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TIGHT UPPER BOUND ON THE EXPONENTS OF A CLASS OF TWO-COLORED DIGRAPHS

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ABSTRACT. A two-colored digraph D is primitive if there exist nonnegative integers h and k with $h+k>0$ such that for each pair (i,j) of vertices there exists an (h,k)-walk in D from i to j. The exponent of the primitive two-colored digraph D is the minimum value of $h+k$ taken over all such h and k. In this paper, we give the tight upper bound on the exponents of a class of primitive two-colored digraphs with $(s+1)$ $n-$cycles and one $(n-1)-$cycle, and the characterizations of the extremal two-colored digraphs.

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1. Introduction

A *two-colored digraph* is a digraph whose arcs are colored red or blue. We allow loops and both a red arc and blue arc from i to j for all pairs (i,j) of vertices. The two-colored digraph D is *strongly connected* provided for each pair (i,j) of vertices there is a walk in D from i to j. Given a walk w in D, let $r(w)$ (respectively, $b(w)$) denote the number of red arcs (respectively, blue arcs) of w. We call w a $(r(w),b(w))$*-walk*, and define the *composition* of w to be the vector $(r(w),b(w))$ or

$$\begin{bmatrix} r(w) \\ b(w) \end{bmatrix}.$$

A two-colored digraph D is primitive if there exist nonnegative integers h and k with $h+k>0$ such that for each pair (i,j) of vertices there exists an (h,k)-walk in D from i to j. The *exponent* of the primitive two-colored digraph D is the minimum value of $h+k$ taken over all such h and k.

Let $C=\{\gamma_1,\gamma_2,\dots,\gamma_l\}$ be the set of cycles of a two-colored digraph D. Set M to be the $2\times l$ matrix whose ith column is the composition of γ_i. We call M

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the *cycle matrix* of D. The *content* of M, denoted content(M), is defined to be 0 if the rank of M is less than 2 and the greatest common divisor (i.e., g.c.d) of the determinants of all 2×2 submatrices of M, otherwise.

Lemma 1. ([1]) *Let D be a two-colored digraph with cycle matrix M. Then D is primitive if and only if D is strongly connected and* content$(M) = 1$.

There is a natural correspondence between two-colored digraphs and nonnegative matrix pairs (see[1]). The concept of the exponent of two-colored digraph arises in the study of finite Markov chains (see [1, 2]), and some results have already been obtained (see[1, 4, 5, 6, 7]).

In this paper, we consider the class of two-colored digraphs of order $n+s$, denoted by $D_{n,s}$, obtained by coloring the digraph as in Fig.1, where $n \geq 3$. If $s = 0$, then the digraph is Wielandt digraph of order n that has the largest exponent $(n-1)^2+1$. The paper [1] gives the tight bound on the exponents of families of primitive two-colored digraphs of order n whose uncolored digraph is Wielandt digraph. Motivated by the paper [1], we consider the exponents of $D_{n,s}$, where $s \geq 1$.

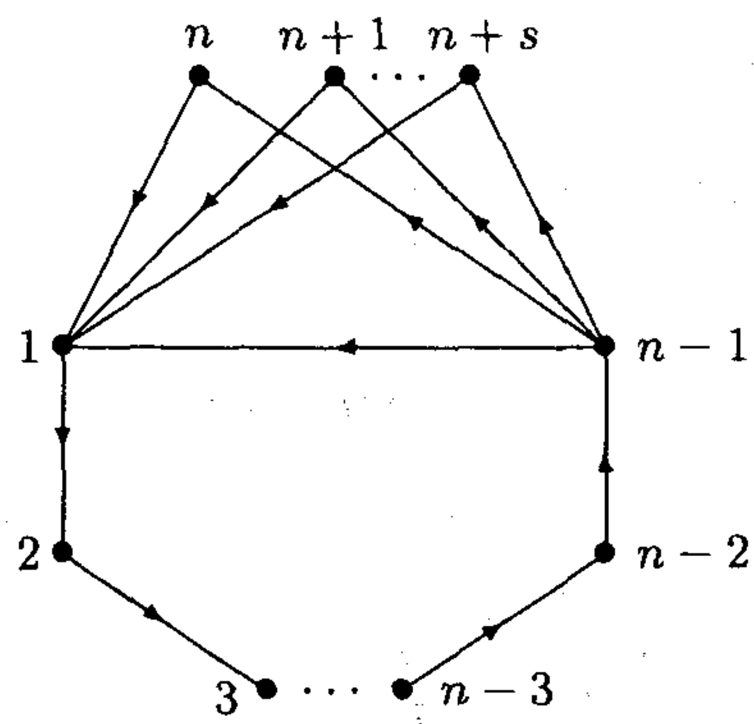

Fig. 1 The digraph

Clearly, for any $D \in D_{n,s}$, D has $(s+1)$ $n-$cycles and one $(n-1)-$cycle. Without loss of generality, we may assume the arc $(n-1) \to 1$ is red. The path $(n-1) \to j \to 1$ has the following four cases for $j = n, n+1, \ldots, n+s$:

Case 1. The arcs $(n-1) \to j$ and $j \to 1$ are all red. In this case, we call the path $(n-1) \to j \to 1$ a red-red path.

Case 2. The arcs $(n-1) \to j$ and $j \to 1$ are all blue. In this case, we call the path $(n-1) \to j \to 1$ a blue-blue path.

Case 3. The arc $(n-1) \to j$ is red and the arc $j \to 1$ is blue. In this case, we call the path $(n-1) \to j \to 1$ a red-blue path.

Case 4. The arc $(n-1) \to j$ is blue and the arc $j \to 1$ is red. In this case, we call the path $(n-1) \to j \to 1$ a blue-red path.

Thus the two-colored digraphs in $D_{n,s}$ have the following fifteen cases:

Case 1. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ are all red-red paths.

Case 2. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ are all blue-blue paths.

Case 3. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ are all red-blue paths.

Case 4. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ are all blue-red paths.

Case 5. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only red-red paths and blue-blue paths.

Case 6. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only red-red paths and red-blue paths.

Case 7. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only red-red paths and blue-red paths.

Case 8. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only blue-blue paths and red-blue paths.

Case 9. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only blue-blue paths and blue-red paths.

Case 10. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only red-blue paths and blue-red paths.

Case 11. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only red-red paths, blue-blue paths, and red-blue paths.

Case 12. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only red-red paths, blue-blue paths, and blue-red paths.

Case 13. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only red-red paths, red-blue paths, and blue-red paths.

Case 14. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only blue-blue paths, red-blue paths, and blue-red paths.

Case 15. The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain exactly red-red paths, blue-blue paths, red-blue paths and blue-red paths.

Throughout the remainder of the paper, for any $D \in D_{n,s}$, we let M be the cycle matrix of D, $\gamma_1, \gamma_2, \ldots, \gamma_{s+1}$ be $(s+1)$ $n-$cycles of D, γ_{s+2} be the $(n-1)-$cycle of D, and the composition of γ_i be the ith column of M for $i = 1, 2, \ldots, s+2$.

2. The primitivity of a two-colored digraph in $D_{n,s}$

Let $D \in D_{n,s}$. Note that D is strongly connected. We assume that the path $1 \to 2 \to \cdots \to (n-2) \to (n-1)$ has a red arcs and $(n-a-2)$ blue arcs. Clearly, $0 \le a \le n-2$.

For *Case 1,* the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & \cdots & a+2 & a+1 \\ n-a-2 & \cdots & n-a-2 & n-a-2 \end{bmatrix}.$$

Then content$(M) = n-a-2$, and so D is primitive if and only if $a = n-3$.

For *Case 2,* the cycle matrix of D is

$$M = \begin{bmatrix} a & \cdots & a & a+1 \\ n-a & \cdots & n-a & n-a-2 \end{bmatrix}.$$

Then content$(M) = n + a$. Since $n \geq 3$ and $a \geq 0$, we have $n + a \geq 3$. So content$(M) \neq 1$, and D is not primitive.

For *Case 3,* the cycle matrix of D is

$$M = \begin{bmatrix} a+1 & \cdots & a+1 & a+1 \\ n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content$(M) = a + 1$, and so D is primitive if and only if $a = 0$.

For *Case 4,* the cycle matrix of D is

$$M = \begin{bmatrix} a+1 & \cdots & a+1 & a+1 \\ n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content$(M) = a + 1$, and so D is primitive if and only if $a = 0$.

For *Case 5,* the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & \cdots & a+2 & a & \cdots & a & a+1 \\ n-a-2 & \cdots & n-a-2 & n-a & \cdots & n-a & n-a-2 \end{bmatrix}.$$

Then content$(M) =$ g.c.d$\{2n, n-a-2, -n-a\}$, and so D is primitive if and only if $n - a$ is odd.

For *Case 6,* the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & \cdots & a+2 & a+1 & \cdots & a+1 & a+1 \\ n-a-2 & \cdots & n-a-2 & n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content$(M) =$ g.c.d$\{n, n-a-2, -a-1\} = 1$, and so D is primitive.

For *Case 7,* the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & \cdots & a+2 & a+1 & \cdots & a+1 & a+1 \\ n-a-2 & \cdots & n-a-2 & n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content$(M) =$ g.c.d$\{n, n-a-2, -a-1\} = 1$, and so D is primitive.

For *Case 8,* the cycle matrix of D is

$$M = \begin{bmatrix} a & \cdots & a & a+1 & \cdots & a+1 & a+1 \\ n-a & \cdots & n-a & n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content$(M) =$ g.c.d$\{-n, -n-a, -a-1\} = 1$, and so D is primitive.

For *Case 9,* the cycle matrix of D is

$$M = \begin{bmatrix} a & \cdots & a & a+1 & \cdots & a+1 & a+1 \\ n-a & \cdots & n-a & n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content$(M) =$ g.c.d$\{-n, -n-a, -a-1\} = 1$, and so D is primitive.

For *Case 10,* the cycle matrix of D is

$$M = \begin{bmatrix} a+1 & \cdots & a+1 & a+1 \\ n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content$(M) = a + 1$, and so D is primitive if and only if $a = 0$.

For *Case 11,* the cycle matrix of D is

$$M=\begin{bmatrix} a+2 & \cdots & a+2 & a & \cdots & a \\ n-a-2 & \cdots & n-a-2 & n-a & \cdots & n-a \end{bmatrix.$$
$$\begin{matrix} a+1 & \cdots & a+1 & a+1 \\ n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content(M) = g.c.d$\{2n, n, n-a-2, -n, -n-a, -a-1\} = 1$, and so D is primitive.

For *Case 12,* the cycle matrix of D is

$$M=\begin{bmatrix} a+2 & \cdots & a+2 & a & \cdots & a \\ n-a-2 & \cdots & n-a-2 & n-a & \cdots & n-a \end{matrix}$$
$$\begin{matrix} a+1 & \cdots & a+1 & a+1 \\ n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content(M) = g.c.d$\{2n, n, n-a-2, -n, -n-a, -a-1\} = 1$, and so D is primitive.

Table 1

Classification	All kinds of paths in the paths $(n-1) \to j \to 1$ for $j = n, n+1, \dots, n+s$	a
Type 1	red-red paths	$a = n-3$
Type 2	red-blue paths	$a = 0$
Type 3	blue-red paths	$a = 0$
Type 4	red-red paths and blue-blue paths	$n-a$ is odd
Type 5	red-red paths and red-blue paths	
Type 6	red-red paths and blue-red paths	
Type 7	blue-blue paths and red-blue paths	
Type 8	blue-blue paths and blue-red paths	
Type 9	red-blue paths and blue-red paths	$a = 0$
Type 10	red-red paths, blue-blue paths and red-blue paths	
Type 11	red-red paths, blue-blue paths and blue-red paths	
Type 12	red-red paths, red-blue paths and blue-red paths	
Type 13	blue-blue paths, red-blue paths and blue-red paths	
Type 14	red-red paths, blue-blue paths, red-blue paths and blue-red paths	

For *Case 13,* the cycle matrix of D is

$$M=\begin{bmatrix} a+2 & \cdots & a+2 & a+1 & \cdots & a+1 & a+1 \\ n-a-2 & \cdots & n-a-2 & n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content(M) = g.c.d$\{n, n-a-2, -a-1\} = 1$, and so D is primitive.

For *Case 14*, the cycle matrix of D is

$$M = \begin{bmatrix} a & \cdots & a & a+1 & \cdots & a+1 & a+1 \\ n-a & \cdots & n-a & n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content$(M) = $ g.c.d$\{-n, -n-a, -a-1\} = 1$, and so D is primitive.

For *Case 15*, the cycle matrix of D is

$$M = \begin{bmatrix} a+2 & \cdots & a+2 & a & \cdots & a \\ n-a-2 & \cdots & n-a-2 & n-a & \cdots & n-a \end{bmatrix.$$

$$\begin{matrix} a+1 & \cdots & a+1 & a+1 \\ n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix}.$$

Then content$(M) = $ g.c.d$\{2n, n, n-a-2, -n, -n-a, -a-1\} = 1$, and so D is primitive.

To combine above discussion, we have the following result.

Theorem 1. *Let $D \in D_{n,s}$. Then D is primitive if and only if D is one of the fourteen types in Table 1.*

3. The tight bound on the exponents

In this section, we give the tight upper bound on the exponents of primitive two-colored digraphs in $D_{n,s}$, and the characterizations of the extremal two-colored digraphs. The main result is Theorem 2.

Lemma 2. *Let $D \in D_{n,s}$ be primitive. If D is Type 1 in Table* 1, *then* $\exp(D) \leq 2n^2 - 4n + 1$.

Proof. The cycle matrix of D is

$$M = \begin{bmatrix} n-1 & \cdots & n-1 & n-2 \\ 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Clearly, D has only one blue arc, and the blue arc is in the path $1 \to 2 \to \cdots \to (n-2) \to (n-1)$.

For any pair (i, j) of vertices of D, we prove that there is a $(2n^2-6n+4, 2n-3)$–walk from i to j in D. Let p_{ij} be the shortest walk from i to j containing the blue arc. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $b = 1$ and $0 \leq r \leq 2n-4$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(2n-4-r)$ times around the $(n-1, 1)$–cycle and r times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} r \\ 1 \end{bmatrix} + (2n-4-r)\begin{bmatrix} n-1 \\ 1 \end{bmatrix} + r\begin{bmatrix} n-2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2n^2-6n+4 \\ 2n-3 \end{bmatrix}.$$

Hence $\exp(D) \leq 2n^2 - 4n + 1$. □

Lemma 3. *Let $D \in D_{n,s}$ be primitive. If D is Type 2 in Table* 1, *then* $\exp(D) \leq 2n^2 - 3n + 1$.

Proof. The cycle matrix of D is

$$M = \begin{bmatrix} 1 & \cdots & 1 & 1 \\ n-1 & \cdots & n-1 & n-2 \end{bmatrix}.$$

For any pair (i, j) of vertices of D, we prove that there is a $(2n-2, 2n^2-5n+3)$−walk from i to j in D. Let p_{ij} be the shortest walk from i to j containing one red arc. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $r = 1$ and $0 \leq b \leq 2n-3$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(2n-3-b)$ times around the $(1, n-1)$−cycle and b times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} 1 \\ b \end{bmatrix} + (2n-3-b)\begin{bmatrix} 1 \\ n-1 \end{bmatrix} + b\begin{bmatrix} 1 \\ n-2 \end{bmatrix} = \begin{bmatrix} 2n-2 \\ 2n^2-5n+3 \end{bmatrix}.$$

Hence $\exp(D) \leq 2n^2 - 3n + 1$. □

Lemma 4. *Let $D \in D_{n,s}$ be primitive. If D is Type 3 in Table* 1, *then* $\exp(D) \leq 2n^2 - 3n + 1$.

Proof. The proof is similar to the proof of Lemma 3, and we omit it. □

Lemma 5. *Let $D \in D_{n,s}$ be primitive. If D is Type 4 in Table* 1, *then $exp(D) \leq 2n^2 - 2n$.*

Proof. The cycle matrix of D is

$$M = \begin{bmatrix} a+2 & \cdots & a+2 & a & \cdots & a & a+1 \\ n-a-2 & \cdots & n-a-2 & n-a & \cdots & n-a & n-a-2 \end{bmatrix},$$

where $n-a$ is odd, and $0 \leq a \leq n-3$.

For any pair (i, j) of vertices of D, we prove that there is a $(2na+2n, 2n^2-2na-4n)$−walk from i to j in D. Let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $0 \leq r \leq a+2$ and $0 \leq b \leq n-a$. We consider the following two cases:

Case 1. b is even. Then $0 \leq b \leq n-a-1$ and $0 \leq r + \frac{b}{2} \leq \frac{1}{2}(n+a+3)$, and thus $n+a-r-\frac{b}{2} \geq 0$ and $n-a-2-\frac{b}{2} \geq 0$. The walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $\left(n+a-r-\frac{b}{2}\right)$ times around the $(a+2, n-a-2)$−cycle, $\left(n-a-2-\frac{b}{2}\right)$ times around the $(a, n-a)$−cycle and $(r+b)$ times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + \left(n+a-r-\frac{b}{2}\right)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + \left(n-a-2-\frac{b}{2}\right)\begin{bmatrix} a \\ n-a \end{bmatrix}$$
$$+(r+b)\begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix} = \begin{bmatrix} 2na+2n \\ 2n^2-2na-4n \end{bmatrix}.$$

Case 2. b is odd. we consider the following three subcases:

Subcase 1. $r \neq a+2$ and $b \neq n-a$. Then $0 \leq r \leq a+1$ and $0 \leq b \leq n-a-2$, and thus $\dfrac{n+a}{2} - r - \dfrac{b}{2} \geq 0$ and $\dfrac{n-a-2}{2} - \dfrac{b}{2} \geq 0$. The walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $\left(\dfrac{n+a}{2} - r - \dfrac{b}{2}\right)$ times around the $(a+2, n-a-2)$–cycle, $\left(\dfrac{n-a-2}{2} - \dfrac{b}{2}\right)$ times around the $(a, n-a)$–cycle and $(n+r+b)$times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + \left(\frac{n+a}{2} - r - \frac{b}{2}\right)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + \left(\frac{n-a-2}{2} - \frac{b}{2}\right)\begin{bmatrix} a \\ n-a \end{bmatrix}$$

$$+(n+r+b)\begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix} = \begin{bmatrix} 2na+2n \\ 2n^2-2na-4n \end{bmatrix}.$$

Subcase 2. $r = a+2$. Then $b = n-a-2$. The walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(n+a-1)$ times around the $(a+2, n-a-2)$–cycle and $(n-a-2)$ times around the $(a, n-a)$–cycle. Such a walk has composition

$$\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (n+a-1)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (n-a-2)\begin{bmatrix} a \\ n-a \end{bmatrix}$$

$$= \begin{bmatrix} 2na+2n \\ 2n^2-2na-4n \end{bmatrix}.$$

Subcase 3. $b = n-a$. Then $r = a$. The walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(n+a)$ times around the $(a+2, n-a-2)$–cycle and $(n-a-3)$ times around the $(a, n-a)$–cycle. Such a walk has composition

$$\begin{bmatrix} a \\ n-a \end{bmatrix} + (n+a)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (n-a-3)\begin{bmatrix} a \\ n-a \end{bmatrix} = \begin{bmatrix} 2na+2n \\ 2n^2-2na-4n \end{bmatrix}.$$

Hence $\exp(D) \leq 2n^2 - 2n$. □

Lemma 6. *Let $D \in D_{n,s}$ be primitive. If D is Type 5 in Table* 1, *then* $\exp(D) \leq n^2 + n$.

Proof. The cycle matrix of D is

$$M = \begin{bmatrix} a+2 & \cdots & a+2 & a+1 & \cdots & a+1 & a+1 \\ n-a-2 & \cdots & n-a-2 & n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix},$$

where $0 \leq a \leq n-2$.

For any pair (i, j) of vertices of D, we prove that there is a $(na+n+2a+3, n^2-na-2a-3)$–walk from i to j in D. Let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $0 \leq r \leq a+2$ and $0 \leq b \leq n-a-1$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(a+2-r)$ times around the $(a+2, n-a-2)$–cycle,

$(n-a-1-b)$ times around the $(a+1, n-a-1)$−cycle and $(r+b)$ times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + (a+2-r)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (n-a-1-b)\begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix}$$

$$+(r+b)\begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix} = \begin{bmatrix} na+n+2a+3 \\ n^2-na-2a-3 \end{bmatrix}.$$

Hence $\exp(D) \le n^2 + n$. □

Lemma 7. *Let $D \in D_{n,s}$ be primitive. If D is Type 6 in Table 1, then* $\exp(D) \le n^2 + n$.

Proof. The proof is similar to the proof of Lemma 6, and we omit it. □

Lemma 8. *Let $D \in D_{n,s}$ be primitive. If D is Type 7 in Table 1, then* $\exp(D) \le 2n^2 - 2n$.

Proof. The cycle matrix of D is

$$M = \begin{bmatrix} a & \cdots & a & a+1 & \cdots & a+1 & a+1 \\ n-a & \cdots & n-a & n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix},$$

where $0 \le a \le n-2$.

For any pair (i,j) of vertices of D, we prove that there is a $(na+n+a^2+a, n^2-n-a^2-a)$−walk from i to j in D. Let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $0 \le r \le a+1$, $0 \le r+b \le n$ and $0 \le 2r+b \le n+a+1$.

If $i = j = k \in \{n, n+1, \ldots, n+s\}$ and the path $(n-1) \to k \to 1$ is a blue-blue path. Then the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(a+1)$ times around the $(a, n-a)$−cycle and n times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + (a+1)\begin{bmatrix} a \\ n-a \end{bmatrix} + n\begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix} = \begin{bmatrix} na+n+a^2+a \\ n^2-n-a^2-a \end{bmatrix}.$$

Otherwise, consider the following two cases:

Case 1. $0 \le 2r+b \le n+a$. Then the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes r times around the $(a, n-a)$−cycle, $(n+a-2r-b)$ times around the $(a+1, n-a-1)$−cycle and $(r+b)$ times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + r\begin{bmatrix} a \\ n-a \end{bmatrix} + (n+a-2r-b)\begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix} + (r+b)\begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix}$$

$$= \begin{bmatrix} na+n+a^2+a \\ n^2-n-a^2-a \end{bmatrix}.$$

Case 2. $2r+b=n+a+1$. Then $r=a+1$ and $b=n-a-1$. The walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(n+a-1)$ times around the $(a+1, n-a-1)$−cycle. Such a walk has composition

$$\begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix} + (n+a-1)\begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix} = \begin{bmatrix} na+n+a^2+a \\ n^2-n-a^2-a \end{bmatrix}.$$

Hence $\exp(D) \le n^2+na \le n^2+n(n-2) = 2n^2-2n$. □

Lemma 9. *Let $D \in D_{n,s}$ be primitive. If D is Type 8 in Table* 1, *then* $\exp(D) \le 2n^2-2n$.

Proof. The proof is similar to the proof of Lemma 8, and we omit it. □

Lemma 10. *Let $D \in D_{n,s}$ be primitive. If D is Type 9 in Table* 1, *then* $\exp(D) = 2n^2-n$.

Proof. The cycle matrix of D is

$$M = \begin{bmatrix} 1 & \cdots & 1 & 1 \\ n-1 & \cdots & n-1 & n-2 \end{bmatrix}.$$

First, we prove that $\exp(D) \le 2n^2-n$. Let (i,j) be any pair of vertices of D, and p_{ij} be the shortest path from i to j. Denote $r=r(p_{ij})$ and $b=b(p_{ij})$. It is easy to see that $n+nr-2r-b \ge 0$ and $n-nr+r+b \ge 0$. We consider the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(n+nr-2r-b)$ times around the $(1, n-1)$−cycle and $(n-nr+r+b)$ times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + (n+nr-2r-b)\begin{bmatrix} 1 \\ n-1 \end{bmatrix} + (n-nr+r+b)\begin{bmatrix} 1 \\ n-2 \end{bmatrix} = \begin{bmatrix} 2n \\ 2n^2-3n \end{bmatrix}.$$

Hence $\exp(D) \le 2n^2-n$.

Next, we prove that $\exp(D) \ge 2n^2-n$. Note that the compositions of cycles $\gamma_1, \gamma_2, \ldots, \gamma_s, \gamma_{s+1}$ are the same. Now we set

$$N = \begin{bmatrix} 1 & 1 \\ n-1 & n-2 \end{bmatrix}.$$

Suppose that (h,k) is a pair of nonnegative integers such that for all pairs (i,j) of vertices there is an (h,k)−walk from i to j. By considering $i=j=1$, we see that there exist nonnegative integers u and v with

$$\begin{bmatrix} h \\ k \end{bmatrix} = N \begin{bmatrix} u \\ v \end{bmatrix}.$$

Without loss of generality, We assume that the the path $(n-1) \to n \to 1$ is a red-blue path and the path $(n-1) \to (n+1) \to 1$ is a blue-red path. Taking $i=n$ and $j=n+1$, then there is a unique path from i to j, and this path has composition $(0,n)$. Hence

$$Nz = \begin{bmatrix} h \\ k-n \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = N^{-1}\begin{bmatrix} h \\ k-n \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - N^{-1}\begin{bmatrix} 0 \\ n \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -n+2 & 1 \\ n-1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ n \end{bmatrix}$$

$$= \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} n \\ -n \end{bmatrix} \geq 0.$$

So $u \geq n$. Taking $i = n+1$ and $j = n$, then there is a unique path from i to j, and this path has composition $(2, n-2)$. Hence

$$Nz = \begin{bmatrix} h-2 \\ k-(n-2) \end{bmatrix}$$

has a nonnegative integer solution. Then

$$z = N^{-1}\begin{bmatrix} h-2 \\ k-(n-2) \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - N^{-1}\begin{bmatrix} 2 \\ n-2 \end{bmatrix}$$

$$= \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -n+2 & 1 \\ n-1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ n-2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} -n+2 \\ n \end{bmatrix} \geq 0.$$

So $v \geq n$. Thus

$$h+k = \begin{bmatrix} 1 & 1 \end{bmatrix} N \begin{bmatrix} u \\ v \end{bmatrix} \geq \begin{bmatrix} n & n-1 \end{bmatrix}\begin{bmatrix} n \\ n \end{bmatrix} = 2n^2 - n.$$

Hence $\exp(D) \geq 2n^2 - n$. The lemma follows. □

Lemma 11. *Let $D \in D_{n,s}$ be primitive. If D is Type 10 in Table 1, then* $\exp(D) \leq n^2 + n$.

Proof. The cycle matrix of D is

$$M = \begin{bmatrix} a+2 & \cdots & a+2 & a & \cdots & a \\ n-a-2 & \cdots & n-a-2 & n-a & \cdots & n-a \end{bmatrix.$$
$$\begin{matrix} a+1 & \cdots & a+1 & a+1 \\ n-a-1 & \cdots & n-a-1 & n-a-2 \end{matrix}\Bigg],$$

where $0 \leq a \leq n-2$.

For any pair (i,j) of vertices of D, we prove that there is a $(na+n+2a+2, n^2-na-2a-2)$−walk from i to j in D. Let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $0 \leq r \leq a+2$ and $0 \leq b \leq n-a$.

If $i = j = k \in \{n, n+1, \ldots, n+s\}$ and the path $(n-1) \to k \to 1$ is a blue-blue path. Then the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(a+2)$ times around the $(a+2, n-a-2)$−cycle, one time around the $(a, n-a)$−cycle and $(n-a-2)$ times around the $(a+1, n-a-1)$−cycle. Such a walk has composition

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + (a+2)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + \begin{bmatrix} a \\ n-a \end{bmatrix} + (n-a-2)\begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix}$$

$$= \begin{bmatrix} na+n+2a+2 \\ n^2-na-2a-2 \end{bmatrix}.$$

Otherwise, consider the following two cases:

Case 1. $0 \le r \le a+1$. Then the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(a+1-r)$ times around the $(a+2, n-a-2)-$cycle, $(n-a-b)$ times around the $(a+1, n-a-1)-$cycle and $(r+b)$ times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + (a+1-r)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (n-a-b)\begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix} + (r+b)\begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix}$$

$$= \begin{bmatrix} na+n+2a+2 \\ n^2-na-2a-2 \end{bmatrix}.$$

Case 2. $r = a+2$. Then $b = n-a-2$. The walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(a+1)$ times around the $(a+2, n-a-2)-$cycle, one time around the $(a, n-a)-$cycle and $(n-a-2)$ times around the $(a+1, n-a-1)-$cycle. Such a walk has composition

$$\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (a+1)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + \begin{bmatrix} a \\ n-a \end{bmatrix} + (n-a-2)\begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix}$$

$$= \begin{bmatrix} na+n+2a+2 \\ n^2-na-2a-2 \end{bmatrix}.$$

Hence $\exp(D) \le n^2+n$. □

Lemma 12. *Let $D \in D_{n,s}$ be primitive. If D is Type 11 in Table 1, then* $\exp(D) \le n^2+n$.

Proof. The proof is similar to the proof of Lemma 11, and we omit it. □

Lemma 13. *Let $D \in D_{n,s}$ be primitive. If D is Type 12 in Table 1, then* $\exp(D) \le n^2+n$.

Proof. The cycle matrix of D is

$$M = \begin{bmatrix} a+2 & \cdots & a+2 & a+1 & \cdots & a+1 & a+1 \\ n-a-2 & \cdots & n-a-2 & n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix},$$

where $0 \le a \le n-2$.

For any pair (i, j) of vertices of D, we prove that there is a $(na+n+2a+2, n^2-na-2a-2)-$walk from i to j in D. Let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $0 \le r \le a+2$, $0 \le b \le n-a$. We consider the following two cases:

Case 1. $0 \le r \le a+1$. Then the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(a+1-r)$ times around the $(a+2, n-a-2)-$cycle, $(n-a-b)$ times around the $(a+1, n-a-1)-$cycle and $(r+b)$ times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + (a+1-r)\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (n-a-b)\begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix} + (r+b)\begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix}$$

$$= \begin{bmatrix} na+n+2a+2 \\ n^2-na-2a-2 \end{bmatrix}.$$

Case 2. $r = a+2$. Then $b = n-a-2$. The walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes a times around the $(a+2, n-a-2)$−cycle and $(n-a)$ times around the $(a+1, n-a-1)$−cycle. Such a walk has composition

$$\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + a \begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + (n-a) \begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix} = \begin{bmatrix} na+n+2a+2 \\ n^2-na-2a-2 \end{bmatrix}.$$

Hence $\exp(D) \leq n^2+n$. □

Lemma 14. *Let $D \in D_{n,s}$ be primitive. If D is Type 13 in Table* 1, *then* $\exp(D) \leq 2n^2-2n$.

Proof. The cycle matrix of D is

$$M = \begin{bmatrix} a & \cdots & a & a+1 & \cdots & a+1 & a+1 \\ n-a & \cdots & n-a & n-a-1 & \cdots & n-a-1 & n-a-2 \end{bmatrix},$$

where $0 \leq a \leq n-2$.

For any pair (i, j) of vertices of D, we prove that there is a $(na+n+a^2+a, n^2-n-a^2-a)$−walk from i to j in D. Let p_{ij} be the shortest path from i to j. Denote $r = r(p_{ij})$ and $b = b(p_{ij})$. It is easy to see that $0 \leq r \leq a+2$, $0 \leq r+b \leq n$ and $0 \leq 2r+b \leq n+a+2$.

If $i = j = k \in \{n, n+1, \ldots, n+s\}$ and the path $(n-1) \to k \to 1$ is a blue-blue path. Then the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(a+1)$ times around the $(a, n-a)$−cycle and n times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + (a+1) \begin{bmatrix} a \\ n-a \end{bmatrix} + n \begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix} = \begin{bmatrix} na+n+a^2+a \\ n^2-n-a^2-a \end{bmatrix}.$$

Otherwise, consider the following three cases:

Case 1. $0 \leq 2r+b \leq n+a$. Then the walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes r times around the $(a, n-a)$−cycle, $(n+a-2r-b)$ times around the $(a+1, n-a-1)$−cycle and $(r+b)$ times around γ_{s+2}. Such a walk has composition

$$\begin{bmatrix} r \\ b \end{bmatrix} + r \begin{bmatrix} a \\ n-a \end{bmatrix} + (n+a-2r-b) \begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix} + (r+b) \begin{bmatrix} a+1 \\ n-a-2 \end{bmatrix}$$

$$= \begin{bmatrix} na+n+a^2+a \\ n^2-n-a^2-a \end{bmatrix}.$$

Case 2. $2r+b = n+a+1$. Then $r = a+1$ and $b = n-a-1$. The walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes $(n+a-1)$ times around the $(a+1, n-a-1)$−cycle. Such a walk has composition

$$\begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix} + (n+a-1) \begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix} = \begin{bmatrix} na+n+a^2+a \\ n^2-n-a^2-a \end{bmatrix}.$$

Case 3. $2r+b=n+a+2$. Then $r=a+2$ and $b=n-a-2$. The walk that starts at vertex i, follows p_{ij} to vertex j and along the way goes one time around the $(a,n-a)$−cycle and $(n+a-2)$ times around the $(a+1,n-a-1)$−cycle. Such a walk has composition

$$\begin{bmatrix} a+2 \\ n-a-2 \end{bmatrix} + \begin{bmatrix} a \\ n-a \end{bmatrix} + (n+a-2)\begin{bmatrix} a+1 \\ n-a-1 \end{bmatrix} = \begin{bmatrix} na+n+a^2+a \\ n^2-n-a^2-a \end{bmatrix}.$$

Hence $\exp(D) \le n^2+na \le n^2+n(n-2) = 2n^2-2n$. □

Lemma 15. *Let $D \in D_{n,s}$ be primitive. If D is Type 14 in Table 1, then* $\exp(D) \le n^2+n$.

Proof. The proof is similar to the proof of Lemma 11, and we omit it. □

By Lemmas 2–15, we obtain the tight upper bound on the exponents of primitive two-colored digraphs in $D_{n,s}$, and the characterizations of the extremal two-colored digraphs.

Theorem 2. *Let $D \in D_{n,s}$ be primitive. Then* $\exp(D) \le 2n^2-n$, *and* $\exp(D) = 2n^2-n$ *if and only if*

(1) *The paths $(n-1) \to j \to 1$ for $j = n, n+1, \ldots, n+s$ contain only red-blue paths and blue-red paths; and*

(2) *All arcs in the path $1 \to 2 \to \cdots \to (n-2) \to (n-1)$ are blue.*

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