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EXTREMAL PROBLEM OF A QUADRATICALLY HYPONORMAL WEIGHTED SHIFT

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ABSTRACT. Let W_α be a recursively generated quadratically hyponormal weighted shift with a weight sequence $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$. In [4] Curto-Jung showed that $\mathcal{R} = \{(x,y) : W_{1,(1,\sqrt{x},\sqrt{y})^\wedge}$ is quadratically hyponormal $\}$ is a closed convex with nonempty interior, which guarantees that there are a lot of quadratically hyponormal weighted shifts with first two equal weights. They suggested a problem computing expressions of certain extremal points of $\mathcal{R}$. In this note we obtain a partial answer of their extremal problem.

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1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For A, $B \in \mathcal{L}(\mathcal{H})$, let $[A,B] := AB - BA$. We say that an n-tuple $T = (T_1, ..., T_n)$ of operators in $\mathcal{L}(\mathcal{H})$ is *hyponormal* if the operator matrix $\left([T_j^*, T_i]\right)_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For $k \geq 1$ and $T \in \mathcal{L}(\mathcal{H})$, $T = (T_1, ..., T_k)$ is *weakly-hyponormal* if $\lambda_1 T_1 + \cdots + \lambda_k T_k$ is hyponormal for every $\lambda_i \in \mathbb{C}$, $i = 1, ..., k$, where $\mathbb{C}$ is the set of complex numbers.

An operator T is *weakly k-hyponormal* if $(T, ..., T^k)$ is weakly hyponormal. In particular, weak 2-hyponormality, often referred to as *quadratic hyponormality*, was discussed in [1], [2], and [3]. To detect the quadratic hyponormality of weighted shifts, Fialkow-Curto introduced the concept of positively quadratically hyponormal weighted shifts whose definition appears in [3].

Also it was shown in [8] that two notions of quadratic hyponormality and positively quadratic hyponormality are equivalent in the one-step extended weighted shifts $W_{\widehat{\alpha}}$ with a tail induced recursively by three numbers $0 < b < c < d$, where

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$\widehat{\alpha} : \sqrt{a}, (\sqrt{b}, \sqrt{c}, \sqrt{d})^\wedge$ (which will be defined below). Furthermore, the flatness of weighted shifts makes an important role to study the quadratic hyponormality. As one of such models for studying its flatness, in [4] they considered the recursively weighted shift $\widehat{\alpha}(x,y) : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ with $1 \le x \le y$ and obtain that the set

$$\mathcal{R} = \{(x,y) : W_{\widehat{\alpha}(x,y)} \text{ is quadratically hyponormal}\}$$

is a closed convex set with nonempty interior and there exist unique maximum values x_M and y_M of x and y such that $\mathcal{R} \cap (\{x_M\} \times \mathbb{R})$ and $\mathcal{R} \cap (\mathbb{R} \times \{y_M\})$ are singletons, where $\mathbb{R}$ is the set of real numbers. And they suggested the following extremal value problem.

Problem 1.1([Problem 5.1][4]). *Find a concrete expression for x_M and y_M.*

According to Corollary 2.2 below, it is worthwhile to consider only the case of weighted shift W_α with $\|W_\alpha\| = 1$ to detect the quadratic hyponormality. For a given $a \in (0,1)$, let $\widehat{\alpha}(x,y) : \sqrt{a}, (\sqrt{a}, \sqrt{x}, \sqrt{y})^\wedge$ be a weight sequence with $a \le x \le y$. In this note we solve Problem 1.1 for a weighted shift $W_{\widehat{\alpha}(x,y)}$ with $\left\|W_{\widehat{\alpha}(x,y)}\right\| = 1$.

We now recall [2] that a weighted shift W_α is said to be *recursively generated* if there exist $i \ge 1$ and $\Psi = (\Psi_0, \cdots, \Psi_{i-1}) \in \mathbb{C}^i$ such that

$$\gamma_n = \Psi_{i-1}\gamma_{n-1} + \cdots + \Psi_0\gamma_{n-i} \quad (n \ge i), \tag{1.1}$$

where $\gamma_n (n \ge 0)$ is the moment sequence of W_α, i.e., $\gamma_0 := 1$, $\gamma_{n+1} := \alpha_n^2 \gamma_n$ $(n \ge 0)$. Furthermore, (1.1) is equivalent to

$$\alpha_n^2 = \Psi_{i-1} + \frac{\Psi_{i-2}}{\alpha_{n-1}^2} + \cdots + \frac{\Psi_0}{\alpha_{n-1}^2 \cdots \alpha_{n-i+1}^2} \quad (n \ge i).$$

Given an initial segment of weights $\alpha : \alpha_0, \cdots, \alpha_{2k}$ $(k \ge 0)$, there is a canonical procedure to generate a sequence (denote $\widehat{\alpha}$) in such a way that $W_{\widehat{\alpha}}$ is a recursively generated shift having α as an initial segment of weights (cf. [2]). We now review this procedure in a special case of $k = 1$. Given $\alpha : \alpha_0, \alpha_1, \alpha_2$ $(0 < \alpha_0 < \alpha_1 < \alpha_2)$, let

$$v_0 := \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}, \quad v_1 := \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad v_2 := \begin{pmatrix} \gamma_2 \\ \gamma_3 \end{pmatrix}.$$

The vectors v_0 and v_1 are linearly independent in $\mathbb{R}^2$, so there exists a unique $\Psi = (\Psi_0, \Psi_1) \in \mathbb{R}^2$ such that $v_2 = \Psi_0 v_0 + \Psi_1 v_1$. In fact,

$$\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2} \quad \text{and} \quad \Psi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}.$$

Let $\widehat{\gamma} := \gamma_n$ $(0 \le n \le 1)$ and let $\widehat{\gamma}_n := \Psi_1\widehat{\gamma}_{n-1} + \Psi_0\widehat{\gamma}_{n-2}$ $(n \ge 2)$. Then

$$\widehat{\alpha}_n := \sqrt{\widehat{\gamma}_{n+1}/\widehat{\gamma}_n}$$

$(n \geq 0)$ (so that $\widehat{\alpha}_n = \alpha_n$ for $0 \leq n \leq 2$) and the coefficients of a recursively generated weighted shift is $\widehat{\alpha}_n^2 = \Psi_1 + \Psi_0/\widehat{\alpha}_{n-1}^2$ $(n \geq 1)$. Such a recursively weight sequence is written by $(\alpha_0, \alpha_1, \alpha_2)^\wedge$ (cf. [5], [7]).

2. Striving extremal values.

We consider a recursively generated weighted shift W_α with a weight sequence $\alpha : \sqrt{a}, (\sqrt{a}, \sqrt{x}, \sqrt{y})^\wedge$ and $0 < a \leq x \leq y$. In special case, we focus on the weighted shift W_α having the norm one which involves the general case.

We begin with the following elementary lemma.

Lemma 2.1. *Let $0 < a \leq b \leq c$. Then $\sqrt{s} \cdot W_{(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge} = W_{(\sqrt{sa}, \sqrt{sb}, \sqrt{sc})^\wedge}$ for any $s \in (0, \infty)$.*

Proof. First we consider two weight sequences $\alpha : (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ and $\alpha' : (\sqrt{sa}, \sqrt{sb}, \sqrt{sc})^\wedge$ with $0 < a \leq b \leq c$ for any fixed $s > 0$. Put

$$\Psi_0 = \frac{ab(b-c)}{b-a} \quad \text{and} \quad \Psi_1 = \frac{b(c-a)}{b-a} \tag{1.2}$$

and let $\gamma_0 = 1, \gamma_1 = a$, and $\gamma_{n+2} = \Psi_1 \gamma_{n+1} + \Psi_0 \gamma_n$ $(n \geq 0)$. Then $\alpha_n := \left(\frac{\gamma_{n+1}}{\gamma_n}\right)^{\frac{1}{2}}$ $(n \geq 0)$ are the coefficients of a recursively generated weighted shift for the recursive sequence $(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$. On the other hand, by similar method, we can find the numbers

$$\Psi_0' = \frac{s^2 ab(b-c)}{b-a} = s^2 \Psi_0 \quad \text{and} \quad \Psi_1' = \frac{sb(c-a)}{b-a} = s\Psi_1$$

about the recursive sequence $(\sqrt{sa}, \sqrt{sb}, \sqrt{sc})^\wedge$. Put $\gamma_0' = 1, \gamma_1' = sa$, and $\gamma_{n+2}' = \Psi_1' \gamma_{n+1}' + \Psi_0' \gamma_n'$ $(n \geq 0)$. Then $\alpha_n' := \left(\frac{\gamma_{n+1}'}{\gamma_n'}\right)^{\frac{1}{2}}$ $(n \geq 0)$ are the coefficients of $(\sqrt{sa}, \sqrt{sb}, \sqrt{sc})^\wedge$. Clearly $\alpha_0' = \sqrt{s}\alpha_0$, and since

$$\alpha_{i+1}' = \sqrt{\frac{\gamma_{i+2}'}{\gamma_{i+1}'}} = \sqrt{s}\sqrt{\frac{\gamma_{i+2}}{\gamma_{i+1}}} = \sqrt{s}\alpha_{i+1} \ (i \geq 0),$$

$\alpha_n' = \sqrt{s}\alpha_n$ for all $n \geq 0$. □

Corollary 2.2. *Let $\alpha : \sqrt{\alpha_0}, \sqrt{\alpha_1}, \cdots, \sqrt{\alpha_{n-1}}, (\sqrt{\alpha_n}, \sqrt{\alpha_{n+1}}, \sqrt{\alpha_{n+2}})^\wedge$ with $0 < \alpha_{i-1} \leq \alpha_i$ for all $i \geq 1$. Then the unilateral weighted shift W_α has norm $\sqrt{\delta}$ if and only if the shift $W_{\alpha'}$ with*

$$\alpha' : \sqrt{\frac{\alpha_0}{\delta}}, \cdots, \sqrt{\frac{\alpha_{n-1}}{\delta}}, \left(\sqrt{\frac{\alpha_n}{\delta}}, \sqrt{\frac{\alpha_{n+1}}{\delta}}, \sqrt{\frac{\alpha_{n+2}}{\delta}}\right)^\wedge$$

has norm 1.

Proof. By Lemma 2.1, $W_{\alpha'} = \frac{1}{\sqrt{\delta}} W_\alpha$. So $\|W_{\alpha'}\| = \left\|\frac{1}{\sqrt{\delta}} W_\alpha\right\| = \frac{1}{\sqrt{\delta}} \|W_\alpha\| = 1$. □

Let $W_{\widehat{\alpha}}$ be a recursively generated weighted shift where $\widehat{\alpha} : (\sqrt{a}, \sqrt{b}, \sqrt{c})^{\wedge}$ and $0 < a < b < c$. Since $\|W_\alpha\|^2 = (\Psi_1 + \sqrt{\Psi_1^2 + 4\Psi_0})/2$, we have that $\|W_{\widehat{\alpha}}\| = 1$ if and only if $c = \dfrac{a(b^2 - b + 1) - b}{(a-1)b}$.

Theorem 2.3. *Let W_α be a recursively generated weighted shift with α : $\sqrt{a}, (\sqrt{a}, \sqrt{x}, \sqrt{y})^{\wedge}$, $0 < a < x < y \leq 1$, and $\|W_\alpha\| = 1$. Then W_α is quadratically hyponormal if and only if $x \in (a, r_a]$ where r_a is the root of $f(x) = 0$, where $f(x) = \sum_{i=0}^{4} c_i x^i$ with*

$$\begin{aligned} c_0 &: = a > 0, \\ c_1 &: = -(a^5 - a^4 - a^3 + 3a^2 + 1) < 0, \\ c_2 &: = a(2a^4 - 3a^3 + a^2 + 3) > 0, \\ c_3 &: = -a^2(a^3 - 2a^2 - a + 3) < 0, \\ c_4 &: = a^3(1-a) > 0. \end{aligned}$$

(Note that $0 < r_a < 1$.)

Proof. Since $\|W_\alpha\| = 1$, by the above remark $y = \dfrac{a(x^2 - x + 1) - x}{(a-1)x}$. Note that W_α is quadratically hyponormal if and only if sW_α is quadratically hyponormal for all $s > 0$. Combining this fact with [8] and Lemma 2.1, W_α is quadratically hyponormal if and only if $W_{\alpha'}$ is positively quadratically hyponormal with weight sequence $\alpha' : 1, \left(1, \sqrt{\dfrac{x}{a}}, \sqrt{\dfrac{a(x^2 - x + 1) - x}{a(a-1)x}}\right)^{\wedge}$. Adjusting numbers from (1.2), we obtain

$$\Psi_0 := -\frac{x-1}{a(a-1)} \quad \text{and} \quad \Psi_1 := \frac{ax-1}{a(a-1)}.$$

And also we consider

$$K := -\frac{\Psi_1^2}{\Psi_0} \cdot \frac{\Psi_1 + \sqrt{\Psi_1^2 + 4\Psi_0}}{2}.$$

By results in [4], $W_{\alpha'}$ is positively quadratically hyponormal if and only if

$$1 \leq h_2^+ := \left(\frac{x[x\Delta + x(\Delta - 1)K + (a\Delta - x)K^2]}{ax[1 + (\Delta - 1)K] + a(a + x\Delta - 2ax)K^2} \right)^{1/2},$$

where

$$K := \frac{(ax-1)^2}{a^2(a-1)(x-1)} \quad \text{and} \quad \Delta := \frac{a(x^2 - x + 1) - x}{a(a-1)x},$$

which is equivalent to $f(x) \geq 0$. Furthermore, since for $a < 1 < \frac{1}{a} < 1 + \frac{1}{a}$,

$$\begin{aligned} f(a) &= (1-a)^3 a^4 (1+2a) > 0, \\ f(1) &= -(1-a)^4 < 0, \\ f\left(\frac{1}{a}\right) &= (1-a)^3 a > 0, \end{aligned}$$

and

$$f\left(1 + \frac{1}{a}\right) = -a^4 < 0,$$

the function f has 4 different positive roots. Hence the smallest root r_a of these positive roots is the required one. □

The following remark provides the concrete expression for r_a.

Remark 2.4. According to simple computation, the root r_a is expressed by

$$r_a = -\frac{c_3}{4c_4} - \frac{1}{2}G - \frac{1}{2}\sqrt{\frac{c_3^2}{2c_4^2} - \frac{4c_2}{3c_4} - A - B - \frac{t}{4G}}, \tag{2.1}$$

where

$$A = \frac{2^{\frac{1}{3}} q}{3c_4 (p + \sqrt{-4q^3 + p^2})^{\frac{1}{3}}}, \tag{2.1a}$$

$$B = \frac{(p + \sqrt{-4q^3 + p^2})^{\frac{1}{3}}}{3 2^{\frac{1}{3}} c_4}, \tag{2.1b}$$

$$G = \sqrt{\frac{c_3^2}{4c_4^2} - \frac{2c_2}{3c_4} + A + B}, \tag{2.1c}$$

$$t = -\frac{c_3^3}{c_4^3} + \frac{4c_2c_3}{c_4^2} - \frac{8c_1}{c_4}, \tag{2.1d}$$

$$p = 2c_2^3 - 9c_1c_2c_3 + 27c_1^2c_4 + 27c_0c_3^2 - 72c_0c_2c_4, \tag{2.1e}$$

$$q = c_2^2 - 3c_1c_3 + 12c_0c_4. \tag{2.1f}$$

The quantity $-4q^3 + p^2$ in (2.1a) and (2.1b) can be negative. In fact, the imaginary parts of A and B are cancelled in computing of $A + B$. Also, by the proof of Theorem 2.3, r_a should be positive real number, which guarantees the positivity of $\frac{c_3^2}{2c_4^2} - \frac{4c_2}{3c_4} - A - B - \frac{t}{4G}$ in (2.1).

Example 2.5. If we consider $a = \frac{1}{2}$, then $f(x) = \frac{1}{16}x^4 - \frac{17}{32}x^3 + \frac{3}{2}x^2 - \frac{51}{32}x + \frac{1}{2}$ and so

$$r_a = \frac{1}{8}\left(17 - \sqrt{17} - \sqrt{2(41 - \sqrt{17})}\right).$$

Hence W_α is quadratically hyponormal if and only if

$$1/2 < x \le \frac{1}{8}\left(17 - \sqrt{17} - \sqrt{2(41 - \sqrt{17})}\right) \approx 0.53611.$$

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