J. Appl. Math. & Informatics Vol. 26(2008), No. 3 - 4, pp. 583 - 590
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SOME NEW RESULTS ON THE RUDIN-SHAPIRO POLYNOMIALS

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ABSTRACT. In this article, we focuss on sequences of polynomials with $\{\pm 1\}$ coefficients constructed by recursive argument that is known as Rudin-Shapiro polynomials. The asymptotic behavior of these polynomials defines as the ratio of their $2q$-norm with 2-norm to be dominated by some number depending on q or "the best" by an absolute constant. In this work we first show the conjecture holds for some finite numbers of m and then introduce a technique that give the result for any positive odd integer m whenever it holds for all pervious even numbers.

AMS Mathematics Subject Classification code: 44
Key words and phrases: Rudin-Shapiro polynomials, asymptotic behavior.

1. Introduction

The Golay polynomial pairs $(A(z), B(z))$ of degree m (see [4]) are of the form

$$A(z) = \sum_{k=0}^{m} a_k z^k \quad , \quad B(z) = \sum_{k=0}^{m} b_k z^k,$$

where $a_k = \pm 1, a_k = \pm 1$, and they satisfy for all real t

$$|A(e^{it})|^2 + |B(e^{it})|^2 = 2m + 2. \tag{1}$$

A special cases of these polynomials are the Rudin-Shapiro polynomials, [7]. They are defined recursively by the following formulas:

$$P_{m+1}(e^{it}) = P_m(e^{it}) + e^{i2^m t} Q_m(e^{it})$$
$$Q_{m+1}(e^{it}) = P_m(e^{it}) - e^{i2^m t} Q_m(e^{it}),$$

where $P_0(e^{it}) = Q_0(e^{it}) = 1$, $m \geq 0$ and $t \in \mathbf{R}$.

2. Preliminary

Received April 29, 2007. Revised October 22, 2007.

We start from some useful definitions and theorems concerning basic properties of Rudin-Shapiro polynomials in this matter as a base for our main result. In the additional references, one finds most of other results and properties on these polynomials. In what follows, unless it mention otherwise, all polynomials are Rudin-Shapiro. In the rest of the article, all z satisfy $|z| = 1$.

Theorem 1. *For* $|j| < 2^m, \widehat{P_{m+1}}(j) = \widehat{P_m}(j)$ *(this means that the Fourier coefficient of* P_m *at* j *equals the Fourier coefficient of* P_{m+1} *at* j*).*

Proof. We have $P_{m+1}(e^{it}) = P_m(e^{it}) + e^{i2^m t}Q_m(e^{it})$ and $deg(e^{i2^m t}Q_m(e^{it})) \geq 2^m$. So for all j such that $|j| < 2^m$

$$\widehat{P_{m+1}}(j) = \widehat{P_m}(j) + e^{i2^m t}\widehat{Q_m}(e^{it})(j) = \widehat{P_m}(j).$$

□

It's clear from the above theorem that a Rudin-Shapiro polynomial P_m has coefficients ± 1, without gaps. The first 2^m coefficients of P_{m+1} are identical with those of P_m and these coefficients do not depend on m. Hence there exist sequences $\{\zeta_m\}_{m=0}^{\infty}, \{\eta_m\}_{m=0}^{\infty}$ both take only the values of $+1$ and -1 such that

$$P_m(z) = \sum_{n=0}^{2^m-1} \zeta_n z^n \ , \ Q_m(z) = \sum_{n=0}^{2^m-1} \eta_n z^n.$$

There is a conjecture that if ζ_n and η_n $(1 \leq n \leq 2^m - 1)$ are coefficients of polynomials P_m and Q_m, then $\sum_0^{2^m-1} \zeta_n \eta_n = 0$. An essential relation between polynomials P_m's and Q_m's is in the next theorem.

Theorem 2. *For every positive integer* m *we have*
(i) $\|P_m\|_2 = \|Q_m\|_2$ *and*
$(ii) |P_m(z)|^2 + |Q_m(z)|^2 = 2^{m+1}$.

Proof. To show (i), by simple calculation we see that

$$|P_m|^2 = |P_{m-1}|^2 + 2Re(z^{2^{m-1}})\overline{P_m}Q_{m-1} + |Q_m|^2$$

and

$$|Q_m|^2 = |P_{m-1}|^2 - 2Re(z^{2^{m-1}})\overline{P_m}Q_{m-1} + |Q_m|^2.$$

So $|P_m|^2 - |Q_m|^2 = 4Re(z^{2^{m-1}})\overline{P_m}Q_{m-1}$. Since $\int_0^{2\pi} Re(z^{2^{m-1}})\overline{P_m}Q_{m-1}$ vanish to zero, integrating of two side of pervious equation proves (i).
For (ii) we use induction. It's clear that the theorem holds for $m = 0$. Assume it holds for $m \geq 1$, then by simple calculation we find that $|P_{m+1}(z)|^2 + |Q_{m+1}(z)|^2 = 2(|P_m(z)|^2 + |Q_m(z)|^2) = 2 \times 2^m = 2^{m+1}$. □

Note that each P_{m+1} has twice as many terms as P_m, and therefore these polynomials are generated by a simple append rule. The ingenuity of these polynomials is the combination of fixed sized coefficients and the alternating signs in the recursive construction of the Rudin-Shapiro polynomials. According to Parseval's Theorem, the former property gives $||P_m||_2^2 = \sum_0^{2^m-1} (\pm 1)^2 = 2^m$ (as each P is the Fourier transform of sequence numbers $+1$ or -1). The norm that we used here is the usual norm and it will be for the rest of this article. That is, if P is a complex polynomial and q a positive real number, then

$$||P||_q = \left(\frac{1}{2\pi}\int_0^{2\pi} |P(e^{it})|^q dt\right)^{1/q}$$

We also define

$$||P||_{+\infty} = \lim_{q\to+\infty} ||P||_q.$$

The relation between norms of polynomials $P_m(z)$ and $Q_m(z)$ has an important role to this article. Up to now, in the last theorem, we showed that 2-norm of these polynomials are equal. But for the general q-norm case, the following theorem give us an elegant result.

Theorem 3. $||P_m||_q = ||Q_m||_q$ *holds for all* $q \in \mathbf{N}$.

Proof. It suffices to show that

$$Q_m(z) = (-1)^m z^{(2^m-1)} P_m\left(-\frac{1}{z}\right) \tag{2}$$

for every $m \geq 0$. Clearly (2) holds for $m = 0$ and $m = 1$. Assume that (2) holds for any $m \geq 1$. Then

$$\begin{aligned}
&(-1)^{m+1} z^{(2^{m+1}-1)} P_{m+1}\left(-\frac{1}{z}\right)\\
&= (-1)^{m+1} z^{(2^{m+1}-1)} \left[P_m\left(-\frac{1}{z}\right) + \left(-\frac{1}{z}\right)^{2^m} Q_m\left(-\frac{1}{z}\right)\right]\\
&= (-1)^{m+1} z^{(2^{m+1}-1)} \Big[(-1)^m Q_m(z)/z^{(2^m-1)}\\
&\qquad + z^{(-2^m)}(-1)^m\left(-\frac{1}{z}\right)^{2^m-1} P_m(z)\Big]\\
&= P_m(z) - z^{2^m} Q_m(z) = Q_{m+1}(z).
\end{aligned}$$

Thus (2) holds for $m+1$ which complete the proof. □

3. Asymptotic behavior

Notation. For $m, q \in N$, we consider two sequences $A(m,q) = \dfrac{||P_m||_{2q}}{||P_m||_2}$ and $A(q) = \sqrt[2q]{\dfrac{2^q}{q+1}}$ (as probably limit point of $\{A(m,q)\}_{m,q}$ when $m \to +\infty$).

The asymptotic behavior of Rudin-Shapiro polynomials is the ratio of their $2q$-norm and 2-norm, i.e.,

$$A(m,q) = \frac{||P_m||_{2q}}{||P_m||_2} \sim \sqrt[2q]{\frac{2^q}{q+1}} = A(q).$$

In this article we focuss on those q's that satisfy the asymptotic behavior. We begin to sate theorems that discuss the behavior on the norms of Rudin-Shapiro polynomials and then find some upper and lower bound for their norms.

Since $||P_m||_2^2 = \sum_0^{2^m-1} (\pm 1)^2 = 2^m$, by Cauchy-Schwartz inequality we have

$$||P_m||_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |P_m|^2 dt \le \frac{1}{2\pi}\left(\int_0^{2\pi} |P_m|^{2q}\right)^{\frac{1}{q}} \left(\int_0^{2\pi} 1^p\right)^{\frac{1}{p}} = ||P_m||_{2q}^2,$$

where $\frac{1}{q} + \frac{1}{p} = 1$. Since $||P_m||_2^2$ tends to infinity when $m \to \infty$, the above inequality implies that $||P_m||_{2q}^{2q}$ also tends to infinity.

Theorem 4. *If $q \in N$ is fixed, Then for every $m \in N$,*
(i) $A(m,q) = O(||P_m||_2)$ and
(*ii*) $1 \le A(m,q) \le \sqrt{2}$.

Proof. For (*i*), note that $P_m(e^{it}) = \sum_0^{2^m-1} \zeta_n e^{int}$ which yields

$$\begin{aligned} ||P_m||_{2q}^{2q} &= \frac{1}{2\pi}\int_0^{2\pi} |P_m(e^{it})|^{2q} dt = \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_0^{2^m-1} \zeta_n e^{int}\right|^{2q} dt \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_0^{2^m-1} |\zeta_n|\right)^{2q} dt \le 2^{2mq} = ||P_m||_2^{4q} = (||P_m||_2^{2q})^2. \end{aligned}$$

So, $\frac{||P_m||_{2q}}{||P_m||_2} \le ||P_m||_2$, this proves one side of above inequality. For the left side in (*ii*), note that

$$\begin{aligned} ||P_m||_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} |P_m(e^{it})|^2 dt \le \frac{1}{2\pi}\left(\int_0^{2\pi} |P_m(e^{it})|^{2q} dt\right)^{\frac{1}{q}} \left(\int_0^{2\pi} 1^p dt\right)^{\frac{1}{p}} \\ &\le \sqrt[q]{2\pi}\left(\int_0^{2\pi} |P_m|^{2q}\right)^{\frac{1}{q}} = ||P_m||_{2q}^2. \end{aligned}$$

Therefore $\frac{||P_m||_{2q}}{||P_m||_2} \ge 1$. For the right side in (*ii*), since $|P_m(z)|^2 \le |P_m(z)|^2 + |Q_m(z)|^2 = 2^{m+1}$, we have $|P_m(z)| \le \sqrt{2} \times 2^{\frac{m}{2}}$ and this is a uniform bound for P_m. Thus

$$\lim_{q\to+\infty} A^2(m,q) = \lim_{q\to+\infty} \frac{\|P_m\|_{2q}^2}{\|P_m\|_2^2} = \frac{\|P_m\|_{+\infty}^2}{\|P_m\|_2^2} \leq 2.$$

Hence the proof is complete. □

Theorem 5. *Suppose that for every* $p, q \in N$, $\lim_{m\to\infty} A(m,q) = A(q)$. *Then*

$$\lim_{m\to\infty} \frac{A(m,q)}{A(m,p)} = \begin{cases} A > 1, & \text{if} \quad p < q; \\ A < 1, & \text{if} \quad p > q; \\ A = 1, & \text{if} \quad p = q. \end{cases}$$

Proof. First note that $\|P_m\|_2 = 2^{\frac{n}{2}} (\neq 0)$. So

$$\begin{aligned} \lim_{m\to+\infty} \frac{A(m,q)}{A(m,p)} &= \lim_{m\to+\infty} \frac{\|P_m\|_{2q}}{\|P_m\|_2} \times \lim_{m\to+\infty} \frac{\|P_m\|_2}{\|P_m\|_{2p}} \\ &= \sqrt[2q]{\frac{2^q}{q+1}} \times \sqrt[2p]{\frac{2^p}{p+1}} \\ &= \frac{\sqrt[2p]{p+1}}{\sqrt[2q]{q+1}}. \end{aligned}$$

One can easily observe that the theorem follows from the fact that if $f(x) = (1+x)^{\frac{1}{2x}}$ for $x > 0$, then $\log f(x) = \frac{1}{2x}\log(1+x)$ and hence

$$\begin{aligned} \frac{f'}{f} &= \frac{-1}{2x^2}\log(1+x) - \frac{1}{2x(x+1)} \\ &= \frac{-1}{2x}\left(\frac{1}{x}\log(x+1) + \frac{1}{x+1}\right) < 0. \end{aligned}$$

This proves our theorem. □

It's clear that the asymptotic behavior of Rudin-Shapiro polynomials holds for $q = 1$. In the following theorem we show that it holds for $q = 2$.

Theorem 6. $\lim_{m\to\infty} A(m,2) = A(2) = \frac{4}{3}$.

Proof. We show that $\lim_{m\to\infty} \frac{\|P_m\|_4^4}{\|P_m\|_2^4} = \frac{4}{3}$. Simple calculation yields

$$|P_m|^2|Q_m|^2 = |P_{m-1}|^4 + |Q_{m-1}|^4 - 2Re\left(e^{2^m it}\overline{P_{m-1}}^2 Q_{m-1}^2\right). \tag{3}$$

Now assume that

$$x_m = \frac{1}{2\pi}\int_0^{2\pi} \left[|P_m(e^{it})|^4 + |Q_m(e^{it})|^4\right] dt = 2\|P_m\|_4^4. \tag{4}$$

Since polynomials P_{m-1}^2 and Q_{m-1}^2 have degree $2^m - 1$, integral of that real part in (3) must be zero. So integrating both sides of (3) yields

$$x_{m-1} = \frac{1}{2\pi}\int_0^{2\pi} |P_m|^2|Q_m|^2 dt.$$

By (1) we have $|P_m|^4 + |Q_m|^4 + 2|P_m|^2|Q_m|^2 = 2^{2m+2}$.
Thus the following recursive formula is concluded

$$x_m + 2x_{m-1} = 2^{2m+2}. \tag{5}$$

On the other hand

$$\frac{2}{3}(2^{2m+2}) - 2^m \leq x_m \leq \frac{2}{3}(2^{2m+2}) + 2^m \qquad m \in \mathbf{N} \cup \{0\} \tag{6}$$

To show this, note that $x_0 = 2$, since $P_0 = Q_0 = 1$ and hence (6) holds for $m = 0$. Assume that (6) hold for $m-1$ then by (5), $x_{m-1} = 2^{2m+1} - \frac{x_m}{2}$ and

$$\frac{2}{3}(2^m) - 2^{m-1} \leq 2^{2m+1} - \frac{x_m}{2} \leq \frac{2}{3}(2^m) + 2^{m-1}$$

which is (6). Now as $\frac{x_m}{2^{2m+1}} = \frac{||P_m||_4^4}{||P_m||_2^4}$, we have

$$\frac{4}{3} - 2^{-m-1} \leq \frac{x_m}{2^{2m+1}} \leq \frac{4}{3} + 2^{-m-1}.$$

This proves Theorem. □

In the following theorem, using the fact that the asymptotic behavior holds for $q = 1$ and for $q = 2$, we'll show that it holds for $q = 3$.

Theorem 7. $\lim_{n\to\infty} A(m,3) = A(3) = \sqrt[6]{2}$.

Proof.

$$\begin{aligned} ||P_m||_6^6 &= \frac{1}{2\pi}\int_0^{2\pi} |P_m(e^{it})|^6 dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} (2^{m+1} - |Q_m(e^{it})|^2)^3 dt \\ &= 2^{3(m+1)} + 3 \times 2^{m+1}||Q_m||_4^4 - 3 \times 2^{2(m+1)}||Q_m||_2^2 - ||Q_m||_6^6. \end{aligned}$$

Since $||P_m||_q = ||Q_m||_q$, for all $q \in \mathbf{N}$, we have

$$2||P_m||_6^6 = 2^{3(m+1)} + 3 \times 2^{m+1}||P_m||_4^4 - 3 \times 2^{2(m+1)}||P_m||_2^2.$$

On the other hand $||P_m||_2^2 = ||Q_m||_2^2 = \sum_0^{2^m-1} (\pm 1)^2 = 2^m$ implies that

$$\begin{aligned} \lim_{m\to\infty} \frac{||P_m||_6^6}{||P_m||_2^6} &= \lim_{m\to\infty} \frac{2^{3m+2}}{2^{3m}} + \lim_{m\to\infty} \frac{3 \times 2^m||P_m||_4^4}{2^m||P_m||_2^4} - \lim_{m\to\infty} \frac{3 \times 2^{2m+1}||P_m||_2^2}{2^{2m}||P_m||_2^2} \\ &= 4 + 3(4/3) - 6 = 2. \end{aligned}$$

Therefore $\lim_{h\to\infty} \frac{||P_n||_6}{||P_n||_2} = \sqrt[6]{2}$.

Thus, theorem is proved. □

Up to now, we have shown that the asymptotic behavior holds for $q = 1, 2, 3$. But, if we want to prove for $q = 5$, we need to assume it holds for $q = 2, 4$. However the $q = 2$ case was shown. Therefore we only need to assume the behavior holds for $q = 4$.

Theorem 8. *If* $\lim_{m\to\infty} A(m,4) = A(4)$, *then* $\lim_{m\to\infty} A(m,5) = A(5)$.

Proof. We must show that $\lim_{n\to\infty} \frac{||P_m||_{10}^{10}}{||P_m||_2^{10}} = \frac{16}{3}$.

$$\begin{aligned}
||P_m||_{10}^{10} &= \frac{1}{2\pi}\int_0^{2\pi} |P_m(e^{it})|^{10} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} (2^{m+1} - |Q_m|^2)^5 dt \\
&= \sum_0^5 (-1)^{5-j} C_j^5 2^{(m+1)j} \left(\frac{1}{2\pi}\int_0^{2\pi} |Q_m(e^{it})|^{2(5-j)} dt\right) \\
&= -||Q_m||_{10}^{10} + \sum_1^5 (-1)^{5-j} C_j^5 2^{(m+1)j} ||Q_m||_{2(5-j)}^{2(5-j)}.
\end{aligned}$$

Since $||P_m||_q = ||Q_m||_q$ for all $q \in N$,

$$\lim_{m\to\infty} \frac{||P_m||_{10}^{10}}{||P_m||_2^{10}} = \sum_1^5 (-1)^{(5-j)} C_j^5 2^{j-1} \lim_{m\to\infty} \frac{2^{mj}||P_m||_{2(5-j)}^{2(5-j)}}{||P_m||_2^{2j}||P_m||_2^{2(5-j)}}.$$

Now, Theorems 6, 7, and the hypothesis of this theorem implies that

$$\lim_{n\to\infty} \frac{||P_n||_{10}^{10}}{||P_n||_2^{10}} = 16 - 40 + \frac{160}{3} - 40 + 16 = \frac{16}{3}.$$

□

Our main goal is to show that the asymptotic behavior is true for any odd positive integer, whenever it is true for all its previous even positive integers.

Theorem 9. *Let q be a positive odd integer and suppose that for every even $n < q$ we have* $\lim_{m\to\infty} A(m,n) = A(n)$. *Then* $\lim_{m\to\infty} A(m,q) = A(q)$.

Proof. Let q be an odd number and let us first suppose that $\lim_{m\to\infty} A(m,n) = A(n)$ for all positive integers n less than q. Then

$$||P_m||_{2q}^{2q} = \frac{1}{2\pi}\int_0^{2\pi} |P_m(e^{it})|^{2q} dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(2^{m+1} - |Q_m(e^{it})|^2)^q dt$$

$$= \sum_0^q (-1)^{q-j} C_j^q 2^{(m+1)j}\left(\frac{1}{2\pi}\int_0^{2\pi}|Q_m(e^{it})|^{2(q-j)}dt\right).$$

Since $||P_m||_q = ||Q_m||_q$ for all $q \in \mathbf{N}\cup\{0\}$, we have

$$||P_m||_{2q}^{2q} = \sum_0^q (-1)^{q-j} C_j^q 2^{(m+1)j} ||P_m||_{2(q-j)}^{2(q-j)}.$$

Hence

$$2\lim_{m\to\infty}\frac{||P_m||_{2q}^{2q}}{||P_m||_2^{2q}} = \sum_1^q (-1)^{q-j} C_j^q \lim_{m\to\infty} 2^{(m+1)j}\frac{||P_m||_{2(q-j)}^{2(m-j)}}{||P_m||_2^{2(q)}}$$

$$= \sum_1^q (-1)^{q-j} C_j^q 2^{j-1}\lim_{m\to\infty}\frac{2^{mj}||P_m||_{2(q-j)}^{2(q-j)}}{||P_m||_2^{2(q-j)}||P_m||_2^{2j}}$$

$$= \sum_1^q (-1)^{q-j} C_j^q \frac{2^q}{(q+1-j)} = 2^q(\frac{2}{q+1}).$$

The last equality comes from well known fact $\sum_0^q \frac{(-1)^{(q-j)}}{(q+1-j)} C_j^q = \frac{1}{q+1}$. □

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