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MATRIX REPRESENTATION FOR MULTI-DEGREE REDUCTION OF BÉZIER CURVES USING CHEBYSHEV POLYNOMIALS

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ABSTRACT. In this paper, we find the matrix representation of multi-degree reduction by L_∞ of Bézier curves with constraints of endpoints continuity. Using the basis transformation between Chebyshev polynomials and Bernstein polynomials we can derive the matrix representation of multi-degree reduction of Bézier with respect to L_∞ norm.

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1. Introduction

Given control points $\{p_i\}_{i=0}^n$, a degree n Bézier curve is defined by

$$p(t) = \sum_{i=0}^{n} p_i B_i^n(t), \ t \in [0,1] \tag{1}$$

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ is the Bernstein polynomial of degree n. The problem of degree reduction is to find control points $\{q_i\}_{i=0}^m$ which define the approximate Bézier curve

$$q(t) = \sum_{i=0}^{m} q_i B_i^m(t), \ t \in [0,1] \tag{2}$$

of degree m $(m < n)$ such that a suitable distance function $d(p,q)$ is minimized. Degree reduction of parametric curves was first proposed as the inverse problem of degree elevation (Forrest [4], Farin [3]). Let

$$\mathbf{p} = (p_0, p_1, \cdots, p_n)^t \text{ and } \mathbf{q} = (q_0, q_1, \cdots, q_m)^t \tag{3}$$

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denote the vectors of the control points of the Bézier curves $p(t)$ and $q(t)$, respectively. The matrix representation of degree redunction of Bézier curves has been studied by many authors. When $m = n - 1$, Sunwoo and Lee [8] propose a unified matrix representation for degree reduction of Bézier curves. They find that the degree reduction matrices for well known methods can be represented in a unified form, namely,

$$\mathbf{q} = A_\lambda \mathbf{p} \tag{4}$$

for a suitable matrix A_λ which is a generalized inverse of the degree elevation matrix. For multi-degree reduction ($m < n - 1$), Lee and Park [5] find the degree reduction matrix with respect to L_2 norm without endpoints continuity. They show that $\mathbf{q} = (T^tQT)^{-1}T^tQ\mathbf{p}$ for some matrix Q and the degree elevation matrix T. Also Ahn et al. [1] prove the constrained polynomial degree reduction in the L_2 norm is equivalent to best weighted Euclidean approximation of Bézier coefficients. They derive the control points $\mathbf{q}$ of the form $\mathbf{q} = (T^tWT)^{-1}T^tW\mathbf{p}$. Although Lee and Park [5] and Ahn et al. [1] propose the matrix representation of the multi-degree reduction, their methods require the computation of matrix inverses.

Sunwoo [7] find the matrix representation of the optimal multi-degree reduction of Bézier curves with high order endpoints continuity by L_2 norm, which is derived by Chen and Wang [2]. In this paper we are interested in multi-degree reduction of Bézier curves with high order endpoints continuity by L_∞ norm $d_\infty(p, q)$ defined by

$$d_\infty(p, q) = \max_{0 \le t \le 1} |p(t) - q(t)|. \tag{5}$$

We derive the basis transformation between Chebyshev polynomials and Bernstein polynomials and then using this result, we find a multi-degree reduction matrix $Q^{(r,s)}_{m\times n}$ satisfying

$$\mathbf{q} = Q^{(r,s)}_{m\times n}\mathbf{p}. \tag{6}$$

where $Q^{(r,s)}_{m\times n}$ is determined only by m, n and the orders (r, s) of endpoints continuity.

The organization of this paper is as follows: We introduce some basic results in section 2. Some results about Chebyshev polynomials as well as the basis transformation between Chebyshev and Bernstein polynomials are stated in section 3. The derivation of the degree reduction matrix $Q^{(r,s)}_{m\times n}$ is presented in section 4.

2. Preliminaries

The optimal multi-degree reduction with constraints of endpoints continuity with respect to L_∞ norm is defined as follows.

Definition 1. Given a degree n Bézier curve $p(t)$, the optimal multi-degree reduction with constraints of endpoints continuity of orders r, s $(r, s \geq 0)$ is to find a degree m $(m < n-1)$ Bézier curve $q(t)$ such that L_∞ norm is minimized and

$$\frac{d^i q(0)}{dt^i} = \frac{d^i p(0)}{dt^i},\ i = 0, 1, \ldots, r;\ \frac{d^j q(1)}{dt^j} = \frac{d^j p(1)}{dt^j},\ j = 0, 1, \ldots, s.$$

The first idea of multi-degree reduction with constraints of endpoints continuity is due to the method given in the results of Chen and Wang [2]. The matrix representation of multi-degree reduction by L_2 norm is derived in Sunwoo [7]. The following theorem presents a key idea to obtain the multi-degree reduced Bézier curve with constraints of endpoints continuity.

Theorem 1. (Chen and Wang [2]) *Given a degree n Bézier curve $p(t) = \sum_{i=0}^{n} p_i B_i^n(t)$, $t \in [0,1]$, and let $r + s < m < n - 1$ and $r, s \leq n - m$. Then the curve can be expressed as*

$$p(t) = \sum_{i=0}^{r} q_i B_i^m(t) + \sum_{i=r+1}^{n-s-1} p_i^I B_i^n(t) + \sum_{i=m-s}^{m} q_i B_i^m(t) \tag{7}$$

where q_i's and p_i^I's are defined as follows:

$$\begin{cases} q_0 = \dfrac{1}{b_{0,0}^{(m,n)}} p_0,\ q_j = \dfrac{1}{b_{j,j}^{(m,n)}} \left(p_j - \displaystyle\sum_{i=0}^{j-1} b_{i,j}^{(m,n)} q_i \right),\ j = 1, 2, \ldots, r, \\ q_m = \dfrac{1}{b_{m,n}^{(m,n)}} p_n, \\ q_{m-j} = \dfrac{1}{b_{m-j,n-j}^{(m,n)}} \left(p_{n-j} - \displaystyle\sum_{i=0}^{j-1} b_{m-i,n-j}^{(m,n)} q_{m-i} \right),\ j = 1, 2, \ldots, s, \end{cases} \tag{8}$$

and

$$b_{i,j}^{(m,n)} = \frac{\binom{m}{i}\binom{n-m}{j-i}}{\binom{n}{j}}. \tag{9}$$

When $m - s > n + r - m$,

$$\begin{cases} p_j^I = p_j - \displaystyle\sum_{i=\max(0,j-(n-m))}^{r} b_{i,j}^{(m,n)} q_i,\ j = r+1, r+2, \ldots, n+r-m, \\ p_j^I = p_j,\quad j = n+r-m+1, n+r-m+2, \ldots, m-s-1, \\ p_j^I = p_j - \displaystyle\sum_{i=\max(0,m-j)}^{s} b_{m-i,j}^{(m,n)} q_{m-i},\ j = m-s, \ldots, n-s-1. \end{cases} \tag{10}$$

When $m - s \le n + r - m$,

$$\begin{cases} p_j^I = p_j - \displaystyle\sum_{i=\max(0,j-(n-m))}^{r} b_{i,j}^{(m,n)} q_i, \quad j = r+1, r+2, \ldots, m-s-1, \\ p_j^I = p_j - \displaystyle\sum_{i=\max(0,j-(n-m))}^{r} b_{i,j}^{(m,n)} q_i - \sum_{i=\max(0,m-j)}^{s} b_{m-i,j}^{(m,n)} q_{m-i}, \\ \qquad\qquad j = m-s, m-s+1, \ldots, n+r-m, \\ p_j^I = p_j - \displaystyle\sum_{i=\max(0,m-j)}^{s} b_{m-i,j}^{(m,n)} q_{m-i}, \\ \qquad\qquad j = n+r-m+1, \ldots, n-s-1. \end{cases} \tag{11}$$

Also, $\{q_i\}_{i=0}^{r}$ *and* $\{q_i\}_{i=m-s}^{m}$ *are the part of the control points of the degree reduced curve* $q(t)$ *of degree* m *satisfying*

$$\frac{d^\lambda q(0)}{dt^\lambda} = \frac{d^\lambda p(0)}{dt^\lambda}, \ \lambda = 0, 1, \ldots, r; \ \frac{d^\mu q(1)}{dt^\mu} = \frac{d^\mu p(1)}{dt^\mu}, \ \mu = 0, 1, \ldots, s. \tag{12}$$

As seen in equation (12), r and s are the orders of endpoints continuity at $t = 0$ and $t = 1$, respectively.

3. Chebyshev polynomials

Chebyshev polynomial $T_n(x) = \cos(n \cdot \arccos(x))$ can be represented in Bernstein forms as

$$T_n(2t-1) = \sum_{i=0}^{n} (-1)^{n+i} \frac{\binom{2n}{2i}}{\binom{n}{i}} B_i^n(t), \tag{13}$$

where $t \in [0, 1]$. These Chebyshev polynomials are orthogonal on $[0, 1]$ with respect to the weight function

$$w(t) = \frac{1}{\sqrt{t(1-t)}} \tag{14}$$

and it is well known that

$$\int_0^1 (T_k(2t-1))^2 w(t) dt = \begin{cases} \pi, & k = 0 \\ \frac{\pi}{2}, & k \neq 0. \end{cases} \tag{15}$$

Let

$$\begin{aligned} \mathbf{T}_n &= (T_0(2t-1), T_1(2t-1), \cdots, T_n(2t-1))^t \text{ and} \\ \mathbf{B}_n &= (B_0^n(t), B_1^n(t), \cdots, B_n^n(t))^t \end{aligned} \tag{16}$$

be vectors of Chebyshev polynomials and Bernstein polynomials, respectively.

In the following lemma we can find a transformation matrix between Chebyshev and Bernstein polynomials.

Lemma 1. (Chen and Wang [2]) $\mathbf{T}_n = \mathbf{A}_n\mathbf{B}_n$
where $\mathbf{A}_n = (A_{k,j})_{(n+1)\times(n+1)}$ *and*

$$A_{k,j} = \sum_{i=\max(0,j+k-n)}^{\min(j,k)} (-1)^{k+i} \frac{\binom{2k}{2i}\binom{n-k}{j-i}}{\binom{n}{j}}. \tag{17}$$

In order to complete the multi-degree reduction of a Bézier curve it is required the inverse matrix of $\mathbf{A}_n$.

Now we derive the inverse transformation between Jacobi and Bernstein polynomials. Chen and Wang [2] suggest the inverse transformation can be found using the basis conversion process given in Li and Zhang [6]. But we find an explicit form of the inverse transformation matrix $\mathbf{A}_n^{-1}$ in the following theorem.

Theorem 2. $\mathbf{B}_n = \mathbf{A}_n^{-1}\mathbf{T}_n$

$$A^{-1}_{k,j} = \frac{\delta}{4^{n+j}}\binom{n}{k}\sum_{i=0}^{j}(-1)^{j+i}\frac{\binom{2j}{2i}\binom{2k+2i}{k+i}\binom{2n+2j-2k-2i}{n+j-k-i}}{\binom{n+j}{k+i}}. \tag{18}$$

where

$$\delta = \begin{cases} 1, & \textit{if } j = 0 \\ 2, & \textit{if } j \neq 0. \end{cases}$$

Proof. We write the transformation of the Chebyshev polynomials on $[0,1]$ into the degree n Bernstein polynomials as

$$B^n_k(t) = \sum_{i=0}^{n} A^{-1}_{k,j} T_i(2t-1) \text{ for } k = 0, 1, \dots, n. \tag{19}$$

In order to find $A^{-1}_{k,j}$, we multiply the above equation by $\dfrac{T_j(2t-1)}{\sqrt{t(1-t)}}$, integrate over $t \in [0,1]$, and invoke the orthogonality of Chebyshev polynomials to obtain

$$A^{-1}_{k,j} = \frac{\delta}{\pi}\int_0^1 B^n_k(t) T_j(2t-1)\frac{1}{\sqrt{t(1-t)}}dt, \tag{20}$$

$$\delta = \begin{cases} 1, & \text{if } j = 0 \\ 2, & \text{if } j \neq 0. \end{cases}$$

Using equation (13), the integral part of (20) becomes

$$\begin{aligned}
&\int_0^1 B_k^n(t)T_j(2t-1)\frac{1}{\sqrt{t(1-t)}}dt \\
&= \sum_{i=0}^{j}(-1)^{j+i}\frac{\binom{2j}{2i}}{\binom{j}{i}}\int_0^1 B_i^j(t)B_k^n(t)\frac{1}{\sqrt{t(1-t)}}dt \\
&= \sum_{i=0}^{j}(-1)^{j+i}\binom{2j}{2i}\binom{n}{k}\int_0^1 t^{k+i-1/2}(1-t)^{n+j-k-i-1/2}dt \\
&= \sum_{i=0}^{j}(-1)^{j+i}\binom{2j}{2i}\binom{n}{k}B(k+i+1/2, n+j-k-i+1/2) \\
&= \sum_{i=0}^{j}(-1)^{j+i}\binom{2j}{2i}\binom{n}{k}\frac{\Gamma(k+i+1/2)\Gamma(n+j-k-i+1/2)}{\Gamma(n+j+1)}
\end{aligned}$$

where $B(\cdot)$ and $\Gamma(\cdot)$ are Beta function and Gamma function, respectively. Since for an integer m

$$\Gamma(m+1/2) = \frac{\sqrt{\pi}(2m)!}{4^m m!} \tag{21}$$

we have the result.

4. Matrix representation of MDR by L_∞

Chen and Wang [2] derive the nearly best uniform multi-degree reduction of Bézier curves with constraints of end point continuity with respect to L_∞ norm, namely, MDR by L_∞. The procedure of MDR by L_∞ is summarized in Algorithm 1.

Algorithm 1 MDR by L_∞

Step 1: Compute $\{q_i\}_{i=0}^{r}$ and $\{q_i\}_{i=m-s}^{m}$ as shown in (8).

Step 2: Compute $\mathbf{p}^I = \left(p_{r+1}^I, p_{r+2}^I, \cdots, p_{n-s-1}^I\right)^t$ as shown in (10) and (11).

Step 3: Compute $\mathbf{p}_N^{II} = \left(p_0^{II}, p_1^{II}, \cdots, p_N^{II}\right)^t$ where $N = n-(r+s+2)$ and

$$p_i^{II} = \frac{\binom{n}{r+1+i}}{\binom{N}{i}} p_{r+1+i}^I, \quad i = 0, 1, \ldots, N. \tag{22}$$

Step 4: Compute $\mathbf{p}_N^{III} = \left(p_0^{III}, p_1^{III}, \cdots, p_N^{III}\right)^t$ as

$$\mathbf{p}_N^{III} = \left(\mathbf{A}_N^{-1}\right)^t \mathbf{p}_N^{II}. \tag{23}$$

where $\mathbf{A}_N^{-1}$ is derived in Theorem 2. Let $\mathbf{p}_M^{III} = (p_0^{III}, p_1^{III}, \cdots, p_M^{III})^t$ where $M = m - (r + s + 2)$.

Step 5: Compute $\mathbf{p}_M^{IV} = \left(p_0^{IV}, p_1^{IV}, \cdots, p_M^{IV}\right)^t$ as

$$\mathbf{p}_M^{IV} = (\mathbf{A}_M)^t \mathbf{p}_M^{III}. \tag{24}$$

where $\mathbf{A}_M$ is given in Lemma 1.

Step 6: Compute $\{q_i\}_{i=r+1}^{m-s-1}$ where

$$q_i = \frac{\binom{M}{i-r-1}}{\binom{m}{i}} p_{i-r-1}^{IV}, \quad i = r+1, r+2, \ldots, m-s-1. \tag{25}$$

Sunwoo [7] find a multi-degree reduction matrix $Q_{m\times n}^{(r,s)}$ according to MDR by L_2 such that $\mathbf{q} = Q_{m\times n}^{(r,s)}\mathbf{p}$. Algorithm 1 is very similar to the multi-degree reduction of Bézier curves with respect to L_2 norm. The only difference is that we have to use the basis transformation matrix between Chebyshev and Bernstein polynomials in Step 4 and 5.

As noted in Sunwoo [7] the control vector $\mathbf{q}$ can be separated into two parts, that is, $\mathbf{q} = \mathbf{q}^I + \mathbf{q}^{II}$, where $\mathbf{q}^I = (q_0, \ldots, q_r, 0, \ldots, 0, q_{m-s}, \ldots, q_m)^t$ and $\mathbf{q}^{II} = (\ldots, q_{r+1}, \ldots, q_{m-s-1}, \ldots)^t$. Let Q^I and Q^{II} such that

$$\mathbf{q}^I = Q^I\mathbf{p} \quad \text{and} \quad \mathbf{q}^{II} = Q^{II}\mathbf{p}. \tag{26}$$

Then the matrix Q^I can be found in Sunwoo [7].

Lemma 2. (Sunwoo [7]) $\mathbf{q}^I = Q^I\mathbf{p}$ *where* $Q^I = (q_{jk})_{(m+1)\times(n+1)}$ *and* q_{jk} *is defined by*

$$q_{jk} = \begin{cases} \dfrac{\binom{n}{k}}{\binom{m}{j}} a_{j-k}, & j = 0, 1, \ldots, r; k = 0, 1, \ldots, j, \\[2ex] \dfrac{\binom{n}{k}}{\binom{m}{j}} a_{k-j-(n-m)}, & \begin{cases} j = m-s, m-s+1, \ldots, m; \\ k = j + (n-m), \ldots, n, \end{cases} \\[2ex] 0, & elsewhere \end{cases} \tag{27}$$

where $a_0 = 1$ *and*

$$a_k = -\sum_{i=0}^{k-1} \binom{n-m}{k-i} a_i, \ k = 1, 2, \ldots, r. \tag{28}$$

The goal of Step 2 to Step 6 is to find the vector $\mathbf{q}^{II}$m that is, the matrix Q^{II}. This matrix can be obtained by a product of a series of matrices. Some of these matrices can be found in Sunwoo [7] and the explicit forms are given in Appendix.

Lemma 3. *For matrices* $\tilde{D}$, $I_{M\times N}$, D; *and* C *given in Appendix and* $\mathbf{A}_M$ *defined in Lemma 1 we have*

$$Q^{II} = \tilde{D}\,(\mathbf{A}_M)^t\, I_{M\times N}\,(\mathbf{A}_N^{-1})^t\, DC \tag{29}$$

so that $\mathbf{q}^{II} = Q^{II}\mathbf{p}$, *where* $N = n - (r+s+2)$ *and* $M = m - (r+s+2)$.

Note that the inverse matrix $\mathbf{A}_N^{-1}$ is given in Theorem 2.

From the Lemma 1 and 2, we have the matrix representation of the vectors $\mathbf{q}^I$ and $\mathbf{q}^{II}$ as $\mathbf{q}^I = Q^I\mathbf{p}$ and $\mathbf{q}^{II} = Q^{II}\mathbf{p}$. Since $\mathbf{q} = \mathbf{q}^I + \mathbf{q}^{II}$, we have $\mathbf{q} = (Q^I + Q^{II})\mathbf{p}$. Hence we have the following result.

Theorem 3. *Let* Q^I *and* Q^{II} *be matrices given in Lemma 2 and 3, respectively. Given control points* $\mathbf{p}$ *of degree* n *Bézier curve* $p(t)$, *when* $\mathbf{q} = Q^{(r,s)}_{m\times n}\mathbf{p}$, *where*

$$Q^{(r,s)}_{m\times n} = Q^I + Q^{II}, \tag{30}$$

the Bézier curve $q(t) = \sum_{i=0}^{m} B_i^m(t)$ *of degree* m *is the nearly best uniform* L_∞ *approximation with constraints of endpoints continuity.*

5. Conclusions

In this paper we have derived the matrix representation of multi-degree reduction of Bézier curves with endpoints continuity with respect to L_∞ norm using the Chebyshev polynomials. Doing this we have completed the matrix representation of multi-degree reduction of Bézier curves with endpoints continuity with respect to L_2 norm as well as L_∞ norm.

Appendix

In Lemma 3 we introduced some matrices such as $\tilde{D}$, $I_{M\times N}$, D, and C. Each matrix plays a role to represent each step described in Algorithm 1 in a matrix form. We give the explicit form of each matrix in the following lemmas. The proofs can be found in Sunwoo [7].

The matrix C stands for STEP2.

Lemma 4. $\mathbf{p}^I = C\mathbf{p}$ *where* $C = (c_{j-(r+1),k})_{(N+1)\times(n+1)}$ *for* $j = r+1, r+2, \ldots, n-s-1$; $k = 0, 1, \ldots, n$, *is given by*

$$c_{j-(r+1),k} = \begin{cases} -\dfrac{\binom{n}{k}}{\binom{n}{j}} \cdot \displaystyle\sum_{i=\max(k,j-(n-m))}^{r} \binom{n-m}{j-i} a_{i-k}, \\ \qquad j = r+1, r+2, \ldots, n+r-m; k = 0, 1, \ldots, r, \\ \delta_{jk}, \qquad j, k = r+1, r+2, \ldots, n-s-1, \\ -\dfrac{\binom{n}{k}}{\binom{n}{j}} \cdot \displaystyle\sum_{i=\max(n-k,m-j)}^{s} \binom{n-m}{j-m+i} a_{i+k-n}, \\ \qquad j = m-s, m-s+1, \ldots, n-s-1; k = n-s, \ldots, n, \\ 0, \qquad elsewhere. \end{cases} \tag{31}$$

where $\delta_{jk} = \begin{cases} 1, & if\, j = k \\ 0, & if\, j \neq k \end{cases}$

The matrix D stands for STEP3.

Lemma 5. $\mathbf{p}_N^{II} = D\mathbf{p}^I$ *where* $D = diag(d_0, d_1, \ldots, d_N)$ *is a diagonal matrix with* $d_i = \dfrac{\binom{n}{r+1+i}}{\binom{N}{i}}$, *for* $i = 0, 1, \ldots, N$.

The matrix $I_{M\times N}$ is defined by

$$I_{M\times N} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{32}$$

The matrix $\tilde{D}$ stands for STEP6.

Lemma 6. $\mathbf{q}^{II} = \tilde{D}\mathbf{p}_M^{IV}$ *where* $\tilde{D} = (\tilde{d}_{ij})_{(m+1)\times(M+1)}$ *is defined by*

$$\tilde{d}_{ij} = \begin{cases} \dfrac{\binom{M}{i-r-1}}{\binom{m}{i}}, & for\ i = r+1, r+2, \ldots, m-s-1; j = i-r-1, \\ 0, & elsewhere. \end{cases} \tag{33}$$

Note that the first $r+1$ and the last $s+1$ elements of $\mathbf{q}^{II}$ are zero. Hence the first $r+1$ rows and the last $s+1$ rows of $\tilde{D}$ are zero and the middle part of $\tilde{D}$ is a diagonal matrix.

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