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ON FORMANEK'S CENTRAL POLYNOMIALS

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ABSTRACT. Formanek([2]) proved that $M_n(K)$, the matrix algebra has a nontrivial central polynomial when char$K = 0$. Also Razmyslov([3]) showed the same result using the essential weak identity. In this article we explicitly compute Formanek's central polynomial for $M_2(\mathbb{C})$ and $M_3(\mathbb{C})$ and classify the coefficients of the central polynomial.

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1. Introduction.

Let K be a field of characteristic 0, A an K-algebra and $R = K < x_1, \ldots, x_n >$ the polynomial ring over K in noncommutative variables $x_1, \ldots, x_n$

Definition 1.1. A polynomial $F(x_1, \ldots, x_n) \in R$ is called a polynomial identity of A if $F(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$

Here are some examples.

Example 1.2. The ring of upper triangular $n \times n$ matrices over a field of characteristic 0 satisfies $[x_1, y_1] \cdots [x_n, y_n]$ where $[x, y] = xy - yx$.

Example 1.3. The matrix algebra $M_2(K)$ satisfies $[[x, y]^2, z] = (xy - yx)^2 \cdot z - z \cdot (xy - yx)^2$.

Definition 1.4. A polynomial $F(x_1, \ldots, x_n) \in R$ is called a central polynomial for an algebra A if

(1) $F(a_1, \ldots, a_n)$ belongs to the center of A for all $a_1, \ldots, a_n \in A$,
(2) $F(x_1, \ldots, x_n)$ is not a polynomial identity for A (i.e. $F(a_1, \ldots, a_n) \neq 0$ for some $a_1, \ldots, a_n \in A$),
(3) The constant term of $F(x_1, \ldots, x_n)$ is 0.

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Remark 1.1. If an algebra A has a central polynomial $F(x_1,\ldots,x_n)$, then $F(x_1,\ldots,x_n)\cdot x_{n+1} - x_{n+1}\cdot F(x_1,\ldots,x_n)$ is a polynomial identity for A.

Amitsur-Levitzki([1])showed that $M_n(K)$ has the polynomial identity of degree $2n$, so called the standard polynomial. This standard polynomial is of minimal degree. In other words, there is no polynomial identity of degree $2n-1$ or less for $M_n(K)$.

Example 1.5. The ring of upper triangular $n \times n$ matrices over K for $n \geq 2$ in Example1.2 does not have central polynomials.

Example 1.6. The matrix algebra $M_2(K)$ has a well known central polynomial $[x,y]^2$.

2. Formanek's central polynomials.

In the section, we briefly review Formanek's construction of central polynomial for $M_n(K)$.

Theorem 2.1. (Formanek [2], Razmyslov [3]) *The matrix algebra $M_n(K)$ has a central polynomial.*

Proof. It's sufficient to show that a polynomial $\mathcal{P}(X,Y_1,\ldots,Y_n)$ is a scalar matrix when X is an $n\times n$ generic matrix over K and $Y_1,\ldots,Y_n$ are arbitrary $n\times n$ matrices. Let $K[u_1,\ldots,u_n,u_{n+1}]$ be a commutative polynomial ring over K and let $K<x,y_1,\ldots,y_n>$ be a noncommutative algebra over K. Define a K-linear map

$$\phi : K[u_1,\ldots,u_n,u_{n+1}] \to K<x,y_1,\ldots,y_n>$$

on monomials by

$$\phi\Big(u_1^{r_1}\cdots u_{n+1}^{r_{n+1}}\Big) = x^{r_1}y_1x^{r_2}y_2x^{r_3}y_3\cdots y_{n-1}x^{r_n}y_nx^{r_{n+1}}$$

and extend by K-linearity. Set $G(x,y_1,\ldots,y_n) = \phi(g(u_1,\ldots,u_n,u_{n+1}))$, where

$$g(u_1,\ldots,u_n,u_{n+1}) = \prod_{2\leq i\leq n}(u_1-u_i)(u_{n+1}-u_i)\prod_{2\leq j<k\leq n}(u_j-u_k)^2$$

Now the central polynomial is

$$\begin{aligned}\mathcal{P}(x,y_1,\ldots,y_n) =& G(x,y_1,\ldots,y_n) + G(x,y_2,\ldots,y_n,y_1)\\ &+\cdots + G(x,y_n,y_1,\ldots,y_{n-1})\end{aligned} \tag{1}$$

We only need to verify that $\mathcal{P}(x,y_1,\ldots,y_n)$ is central when $x = \mathrm{diag}(v_1,\ldots,v_n)$ and $y_k = e_{i_k,j_k}$. Then $x\cdot e_{ij} = v_ie_{ij}$ and $e_{ij}\cdot x = v_je_{ij}$ implies

$$x^{r_1}y_1x^{r_2}y_2x^{r_3}y_3\cdots y_{n-1}x^{r_n}y_nx^{r_{n+1}} = v_{i_1}^{r_1}v_{i_2}^{r_2}\cdots v_{i_n}^{r_n}v_{j_n}^{r_{n+1}}e_{i_1,j_1}e_{i_2,j_k2}\cdots e_{i_n,j_n}.$$

Thus $G(x, e_{i_1,j_1}, e_{i_2,j_k2}\dots e_{i_n,j_n}) = g(v_{i_1},\dots v_{i_n}, v_{j_n})e_{i_1,j_1}e_{i_2,j_k2}\cdots e_{i_n,j_n}$.

But $g(v_{i_1},\dots v_{i_n}, v_{j_n}) = 0$ unless

$$(i_1,\dots,i_n) \text{ is a permutation of } (1,\dots,n) \text{ and } i_1 = j_n. \tag{2}$$

If (2) holds, then

$$g(v_{i_1},\dots v_{i_n}, v_{j_n}) = \prod_{1\le js<t\le n}(v_s - v_t)^2 = \Delta,$$

where $\Delta = \Delta(v_1,\dots,v_n)$ is the discriminant of $v_1,\dots,v_n$.

Furthermore $e_{i_1,j_1}e_{i_2,j_k2}\cdots e_{i_n,j_n} = 0$ unless $j_1 = i_2$, $j_2 = i_3$, $\dots$, $j_{n-1} = i_n$. Therefore

$$G(x, e_{i_1,j_1}, e_{i_2,j_k2}\dots e_{i_n,j_n}) = \begin{cases}\Delta e_{i_1,i_1} & \text{if the matrix units are a cycle;}\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\mathcal{P}\Big(x, e_{i_1,j_1}, e_{i_2,j_k2},\dots, e_{i_n,j_n}\Big) = \begin{cases}\Delta I & \text{if the matrix units are a cycle;}\\ 0 & \text{otherwise.}\end{cases}$$

This implies that $\mathcal{P}(x, y_1,\dots,y_n)$ is a central polynomial for $M_n(K)$. □

3. The coefficients of the central polynomials.

Now we are ready investigate the coefficients of the central polynomials. First of all, for $n = 2$, $g(u_1,u_2,u_3) = (u_1-u_2)(u_3-u_2) = u_1u_3 - u_1u_2 - u_2u_3 + u_2^2$. Then

$$G(x, y_1, y_2) = xy_1y_2x - xy_1xy_2 - y_1xy_2x + y_1x^2y_2 \tag{3}$$

Lemma 3.1. *The central polynomial of* 2 *variables for* $M_2(K)$ *is* $[X,Y]^2$.

Proof. In (3), set $y = y_1 = y_2$. Then

$$\begin{aligned}G(x,y,y) &= xy^2x - xyxy - yxyx + yx^2y\\ &= (xy-yx)^2 = [X,Y]^2.\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{P}(x,y,y) &= G(x,y_1,y_2) + G(x,y_2,y_1)\\ &= 2G(x,y,y) = 2[X,Y]^2.\end{aligned}$$

□

Lemma 3.2. *The central polynomial of* 2 *variables for* $M_3(K)$ *is (4).*

Proof. If $n = 3$, then

$$g((u_1, u_2, u_3, u_4) = (u_1 - u_2)(u_4 - u_2)(u_1 - u_3)(u_4 - u_3)(u_2 - u_3)^2.$$

By expanding we have

$$\begin{aligned} g(u_1, u_2, u_3, u_4) = {} & u_1^2 u_2^3 u_3 - u_1 u_2^4 u_3 - 2u_1^2 u_2^2 u_3^2 + u_1 u_2^3 u_3^2 \\ & + u_2^4 u_3^2 + u_1^2 u_2 u_3^3 + u_1 u_2^2 u_3^3 - 2u_2^3 u_3^3 \\ & - u_1 u_2 u_3^4 + u_2^2 u_3^4 - u_1^2 u_2^3 u_4 + u_1 u_2^4 u_4 \\ & + u_1^2 u_2^2 u_3 u_4 - u_2^4 u_3 u_4 + u_1^2 u_2 u_3^2 u_4 - 2u_1 u_2^2 u_3^2 u_4 \\ & + u_2^3 u_3^2 u_4 - u_1^2 u_3^3 u_4 + u_2^2 u_3^3 u_4 + u_1 u_3^4 u_4 \\ & - u_2 u_3^4 u_4 + u_1^2 u_2^2 u_4^2 - u_1 u_2^3 u_4^2 - 2u_1^2 u_2 u_3 u_4^2 \\ & + u_1 u_2^2 u_3 u_4^2 + u_2^3 u_3 u_4^2 + u_1^2 u_3^2 u_4^2 + u_1 u_2 u_3^2 u_4^2 \\ & - 2u_2^2 u_3^2 u_4^2 - u_1 u_3^3 u_4^2 + u_2 u_3^3 u_4^2. \end{aligned}$$

Thus

$$\begin{aligned} G(x, y_1, y_2, y_3) = {} & x^2 y_1 x^3 y_2 x y_3 - x y_1 x^4 y_2 x y_3 - 2x^2 y_1 x^2 y_2 x^2 y_3 + x y_1 x^3 y_2 x^2 y_3 \\ & + y_1 x^4 y_2 x^2 y_3 + x^2 y_1 x y_2 x^3 y_3 + x y_1 x^2 y_2 x^3 y_3 - 2y_1 x^3 y_2 x^3 y_3 \\ & - x y_1 x y_2 x^4 y_3 + y_1 x^2 y_2 x^4 y_3 - x^2 y_1 x^3 y_2 y_3 x + x y_1 x^4 y_2 y_3 x \\ & + x^2 y_1 x^2 y_2 x y_3 x - y_1 x^4 y_2 x y_3 x + x^2 y_1 x y_2 x^2 y_3 x - 2x y_1 x^2 y_2 x^2 y_3 x \\ & + y_1 x^3 y_2 x^2 y_3 x - x^2 y_1 y_2 x^3 y_3 x + y_1 x^2 y_2 x^3 y_3 x + x y_1 y_2 x^4 y_3 x \\ & - y_1 x y_2 x^4 y_3 x + x^2 y_1 x^2 y_2 y_3 x^2 - x y_1 x^3 y_2 y_3 x^2 - 2x^2 y_1 x y_2 x y_3 x^2 \\ & + x y_1 x^2 y_2 x y_3 x^2 + y_1 x^3 y_2 x y_3 x^2 + x^2 y_1 y_2 x^2 y_3 x^2 + x y_1 x y_2 x^2 y_3 x^2 \\ & - 2y_1 x^2 y_2 x^2 y_3 x^2 - x y_1 y_2 x^3 y_3 x^2 + y_1 x y_2 x^3 y_3 x^2 \end{aligned}$$

If we set $y = y_1 = y_2 = y_3$, then

$$\begin{aligned} \frac{1}{3}\mathcal{P}(x, y, y, y) = {} & G(x, y, y, y) \\ = {} & x^2 y x^3 y x y - x y x^4 y x y - 2x^2 y x^2 y x^2 y + x y x^3 y x^2 y \\ & + y x^4 y x^2 y + x^2 y x y x^3 y + x y x^2 y x^3 y - 2y x^3 y x^3 y \\ & - x y x y x^4 y + y x^2 y x^4 y - x^2 y x^3 y^2 x + x y x^4 y^2 x \\ & + x^2 y x^2 y x y x - y x^4 y x y x + x^2 y x y x^2 y x - 2x y x^2 y x^2 y x \\ & + y x^3 y x^2 y x - x^2 y^2 x^3 y x + y x^2 y x^3 y x + x y^2 x^4 y x \\ & - y x y x^4 y x + x^2 y x^2 y^2 x^2 - x y x^3 y^2 x^2 - 2x^2 y x y x y x^2 \\ & + x y x^2 y x y x^2 + y x^3 y x y x^2 + x^2 y^2 x^2 y x^2 + x y x y x^2 y x^2 \\ & - 2y x^2 y x^2 y x^2 - x y^2 x^3 y x^2 + y x y x^3 y x^2. \end{aligned} \tag{4}$$

The absolute value of the coefficients of the central polynomials in Lemma 3.1, 3.2 are less than or equal to n. □

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