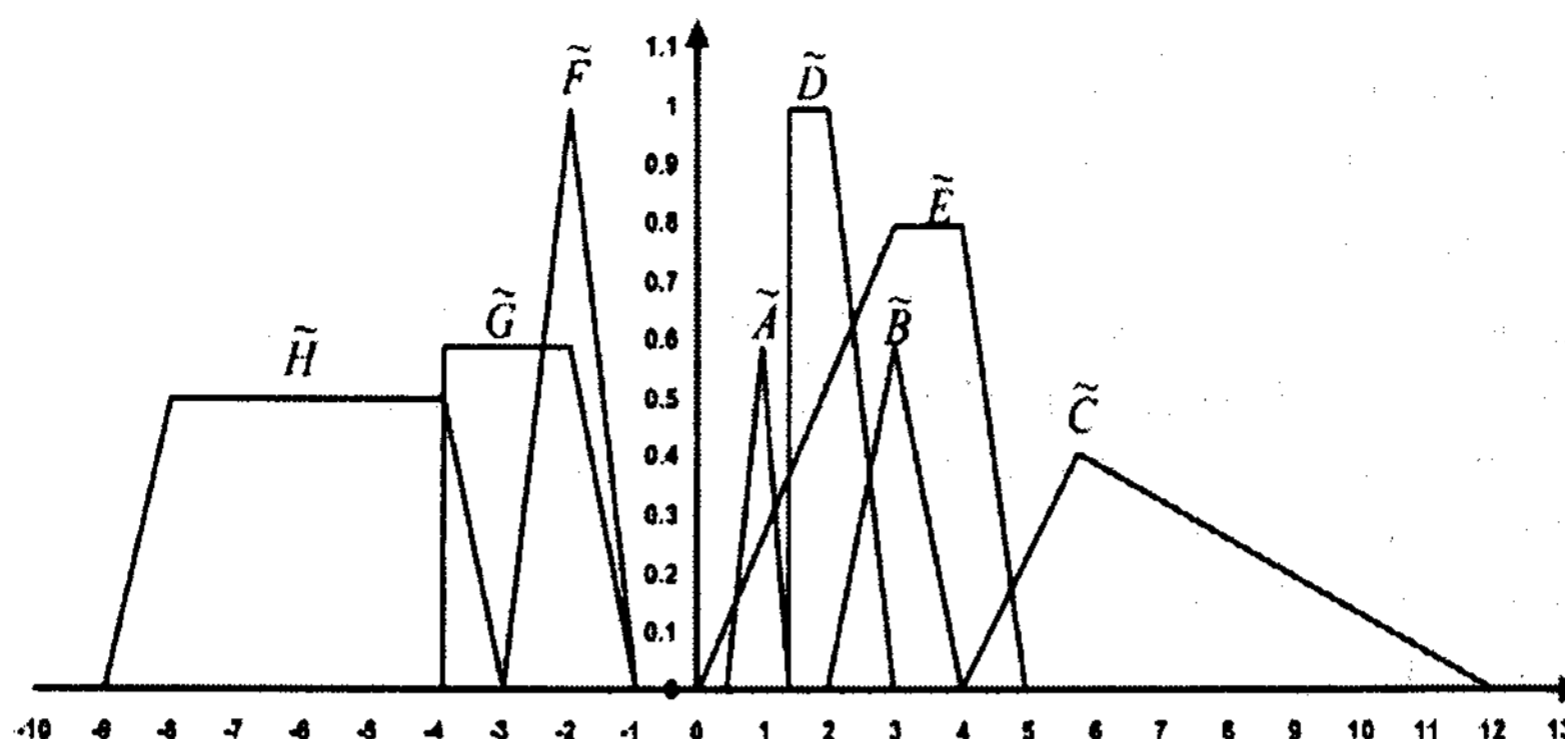


FIGURE 10

$$\begin{aligned}
 Q_\alpha(\tilde{A}) &= 2(0.6 - \alpha), & Q_\alpha(\tilde{B}) &= 6(0.6 - \alpha), & Q_\alpha(\tilde{C}) &= 12(0.4 - \alpha) + 5(0.4 - \alpha)^2, \\
 Q_\alpha(\tilde{D}) &= 3.5(1 - \alpha) + 0.5(1 - \alpha)^2, & Q_\alpha(\tilde{E}) &= 7(0.8 - \alpha) - 1.25(0.8 - \alpha)^2, \\
 Q_\alpha(\tilde{F}) &= -4(1 - \alpha), & Q_\alpha(\tilde{G}) &= -6(0.6 - \alpha) + 0.83(0.6 - \alpha)^2, & Q_\alpha(\tilde{H}) &= \\
 & & & & -12(0.5 - \alpha).
 \end{aligned}$$

To compare fuzzy numbers in each decision level of interest, we can simply plot Q_α for each fuzzy number, in the following system:

As shown in Fig.10 we have:

α	Ranking fuzzy numbers
0.1	$\tilde{H} < \tilde{F} < \tilde{G} < \tilde{A} < \tilde{B} < \tilde{D} < \tilde{C} < \tilde{E}$
0.2	$\tilde{H} < \tilde{F} < \tilde{G} < \tilde{A} < \tilde{B} < \tilde{C} < \tilde{D} < \tilde{E}$
0.3	$\tilde{F} < \tilde{H} < \tilde{G} < \tilde{A} < \tilde{C} < \tilde{B} < \tilde{D} < \tilde{E}$
0.4	$\tilde{F} < \tilde{H} < \tilde{G} < \tilde{C} < \tilde{A} < \tilde{B} < \tilde{D} < \tilde{E}$
0.5	$\tilde{F} < \tilde{G} < \tilde{H} = \tilde{C} < \tilde{A} < \tilde{B} < \tilde{D} < \tilde{E}$
0.6	$\tilde{F} < \tilde{G} = \tilde{H} = \tilde{C} = \tilde{A} = \tilde{B} < \tilde{E} < \tilde{D}$
0.7	$\tilde{F} < \tilde{G} = \tilde{H} = \tilde{C} = \tilde{A} = \tilde{B} < \tilde{E} < \tilde{D}$
0.8	$\tilde{F} < \tilde{G} = \tilde{H} = \tilde{C} = \tilde{A} = \tilde{B} = \tilde{E} < \tilde{D}$
0.9	$\tilde{F} < \tilde{G} = \tilde{H} = \tilde{C} = \tilde{A} = \tilde{B} = \tilde{E} < \tilde{D}$

Table 2

4. Conclusion

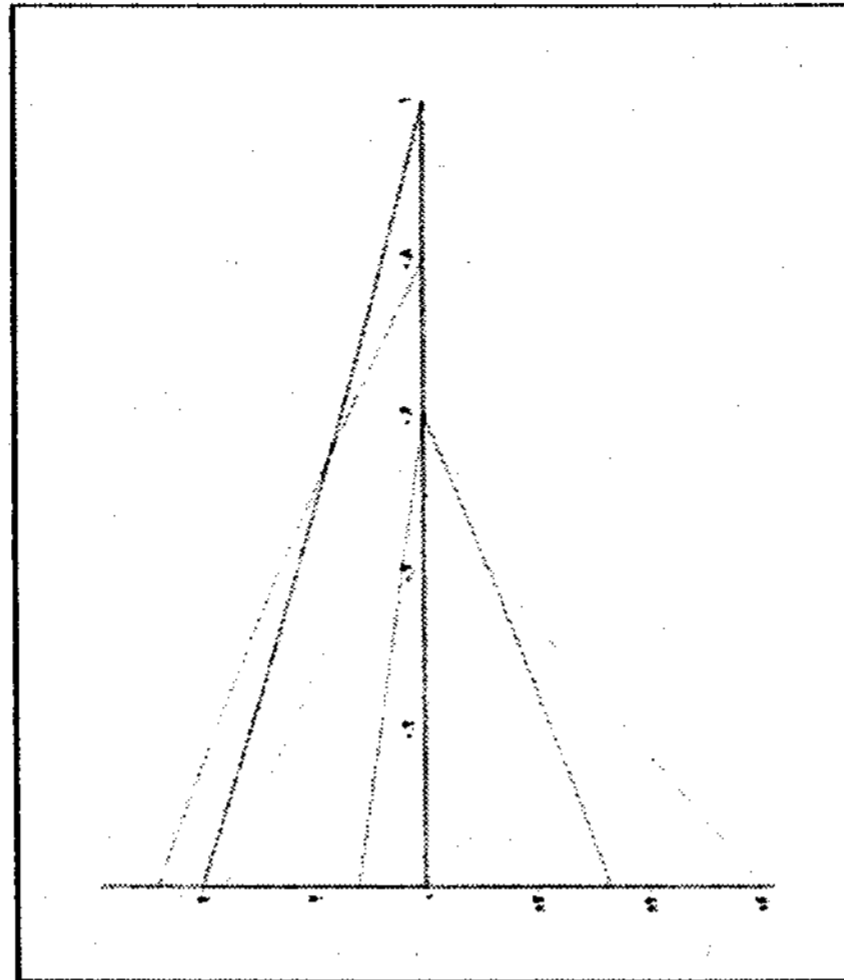


FIGURE 11

In this paper, a simple yet effective parametric method was introduced for comparison and ranking of fuzzy numbers. This method can be used for all kinds of fuzzy numbers, whether normal or abnormal. Due to the importance of ranking fuzzy numbers, specially in the theory of fuzzy decision making, it seems that the access to a simple parametric method of the comparison between fuzzy numbers paves the way for better study of phenomena in fuzzy environments.

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