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DIMENSIONS OF THE SUBSETS IN THE SPECTRAL CLASSES OF A SELF-SIMILAR CANTOR SET

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ABSTRACT. Using an information of dimensions of divergence points, we give full information of dimensions of the completely decomposed class of the lower(upper) distribution sets of a self-similar Cantor set. Further using a relationship between the distribution sets and the subsets generated by the lower(upper) local dimensions of a self-similar measure, we give full information of dimensions of the subsets by the local dimensions.

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1. Introduction

Many authors([4],[5],[8],[10],[11]) have studied multifractals of a self-similar set associated with self-similar measures. A self-similar Cantor set is a typical example of such self-similar set. A spectral class of a self-similar Cantor set, a class of subsets derived from the local dimensions of a self-similar measure on a self-similar Cantor set, has been investigated in [5, 6, 8] to study its geometrical properties. In [6, 8], the Hausdorff and packing dimensions of subsets composing a spectral class were calculated using power equations related to contraction ratios and an associated probability of a self-similar measure. In [2], we related a spectral class by the lower(upper) local dimensions of a self-similar measure with the class by the lower or upper distribution sets(cf. [7]). It gives the comparison of a subset in a spectral class with another subset in a different spectral class via a distribution set. Using these results with the relationship, we compute the values of dimensions of the subsets composing a spectral class generated by a self-similar measure and its lower(upper) local dimensions. However we could not find packing dimensions of some subsets. In fact, the accumulation points

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of some frequency sequence of the point in the subset contain a singular point. We use a recent result([3]) of packing dimension about the divergence points in a self-similar set to get the packing dimension of the subset. Further we make the previous results([2]) simpler for better understanding.

2. Preliminaries

We denote F a self-similar Cantor set, which is the attractor of the similarities $f_1(x) = ax$ and $f_2(x) = bx + (1-b)$ on $I = [0,1]$ with $a > 0$, $b > 0$ and $1-(a+b) > 0$. Let $I_{i_1,\cdots,i_k} = f_{i_1} \circ \cdots \circ f_{i_k}(I)$ where $i_j \in \{1,2\}$ and $1 \le j \le k$. We note that if $x \in F$, then there is $\sigma \in \{1,2\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1, i_2, \cdots, i_k$ where $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$). If $x \in F$ and $x \in I_\sigma$ where $\sigma \in \{1,2\}^k$, $c_k(x)$ denotes I_σ and $|c_k(x)|$ denotes the diameter of $c_k(x)$ for each $k = 0,1,2,\cdots$. Let $p \in (0,1)$ and we denote γ_p a self-similar Borel probability measure on F satisfying $\gamma_p(I_1) = p$(cf. [6]). $\dim(E)$ denotes the Hausdorff dimension of E and $\mathrm{Dim}(E)$ denotes the packing dimension of E([6]). We note that $\dim(E) \le \mathrm{Dim}(E)$ for every set E([6]). We denote $n_1(x|k)$ the number of times the digit 1 occurs in the first k places of $x = \sigma$(cf. [7]).

For $r \in [0,1]$, we define the lower(upper) distribution set $\underline{F}(r)(\overline{F}(r))$ containing the digit 1 in proportion r by

$$\underline{F}(r) = \left\{ x \in F : \liminf_{k \to \infty} \frac{n_1(x|k)}{k} = r \right\},$$

$$\overline{F}(r) = \left\{ x \in F : \limsup_{k \to \infty} \frac{n_1(x|k)}{k} = r \right\}.$$

We call $\{\underline{F}(r) : 0 \le r \le 1\}$ the lower distribution class and $\left\{\overline{F}(r) : 0 \le r \le 1\right\}$ the upper distribution class. We write $\underline{E}_\alpha^{(p)}$ $\left(\overline{E}_\alpha^{(p)}\right)$ for the set of points at which the lower(upper) local dimension of γ_p on F is exactly α, so that

$$\underline{E}_\alpha^{(p)} = \left\{ x : \liminf_{r \to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha \right\},$$

$$\overline{E}_\alpha^{(p)} = \left\{ x : \limsup_{r \to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha \right\}.$$

We call $\{\underline{E}_\alpha^{(p)}(\neq \phi) : \alpha \in \mathbb{R}\}$ the spectral class generated by the lower local dimensions of a self-similar measure γ_p and $\left\{\overline{E}_\alpha^{(p)}(\neq \phi) : \alpha \in \mathbb{R}\right\}$ the spectral class generated by the upper local dimensions of a self-similar measure γ_p. We call α satisfying $\underline{E}_\alpha^{(p)}(\neq \phi)$ $\left(\overline{E}_\alpha^{(p)}(\neq \phi)\right)$ an associated lower(upper) local dimension of γ_p.

In this paper, we assume that $0 \log 0 = 0$ for convenience. We define for $r \in [0, 1]$

$$g(r,p) = \frac{r \log p + (1-r)\log(1-p)}{r \log a + (1-r)\log b}.$$

From now on we will use $g(r, p)$ as the above definition.

3. Main results

Lemma 1. *Let a real number s satisfy $a^s + b^s = 1$. If $0 < p < a^s$, then $\frac{\log(1-p)}{\log b} < \frac{\log p}{\log a}$. Similarly if $a^s < p < 1$, then $\frac{\log p}{\log a} < \frac{\log(1-p)}{\log b}$.*

Proof. Let $0 < p < a^s$. Since log function is an increasing function, we easily see that $\frac{\log(1-p)}{\log b} < s < \frac{\log p}{\log a}$. The same arguments hold for $a^s < p < 1$. □

Remark 1. If $p \in (0,1)$ and $p \neq a^s$ where s satisfy $a^s + b^s = 1$, then $g(r,p)$ is a stricltly monotone function for $r \in [0,1]$([2]). Hence we find a solution r for the equation $g(r,p) = \alpha$ where $\alpha \in \left[\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}\right]$ or $\alpha \in \left[\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}\right]$.

Proposition 1. *For $0 \le r_1 \le r_2 \le 1$,*

$$\dim\left(\underline{F}(r_1) \cap \overline{F}(r_2)\right) = \inf_{r_1 \le r \le r_2} g(r,r)$$

and

$$\mathrm{Dim}\left(\underline{F}(r_1) \cap \overline{F}(r_2)\right) = \sup_{r_1 \le r \le r_2} g(r,r).$$

Proof. It follow from [3, 9]. □

Corollary 1. *Let a real number s satisfy $a^s + b^s = 1$. For $0 \le r_1 \le a^s \le r_2 \le 1$,*

$$\mathrm{Dim}(\underline{F}(r_1)) = s = \mathrm{Dim}(\overline{F}(r_2)).$$

In particular,

$$\mathrm{Dim}(\underline{F}(0)) = s = \mathrm{Dim}(\overline{F}(1)).$$

Proof. It follow from the fact that $g(a^s, a^s) = s$ and the above Proposition. □

Corollary 2. *Let s be the unique real number satisfying $a^s + b^s = 1$. Then*
(1) $\dim(\underline{F}(r)) = \dim(\overline{F}(r)) = g(r,r)$ *and* $\mathrm{Dim}(\underline{F}(r)) = s$ *and* $\mathrm{Dim}(\overline{F}(r)) = g(r,r)$ *if* $0 \leq r < a^s$,
(2) $\dim(\underline{F}(r)) = \dim(\overline{F}(r)) = g(r,r)$ *and* $\mathrm{Dim}(\underline{F}(r)) = g(r,r)$ *and* $\mathrm{Dim}(\overline{F}(r)) = s$ *if* $a^s < r \leq 1$,
(3) $\dim(\underline{F}(a^s)) = \dim(\overline{F}(a^s)) = s$ *and* $\mathrm{Dim}(\underline{F}(a^s)) = \mathrm{Dim}(\overline{F}(a^s)) = s$

Proof. It follow from the above Corollary and the corollary 5 in [2]. □

Lemma 2. *Let $p \in (0,1)$ and consider a self-similar measure γ_p on F and let $r \in [0,1]$. Then for a real number s satisfying $a^s + b^s = 1$*
(1) $\underline{F}(r) = \underline{E}^{(p)}_{g(r,p)}$ *if* $0 < p < a^s$,
(2) $\underline{F}(r) = \overline{E}^{(p)}_{g(r,p)}$ *if* $a^s < p < 1$,
(3) $\overline{F}(r) = \overline{E}^{(p)}_{g(r,p)}$ *if* $0 < p < a^s$,
(4) $\overline{F}(r) = \underline{E}^{(p)}_{g(r,p)}$ *if* $a^s < p < 1$.

Proof. It follows from [2]. □

Theorem 1. *For $0 < p_1,\ \ p_2 < a^s < p_3,\ \ p_4 < 1$, $\underline{E}^{(p_1)}_{g(r,p_1)} = \underline{E}^{(p_2)}_{g(r,p_2)} = \overline{E}^{(p_3)}_{g(r,p_3)} = \overline{E}^{(p_4)}_{g(r,p_4)}$ for some $r \in [0,1]$.*

Proof. We note that all they are $\underline{F}(r)$. It is immediate from the above Lemma. □

Theorem 2. *For $0 < p_1,\ \ p_2 < a^s < p_3,\ \ p_4 < 1$ and $0 \leq r \leq a^s$ where a real number s satisfies $a^s + b^s = 1$,*

$$\underline{E}^{(p_1)}_{g(r,p_1)} = \underline{E}^{(p_2)}_{g(r,p_2)} = \overline{E}^{(p_3)}_{g(r,p_3)} = \overline{E}^{(p_4)}_{g(r,p_4)}$$

has Hausdorff dimension $g(r,r)$ and packing dimension s.

Proof. If $0 \leq r \leq a^s$, then $\underline{F}(r)$ has Hausdorff dimension $g(r,r)$ and packing dimension s. It is immediate from the above Proposition. □

Theorem 3. *For $0 < p_1,\ \ p_2 < a^s < p_3,\ \ p_4 < 1$ and $a^s \leq r \leq 1$ where a real number s satisfies $a^s + b^s = 1$,*

$$\underline{E}^{(p_1)}_{g(r,p_1)} = \underline{E}^{(p_2)}_{g(r,p_2)} = \overline{E}^{(p_3)}_{g(r,p_3)} = \overline{E}^{(p_4)}_{g(r,p_4)}$$

has Hausdorff dimension and packing dimension $g(r,r)$.

Proof. If $a^s \leq r \leq 1$, then $\underline{F}(r)$ has Hausdorff dimension and packing dimension $g(r,r)$. It is immediate from the above Proposition. □

Theorem 4. *For* $0 < p_1,\ \ p_2 < a^s < p_3,\ \ p_4 < 1$

$$\overline{E}^{(p_1)}_{g(r,p_1)} = \overline{E}^{(p_2)}_{g(r,p_2)} = \underline{E}^{(p_3)}_{g(r,p_3)} = \underline{E}^{(p_4)}_{g(r,p_4)}$$

for some $r \in [0,1]$.

Proof. We note that all they are $\overline{F}(r)$. It is immediate from the above Lemma. □

Theorem 5. *For* $0 < p_1,\ \ p_2 < a^s < p_3,\ \ p_4 < 1$ *and* $0 \le r \le a^s$ *where a real number* s *satisfies* $a^s + b^s = 1$,

$$\overline{E}^{(p_1)}_{g(r,p_1)} = \overline{E}^{(p_2)}_{g(r,p_2)} = \underline{E}^{(p_3)}_{g(r,p_3)} = \underline{E}^{(p_4)}_{g(r,p_4)}$$

has Hausdorff dimension $g(r,r)$ *and packing dimension* s.

Proof. If $0 \le r \le a^s$, $\overline{F}(r)$ has Hausdorff dimension $g(r,r)$ and packing dimension s. It is immediate from the above Proposition. □

Theorem 6. *For* $0 < p_1,\ \ p_2 < a^s < p_3,\ \ p_4 < 1$ *and* $a^s \le r \le 1$ *where a real number* s *satisfies* $a^s + b^s = 1$,

$$\overline{E}^{(p_1)}_{g(r,p_1)} = \overline{E}^{(p_2)}_{g(r,p_2)} = \underline{E}^{(p_3)}_{g(r,p_3)} = \underline{E}^{(p_4)}_{g(r,p_4)}$$

has Hausdorff dimension and packing dimension $g(r,r)$.

Proof. If $a^s \le r \le 1$, $\overline{F}(r)$ has Hausdorff dimension and packing dimension $g(r,r)$. It is immediate from the above Proposition. □

Theorem 7. *For* $p = a^s$ *where a real number* s *satisfies* $a^s + b^s = 1$,

$$\overline{E}^{(p)}_s = \underline{E}^{(p)}_s = E^{(p)}_s = F$$

has Hausdorff dimension and packing dimension s.

Proof. It is immediate from Remark 1 in [2]. □

Remark 2. For any $p \in (0,1)$ and any α such that $\alpha \in \left[\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}\right]$ or $\alpha \in \left[\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}\right]$, $\underline{E}^{(p)}_\alpha$ or $\overline{E}^{(p)}_\alpha$ can be represented by $\underline{F}(r)$ or $\overline{F}(r)$ for some solution r of the equation $g(r,p) = \alpha$. Further we have full information of its Hausdorff dimension and packing dimension.

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