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SOFT SET THEORY APPLIED TO COMMUTATIVE IDEALS IN BCK-ALGEBRAS

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ABSTRACT. Molodtsov [12] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. In this paper we apply the notion of soft sets by Molodtsov to commutative ideals of BCK-algebras. The notions of commutative soft ideals and commutative idealistic soft BCK-algebras are introduced, and their basic properties are investigated. Examples to show that there is no relations between positive implicative idealistic soft BCK-algebras and commutative idealistic soft BCK-algebras are provided.

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1. Introduction

To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [12]. Maji et al. [10] and Molodtsov [12] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [12] introduced the concept of soft set as a new mathematical tool for dealing

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with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [10] described the application of soft set theory to a decision making problem. Maji et al. [9] also studied several operations on the theory of soft sets. Chen et al. [1] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. MV-algebras, introduced by Chang in the 1950s, provided an algebraic semantics for the Lukasiewicz logics ([2]). Effect algebras, introduced in 1993, generalize MV-algebras. They are partial additive algebras modeled upon the Hilbert space quantum effects, which in turn represent the positive outcomes of the yes-no tests performable at some physical system ([3,4]). Y. Imai and K. Iseki introduced two classes of abstract algebras: BCKalgebras and BCI-algebras, which was originated from both set theory and (non-) classical propositional calculus ([5,11]). D. Mundici [13] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras. In this paper, we deal with the algebraic structure of BCK-algebras by applying soft set theory. We discuss the algebraic properties of soft sets in BCK-algebras. We introduce the notions of commutative soft ideals and commutative idealistic soft BCK-algebras, and give several examples. We investigate relations between idealistic soft BCK-algebras and commutative idealistic soft BCK-algebras. We give examples to show that there is no relations between positive implicative idealistic soft BCK-algebras and commutative idealistic soft BCK-algebras. Finally, we establish the intersection, union, "AND" operation, and "OR" operation of commutative soft ideals and commutative idealistic soft BCK-algebras.

2. Basic results on BCK-algebras

A BCK-algebra is an important calss of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,
(a2) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,

(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is said to be *positive implicative* if it satisfies the following identity:

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * (y * z)). \tag{2.1}$$

A BCK-algebra X is s said to be *commutative* if $x \wedge y = y \wedge x$ for all $x, y \in X$ where $x \wedge y = y * (y * x)$. A nonempty subset S of a BCK-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK-algebra X is called an *ideal* of X, denoted by $I \lhd X$, if it satisfies the following axioms:

(I1) $0 \in I$,
(I2) $(\forall x \in X)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

Any ideal I of a BCK-algebra X satisfies the following implication:

$$(\forall x \in X)\,(\forall y \in I)\,(x \leq y \Rightarrow x \in I). \tag{2.2}$$

A subset I of a BCK-algebra X is called a *positive implicative ideal* of X, denoted by $I \lhd_{pi} X$, if it satisfies (I1) and

$$(\forall x, y, z \in X)\,((x * y) * z \in I,\ y * z \in I \Rightarrow x * z \in I). \tag{2.3}$$

A subset I of a BCK-algebra X is called a *commutative ideal* of X, denoted by $I \lhd_c X$, if it satisfies (I1) and

$$(\forall x, y \in X)\,(\forall z \in I)\,((x * y) * z \in I \Rightarrow x * (y * (y * x)) \in I). \tag{2.4}$$

We refer the reader to the book [11] for further information regarding BCK-algebras.

3. Basic results on soft sets

Molodtsov [12] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathrm{P}(U)$ denotes the power set of U and $A \subset E$.

Definition 3.1. [12] A pair (A, A) is called a *soft set* over U, where A is a mapping given by

$$\mathrm{A} : A \to \mathrm{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\mathrm{A}(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (A, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [12] .

Definition 3.2. [9] Let (A, A) and (B, B) be two soft sets over a common universe U. The *intersection* of (A, A) and (B, B) is defined to be the soft set (H, C) satisfying the following conditions:

(i) $C = A \cap B$,

(ii) $(\forall e \in C)$ $(\mathrm{B}(e) = \mathrm{A}(e)$ or $\mathrm{B}(e)$, (as both are same sets)).
In this case, we write $(\mathrm{A}, A)\tilde{\cap}(\mathrm{B}, B) = (\mathrm{H}, C)$.

Definition 3.3. [9] Let (A, A) and (B, B) be two soft sets over a common universe U. The *union* of (A, A) and (B, B) is defined to be the soft set (H, C) satisfying the following conditions:

(i) $C = A \cup B$,
(ii) for all $e \in C$,

$$\mathrm{H}(e) = \begin{cases} \mathrm{A}(e) & \text{if } e \in A \setminus B, \\ \mathrm{B}(e) & \text{if } e \in B \setminus A, \\ \mathrm{A}(e) \cup \mathrm{B}(e) & \text{if } e \in A \cap B. \end{cases}$$

In this case, we write $(\mathrm{A}, A)\tilde{\cup}(\mathrm{B}, B) = (\mathrm{H}, C)$.

Definition 3.4. [9] If (A, A) and (B, B) are two soft sets over a common universe U, then "(A, A) *AND* (B, B)" denoted by $(\mathrm{A}, A)\tilde{\wedge}(\mathrm{B}, B)$ is defined by $(\mathrm{A}, A)\tilde{\wedge}(\mathrm{B}, B) = (\mathrm{H}, A \times B)$, where $\mathrm{H}(x, y) = \mathrm{A}(x) \cap \mathrm{B}(y)$ for all $(x, y) \in A \times B$.

Definition 3.5. [9] If (A, A) and (B, B) are two soft sets over a common universe U, then "(A, A) *OR* (B, B)" denoted by $(\mathrm{A}, A)\tilde{\vee}(\mathrm{B}, B)$ is defined by $(\mathrm{A}, A)\tilde{\vee}(\mathrm{B}, B) = (\mathrm{H}, A \times B)$, where $\mathrm{H}(x, y) = \mathrm{A}(x) \cup \mathrm{B}(y)$ for all $(x, y) \in A \times B$.

Definition 3.6. [9] For two soft sets (A, A) and (B, B) over a common universe U, we say that (A, A) is a *soft subset* of (B, B), denoted by $(\mathrm{A}, A)\tilde{\subset}(\mathrm{B}, B)$, if it satisfies:

(i) $A \subset B$,
(ii) For every $\varepsilon \in A$, $\mathrm{A}(\varepsilon)$ and $\mathrm{B}(\varepsilon)$ are identical approximations.

4. Soft commutative ideas

In what follows let X denote a BCK-algebra unless otherwise specified.

Definition 4.1. [6] Let (A, A) be a soft set over X. Then (A, A) is called a *soft BCK-algebra* over X if $\mathrm{A}(x)$ is a subalgebra of X for all $x \in X$.

Definition 4.2. [7] Let S be a subalgebra of X. A subset I of X is called an *ideal* of X related to S (briefly, *S-ideal* of X), denoted by $I \lhd S$, if it satisfies:

(i) $0 \in I$,
(ii) $(\forall x \in S)\,(\forall y \in I)\,(x * y \in I \Rightarrow x \in I)$.

Definition 4.3. [7] Let (A, A) be a soft BCK-algebra over X. A soft set (B, I) over X is called a *soft ideal* of (A, A), denoted by $(\mathrm{B}, I)\tilde{\triangleleft}(\mathrm{A}, A)$, if it satisfies:

(i) $I \subset A$,
(ii) $(\forall x \in I)\ (\mathrm{B}(x) \triangleleft \mathrm{A}(x))$.

Definition 4.4. Let S be a subalgebra of X. A subset I of X is called a *commutative ideal* of X related to S (briefly, *commutative S-ideal* of X), denoted by $I \triangleleft_c S$, if it satisfies (I1) and

$$(\forall x, y \in S)\,(\forall z \in I)\,((x * y) * z \in I \ \Rightarrow\ x * (y * (y * x)) \in I). \tag{4.1}$$

Note that a commutative X-ideal means a commutative ideal, and every commutative S-ideal of X is an S-ideal of X.

Definition 4.5. Let (A, A) be a soft BCK-algebra over X. A soft set (B, I) over X is called a *commutative soft ideal* of (A, A), denoted by $(\mathrm{B}, I)\tilde{\triangleleft}_c\ (\mathrm{A}, A)$, if it satisfies:

(i) $I \subset A$,
(ii) $(\forall x \in I)\ (\mathrm{B}(x) \triangleleft_c \mathrm{A}(x))$.

For any $a \in X$ and a subset D of X, let

$$\frac{a}{D} := \{x \in X \mid x * a \in D\},\ \frac{a^2}{D} := \{x \in X \mid x * (x * a) \in D\}.$$

Let us illustrate this definition using the following examples.

Example 4.6. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	b
c	c	b	a	0	b
d	d	a	d	a	0

Let (A, A) be a soft set over X, where $A = X$ and $\mathrm{A} : A \to \mathrm{P}(X)$ is a set-valued function defined by $\mathrm{A}(x) = \dfrac{x^2}{\{0, a\}}$ for all $x \in A$. Then $\mathrm{A}(0) = X = \mathrm{A}(a)$, $\mathrm{A}(b) = \mathrm{A}(c) = \{0, a, d\}$ and $\mathrm{A}(d) = \{0, a, b, c\}$, which are subalgerbas of X. Hence (A, A) is a soft BCK-algebra over X. For $I = \{a, c, d\}$, let $\mathrm{B} : I \to \mathrm{P}(X)$ be a set-valued function defined by $\mathrm{B}(x) = \dfrac{x}{\{0, a\}}$ for all $x \in I$. Then $\mathrm{B}(a) = \{0, a, d\} \triangleleft_c \mathrm{A}(a)$, $\mathrm{B}(c) = X \triangleleft_c \mathrm{A}(c)$, and $\mathrm{B}(d) = \{0, a, d\} \triangleleft_c \mathrm{A}(d)$. Hence $(\mathrm{B}, I)\tilde{\triangleleft}_c\ (\mathrm{A}, A)$.

Theorem 4.7. *Let* (B, I) *and* (B, J) *be soft sets over* X *such that* $I \subset J$. *If* (B, J) *is a commutative soft ideal of a soft BCK-algebra* (A, A) *over* X, *then so is* (B, I).

Proof. Straightforward. □

The converse of Theorem 4.7 is not valid in general as seen in the following example.

Example 4.8. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Let (A, A) be a soft set over X, where $A = X$ and $\mathrm{A} : A \to \mathrm{P}(X)$ is a set-valued function defined by $\mathrm{A}(x) = \dfrac{x^2}{\{0, b\}}$ for all $x \in A$. It is easy to verify that (A, A) is a soft BCK-algebra over X. Take $J := \{a, b, c\}$ and let (B, J) be a soft set over X which is given by $\mathrm{B}(x) = \dfrac{x}{\{0\}}$ for all $x \in J$. Then $\mathrm{B}(a) = \{0, a\}$ is not a commutative ideal of X related to $\mathrm{A}(a)$ since $(b * c) * a = 0 * a = 0 \in \mathrm{B}(a)$ and $b * (c * (c * b)) = b * (c * c) = b * 0 = b \notin \mathrm{B}(a)$. Hence (B, J) is not a commutative soft ideal of (A, A). For $I = \{b\} \subseteq J$, let $\mathrm{B} : I \to \mathrm{P}(X)$ be a set-valued function defined by $\mathrm{B}(x) = \dfrac{x}{\{0\}}$ for all $x \in I$. Then $\mathrm{B}(b) = \{0, a, b\} \lhd_c \mathrm{A}(b)$, and so (B, I) is a commutative soft ideal of (A, A).

Theorem 4.9. *Let* (A, A) *be a soft BCK-algebra over* X. *For any soft sets* (B_1, I_1) *and* (B_2, I_2) *over* X *where* $I_1 \cap I_2 \neq \emptyset$, *we have*

$$(\mathrm{B}_1, I_1)\tilde{\lhd}_c(\mathrm{A}, A),\ (\mathrm{B}_2, I_2)\tilde{\lhd}_c(\mathrm{A}, A) \Rightarrow (\mathrm{B}_1, I_1)\tilde{\sqcap}(\mathrm{B}_2, I_2)\tilde{\lhd}_c(\mathrm{A}, A).$$

Proof. Using Definition 3.2, we can write

$$(\mathrm{B}_1, I_1)\tilde{\sqcap}(\mathrm{B}_2, I_2) = (\mathrm{B}, I),$$

where $I = I_1 \cap I_2$ and $\mathrm{B}(x) = \mathrm{B}_1(x)$ or $\mathrm{B}_2(x)$ for all $x \in I$. Obviously, $I \subset A$ and $\mathrm{B} : I \to \mathrm{P}(X)$ is a mapping. Hence (B, I) is a soft set over X. Since $(\mathrm{B}_1, I_1)\tilde{\lhd}_c(\mathrm{A}, A)$ and $(\mathrm{B}_2, I_2)\tilde{\lhd}_c(\mathrm{A}, A)$, we know that $\mathrm{B}(x) = \mathrm{B}_1(x) \lhd_c \mathrm{A}(x)$ or $\mathrm{B}(x) = \mathrm{B}_2(x) \lhd_c \mathrm{A}(x)$ for all $x \in I$. Hence

$$(\mathrm{B}_1, I_1)\tilde{\sqcap}(\mathrm{B}_2, I_2) = (\mathrm{B}, I)\tilde{\lhd}_c(\mathrm{A}, A).$$

This completes the proof. □

Corollary 4.10. *Let* (A, A) *be a soft BCK-algebra over* X. *For any soft sets* (B, I) *and* (D, I) *over* X, *we have*

$$(\mathrm{B}, I)\tilde{\lhd}_c(\mathrm{A}, A),\ (\mathrm{D}, I)\tilde{\lhd}_c(\mathrm{A}, A) \Rightarrow (\mathrm{B}, I)\tilde{\cap}(\mathrm{D}, I)\tilde{\lhd}_c(\mathrm{A}, A).$$

Proof. Straightforward. □

Theorem 4.11. *Let* (A, A) *be a soft BCK-algebra over* X. *For any soft sets* (B, I) *and* (D, J) *over* X *in which* I *and* J *are disjoint, we have*

$$(\mathrm{B}, I)\tilde{\lhd}_c(\mathrm{A}, A),\ (\mathrm{D}, J)\tilde{\lhd}_c(\mathrm{A}, A) \Rightarrow (\mathrm{B}, I)\tilde{\cup}(\mathrm{D}, J)\tilde{\lhd}_c(\mathrm{A}, A).$$

Proof. Assume that $(\mathrm{B}, I)\tilde{\lhd}_c(\mathrm{A}, A)$ and $(\mathrm{D}, J)\tilde{\lhd}_c(\mathrm{A}, A)$. By means of Definition 3.3, we can write $(\mathrm{B}, I)\tilde{\cup}(\mathrm{D}, J) = (\mathrm{H}, U)$ where $U = I \cup J$ and for every $x \in U$,

$$\mathrm{H}(x) = \begin{cases} \mathrm{B}(x) & \text{if } x \in I \setminus J, \\ \mathrm{D}(x) & \text{if } x \in J \setminus I, \\ \mathrm{B}(x) \cup \mathrm{D}(x) & \text{if } x \in I \cap J. \end{cases}$$

Since $I \cap J = \emptyset$, either $x \in I \setminus J$ or $x \in J \setminus I$ for all $x \in U$. If $x \in I \setminus J$, then $\mathrm{H}(x) = \mathrm{B}(x)\lhd_c\mathrm{A}(x)$ since $(\mathrm{B}, I)\tilde{\lhd}_c(\mathrm{A}, A)$. If $x \in J \setminus I$, then $\mathrm{H}(x) = \mathrm{D}(x)\lhd_c\mathrm{A}(x)$ since $(\mathrm{D}, J)\tilde{\lhd}_c(\mathrm{A}, A)$. Thus $\mathrm{H}(x)\lhd_c\mathrm{A}(x)$ for all $x \in U$, and so $(\mathrm{B}, I)\tilde{\cup}(\mathrm{D}, J) = (\mathrm{H}, U)\tilde{\lhd}_c(\mathrm{A}, A)$. □

If I and J are not disjoint in Theorem 4.11, then Theorem 4.11 is not true in general as seen in the following example.

Example 4.12. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	d	c	b	0

Let (A, A) be a soft set over X, where $A = X$ and $\mathrm{A} : A \to \mathrm{P}(X)$ is a set-valued function defined by $\mathrm{A}(x) = \dfrac{x^2}{\{0, b\}}$ for all $x \in A$. Then (A, A) is a soft BCK-algebra over X, since $\mathrm{A}(0) = X$, $\mathrm{A}(a) = \mathrm{A}(b) = \{0, b, c, d\}$, and $\mathrm{A}(c) = \mathrm{A}(d) = \{0, b\}$ are subalgebras of X. For $I = \{b, c, d\}$, let $\mathrm{B} : I \to \mathrm{P}(X)$ be a set-valued function defined by $\mathrm{B}(x) = \dfrac{x}{\{0\}}$ for all $x \in I$. Then $\mathrm{B}(b) = \{0, a, b\} \lhd_c \mathrm{A}(b)$, $\mathrm{B}(c) = \{0, a, c\} \lhd_c \mathrm{A}(c)$, and $\mathrm{B}(d) = X \lhd_c \mathrm{A}(d)$, and hence $(\mathrm{B}, I)\tilde{\lhd}_c(\mathrm{A}, A)$. Now, for $J = \{b\}$, let $\mathrm{D} : J \to \mathrm{P}(X)$ be a set-valued function defined by $\mathrm{D}(x) = \dfrac{x^2}{\{0\}}$ for all $x \in J$. Then $\mathrm{D}(b) = \{0, c\} \lhd_c \mathrm{A}(b)$ implies $(\mathrm{D}, J)\tilde{\lhd}_c(\mathrm{A}, A)$.

But $(\mathrm{H}, U) := (\mathrm{B}, I)\tilde{\cup}(\mathrm{D}, J)$ is not a commutative soft ideal of (A, A), since $\mathrm{H}(b) = \mathrm{B}(b) \cup \mathrm{D}(b) = \{0, a, b, c\}$ is not a commutative ideal of X related to $\mathrm{A}(b)$ because $(d * 0) * c = b \in \mathrm{H}(b)$ and $d * (0 * (0 * d)) = d \notin \mathrm{H}(b)$.

5. Commutative idealistic soft BCK-algebras

Definition 5.1. [7] Let (A, A) be a soft set over X. Then (A, A) is called an *idealistic soft BCK-algebra* over X if $\mathrm{A}(x)$ is an ideal of X for all $x \in A$.

Definition 5.2. Let (A, A) be a soft set over X. Then (A, A) is called a *commutative idealistic soft BCK-algebra* over X if it satisfies:

$$(\forall x \in A)\,(\mathrm{A}(x) \lhd_c X). \tag{5.1}$$

Let us illustrate this definition using the following examples.

Example 5.3. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	a
b	b	a	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let (A, A) be a soft set over X, where $A = \{b, c, d\}$ and $\mathrm{A} : A \to \mathcal{P}(X)$ is a set-valued function defined by $\mathrm{A}(x) = \dfrac{x}{\{0, d\}}$ for all $x \in A$. Then $\mathrm{A}(b) = \{0, a, b, d\}$, $\mathrm{A}(c) = \{0, c, d\}$ and $\mathrm{A}(d) = \{0, d\}$, which are commutative ideals of X. Hence (A, A) is a commutative idealistic soft BCK-algebra over X.

Note that every commutative idealistic soft BCK-algebra over X is an idealistic soft BCK-algebra over X, but the converse is not true in general as seen in the following example.

Example 5.4. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	a	0	b	0
c	c	c	c	0	0
d	d	d	d	d	0

Let (A, A) be a soft set over X, where $A = X$ and $\mathrm{A} : A \to \mathcal{P}(X)$ is a set-valued function defined by $\mathrm{A}(x) = \dfrac{x^2}{\{0, a, c\}}$ for all $x \in A$. It is easy to verify that (A, A) is a soft BCK-algebra over X. Now let (B, I) be a soft set over X, where $I = \{0, b, c, d\} \subset A$ and $\mathrm{B} : I \to \mathrm{P}(X)$ is a set-valued function defined by $\mathrm{B}(x) = \dfrac{x}{\{0, c\}}$ for all $x \in I$. Then $\mathrm{B}(0) = \mathrm{B}(c) = \{0, c\} \lhd X$, $\mathrm{B}(b) = \{0, a, b, c\} \lhd X$ and $\mathrm{B}(d) = X \lhd X$. Hence (B, I) is an idealistic soft BCK-algebra over X. Now we have $(a*d)*c = 0*c = 0 \in \{0, c\}$ and $a*(d*(d*a)) = a*(d*d) = a*0 = a \notin \{0, c\}$. Thus $\mathrm{B}(0) = \mathrm{B}(c) = \{0, c\}$ is not a commutative ideal of X, and so (B, I) is not a commutative idealistic soft BCK-algebra over X.

Definition 5.5. [8] Let (A, A) be a soft set over X. Then (A, A) is called a *positive implicative idealistic soft BCK-algebra* over X if it satisfies:

$$(\forall x \in A)\,(\mathrm{A}(x) \lhd_{pi} X). \tag{5.2}$$

In the following examples, we know that there is no relations between positive implicative idealistic soft BCK-algebras over X and commutative idealistic soft BCK-algebras over X.

Example 5.6. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	d	c	b	0

Let (A, A) be a soft set over X, where $A = \{0, b, c, d\}$ and $\mathrm{A} : A \to \mathrm{P}(X)$ is a set-valued function defined by $\mathrm{A}(x) = \dfrac{x^2}{\{0, b\}}$ for all $x \in A$. Then (A, A) is a positive implicative idealistic soft BCK-algebra over X (see [6]). But $\mathrm{A}(c) = \mathrm{A}(d) = \{0, b\}$ is not a commutative ideal of X since $(a * c) * 0 = 0 \in \{0, b\}$ and $a * (c * (c * a)) = a \notin \{0, b\}$, and hence (A, A) is not a commutative idealistic soft BCK-algebra over X.

Example 5.7. Recall that the soft set (A, A) which is given in Example 5.3 is a commutative idealistic soft BCK-algebra over X. But $\mathrm{A}(c) = \{0, c, d\}$ and $\mathrm{A}(d) = \{0, d\}$ are not positive implicative ideals of X since $(b*a)*a = 0 \in \{0, d\}$, $a*a = 0 \in \{0, d\}$, and $b*a = a \notin \{0, d\}$. Hence (A, A) is not a positive implicative idealistic soft BCK-algebra over X.

Proposition 5.8. *Let* (A, A) *and* (A, B) *be soft sets over* X *where* $B \subseteq A \subseteq X$. *If* (A, A) *is a commutative idealistic soft BCK-algebra over* X, *then so is* (A, B).

Proof. Straightforward. □

If we take $J := \{b, d\} \subseteq I$ in Example 5.4, then $\mathrm{B}(b) = \{0, a, b, c\} \lhd_c X$ and $\mathrm{B}(d) = X \lhd_c X$. Hence (B, J) is a commutative idealistic soft BCK-algebra over X. But (B, I) is not a commutative idealistic soft BCK-algebra over X (see Example 5.4). This shows that the converse of Proposition 5.8 is not true in general.

Theorem 5.9. *Let* (A, A) *and* (B, B) *be two commutative idealistic soft BCK-algebras over* X. *If* $A \cap B \neq \emptyset$, *then the intersection* $(\mathrm{A}, A)\tilde{\cap}(\mathrm{B}, B)$ *is a commutative idealistic soft BCK-algebra over* X.

Proof. Using Definition 3.2, we can write $(\mathrm{A}, A)\tilde{\cap}(\mathrm{B}, B) = (\mathrm{D}, C)$, where $C = A \cap B$ and $\mathrm{D}(x) = \mathrm{A}(x)$ or $\mathrm{B}(x)$ for all $x \in C$. Note that $\mathrm{D} : C \to \mathrm{P}(X)$ is a mapping, and therefore (D, C) is a soft set over X. Since (A, A) and (B, B) are commutative idealistic soft BCK-algebras over X, it follows that $\mathrm{D}(x) = \mathrm{A}(x)$ is a commutative ideal of X, or $\mathrm{D}(x) = \mathrm{B}(x)$ is a commutative ideal of X for all $x \in C$. Hence $(\mathrm{D}, C) = (\mathrm{A}, A)\tilde{\cap}(\mathrm{B}, B)$ is a commutative idealistic soft BCK-algebra over X. □

Corollary 5.10. *Let* (A, A) *and* (B, A) *be two commutative idealistic soft BCK-algebras over* X. *Then their intersection* $(\mathrm{A}, A)\tilde{\cap}(\mathrm{B}, A)$ *is a commutative idealistic soft BCK-algebra over* X.

Proof. Straightforward. □

Theorem 5.11. *Let* (A, A) *and* (B, B) *be two commutative idealistic soft BCK-algebras over* X. *If* A *and* B *are disjoint, then the union* $(\mathrm{A}, A)\tilde{\cup}(\mathrm{B}, B)$ *is a commutative idealistic soft BCK-algebra over* X.

Proof. Using Definition 3.3, we can write $(\mathrm{A}, A)\tilde{\cup}(\mathrm{B}, B) = (\mathrm{D}, C)$, where $C = A \cup B$ and for every $e \in C$,

$$\mathrm{D}(e) = \begin{cases} \mathrm{A}(e) & \text{if } e \in A \setminus B, \\ \mathrm{B}(e) & \text{if } e \in B \setminus A, \\ \mathrm{A}(e) \cup \mathrm{B}(e) & \text{if } e \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, either $x \in A \setminus B$ or $x \in B \setminus A$ for all $x \in C$. If $x \in A \setminus B$, then $\mathrm{D}(x) = \mathrm{A}(x)$ is a commutative ideal of X since (A, A) is a commutative idealistic soft BCK-algebra over X. If $x \in B \setminus A$, then $\mathrm{D}(x) = \mathrm{B}(x)$ is a commutative ideal of X since (B, B) is a commutative idealistic soft BCK-algebra over X. Hence $(\mathrm{D}, C) = (\mathrm{A}, A)\tilde{\cup}(\mathrm{B}, B)$ is a commutative idealistic soft BCK-algebra over X. □

Theorem 5.12. *If* (A, A) *and* (B, B) *are commutative idealistic soft BCK-algebras over* X, *then* $(\mathrm{A}, A)\tilde{\wedge}(\mathrm{B}, B)$ *is a commutative idealistic soft BCK-algebra over* X.

Proof. By means of Definition 3.4, we know that

$$(\mathrm{A}, A)\tilde{\wedge}(\mathrm{B}, B) = (\mathrm{D}, A \times B),$$

where $\mathrm{D}(x, y) = \mathrm{A}(x) \cap \mathrm{B}(y)$ for all $(x, y) \in A \times B$. Since $\mathrm{A}(x)$ and $\mathrm{B}(y)$ are commutative ideals of X, the intersection $\mathrm{A}(x) \cap \mathrm{B}(y)$ is also a commutative ideal of X. Hence $\mathrm{D}(x, y)$ is a commutative ideal of X for all $(x, y) \in A \times B$, and therefore $(\mathrm{A}, A)\tilde{\wedge}(\mathrm{B}, B) = (\mathrm{D}, A \times B)$ is a commutative idealistic soft BCK-algebra over X. □

Definition 5.13. A commutative idealistic soft BCK-algebra (A, A) over X is said to be *trivial* (resp., *whole*) if $\mathrm{A}(x) = \{0\}$ (resp., $\mathrm{A}(x) = X$) for all $x \in A$.

Example 5.14. Let $X = \{0, a, b, c, d\}$ be the BCK-algebra which is described in Example 5.4. Consider $A = \{b, d\} \subset X$ and a set-valued function $\mathrm{A} : A \to \mathrm{P}(X)$ defined by $\mathrm{A}(x) = \frac{x}{\{0, c, d\}}$ for all $x \in A$. Then $\mathrm{A}(b) = \frac{b}{\{0, c, d\}} = X$ and $\mathrm{A}(d) = \frac{d}{\{0, c, d\}} = X$. Hence (A, A) is a whole commutative idealistic soft BCK-algebra over X.

Example 5.15. Let $X = \{0, a, b, c, d\}$ be the BCK-algebra which is described in Example 4.12. Consider $A = \{c, d\} \subset X$ and a set-valued function $\mathrm{A} : A \to \mathrm{P}(X)$ defined by $\mathrm{A}(x) = \frac{x}{\{0, b\}}$ for all $x \in A$. Then $\mathrm{A}(c) = \frac{c}{\{0, b\}} = X$ and $\mathrm{A}(d) = \frac{d}{\{0, b\}} = X$. Hence (A, A) is a whole commutative idealistic soft BCK-algebra over X.

Example 5.16. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	b	a	0	a
d	d	b	a	a	0

Let $\mathrm{B} : \{0\} \to \mathrm{P}(X)$ be a set-valued function given by $\mathrm{B}(0) = \frac{0}{\{0\}}$. Then $(\mathrm{B}, \{0\})$ is a trivial commutative idealistic soft BCK-algebra over X.

Example 5.17. Let $X = \{0, a, b, c, d\}$ be the BCK-algebra which is given in Example 5.3. Let $\mathrm{B} : \{0\} \to \mathrm{P}(X)$ be a set-valued function given by $\mathrm{B}(0) = \frac{0}{\{0\}}$. Then $(\mathrm{B}, \{0\})$ is a trivial commutative idealistic soft BCK-algebra over X.

The following example shows that there exists a BCK-algebra X such that a soft set $(\mathrm{A}, \{0\})$ may not be a trivial commutative idealistic soft BCK-algebra over X, where $\mathrm{A} : \{0\} \to \mathrm{P}(X)$ is given by $\mathrm{A}(0) = \frac{0}{\{0\}}$.

Example 5.18. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	b	a	0	a
d	d	d	d	d	0

Let $\mathrm{A} : \{0\} \to \mathrm{P}(X)$ be a set-valued function given by $\mathrm{A}(0) = \dfrac{0}{\{0\}}$. Then $\mathrm{A}(0) = \{0\}$ is not a commutative ideal of X since $(b * d) * 0 = 0 \in \mathrm{A}(0)$ and $b * (d * (d * b)) = b * (d * d) = b * 0 = b \notin \mathrm{A}(0)$. Hence $(\mathrm{A}, \{0\})$ is not a trivial commutative idealistic soft BCK-algebra over X.

Proposition 5.19. *Let (A, A) be a soft set over X defined by $\mathrm{A}(x) = \dfrac{x}{\{0\}}$ for all $x \in A$. Then*

(i) *(A, A) is a trivial idealistic soft BCK-algebra over X if and only if $A = \{0\}$.*
(ii) *Assume that $A = \{0\}$. Then (A, A) is commutative if and only if X is commutative.*

Proof. (i) If $A = \{0\}$, then $\mathrm{A}(0) = \{0\} \lhd X$. Hence (A, A) is a trivial idealistic soft BCK-algebra over X. Conversely, assume that $A \neq \{0\}$. Then there exists $a (\neq 0) \in A$, and so $\{0, a\} \subseteq \mathrm{A}(a)$ since $a * a = 0$. This is a contradiction.

(ii) Note that X is commutative if and only if $\{0\}$ is a commutative ideal of X. Hence it is straightforward. □

Let $f : X \to Y$ be a mapping of BCK-algebras. For a soft set (A, A) over X, $(f(\mathrm{A}), A)$ is a soft set over Y where $f(\mathrm{A}) : A \to \mathrm{P}(Y)$ is defined by $f(\mathrm{A})(x) = f(\mathrm{A}(x))$ for all $x \in A$.

Lemma 5.20. *Let $f : X \to Y$ be an onto homomorphism of BCK-algebras. If (A, A) is a commutative idealistic soft BCK-algebra over X, then $(f(\mathrm{A}), A)$ is a commutative idealistic soft BCK-algebra over Y.*

Proof. For every $x \in A$, we have $f(\mathrm{A})(x) = f(\mathrm{A}(x))$ is a commutative ideal of Y since $\mathrm{A}(x)$ is a commutative ideal of X and its onto homomorphic image is also a commutative ideal of Y. Hence $(f(\mathrm{A}), A)$ is a commutative idealistic soft BCK-algebra over Y. □

Theorem 5.21. *Let $f : X \to Y$ be an onto homomorphism of BCK-algebras and let* (A, A) *be a commutative idealistic soft BCK-algebra over* X.

(i) *If* $\mathrm{A}(x) \subset \ker(f)$ *for all* $x \in A$, *then* $(f(\mathrm{A}), A)$ *is a trivial commutative idealistic soft BCK-algebra over* Y.

(ii) *If* (A, A) *is whole, then* $(f(\mathrm{A}), A)$ *is a whole commutative idealistic soft BCK-algebra over* Y.

Proof. (i) By Lemma 5.20, $(f(\mathrm{A}), A)$ is a commutative idealistic soft BCK-algebra over Y. Assume that $\mathrm{A}(x) \subset \ker(f)$ for all $x \in A$. Then $f(\mathrm{A})(x) = f(\mathrm{A}(x)) \subset f(\ker(f)) = \{0\} \subset f(\mathrm{A})(x)$, and so $f(\mathrm{A})(x) = \{0\}$ for all $x \in A$. It follows from Definition 5.13 that $(f(\mathrm{A}), A)$ is a trivial commutative idealistic soft BCK-algebra over Y.

(ii) Suppose that (A, A) is whole. Then $\mathrm{A}(x) = X$ for all $x \in A$, and so $f(\mathrm{A})(x) = f(\mathrm{A}(x)) = f(X) = Y$ for all $x \in A$. It follows from Lemma 5.20 and Definition 5.13 that $(f(\mathrm{A}), A)$ is a whole commutative idealistic soft BCK-algebra over Y. □

6. Conclusions

The concept of soft set, which is introduced by Molodtsov [12], is a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. As an algebraic application, we applied the notion of soft set to BCK-algebras. We introduced the notion of commutative soft ideals and commutative idealistic soft BCK-algebras, and gave several examples. We investigated relations between idealistic soft BCK-algebras and commutative idealistic soft BCK-algebras. We gave examples to show that there is no relations between positive implicative idealistic soft BCK-algebras and commutative idealistic soft BCK-algebras. Finally, we established the intersection, union, "AND" operation, and "OR" operation of commutative soft ideals and commutative idealistic soft BCK-algebras. Based on these results, we will apply the notion of soft set to implicative ideals of BCK-algebras. We will define an implicative soft ideal and an implicative idealistic soft BCK-algebra. We will discuss relationships between positive implicative idealistic soft BCK-algebras, commutative idealistic soft BCK-algebras and implicative idealistic soft BCK-algebras.

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