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A FINITE DIFFERENCE SCHEME FOR RLW-BURGERS EQUATION

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ABSTRACT. In this paper, a finite difference method for a Cauchy problem of RLW-Burgers equation was considered. Although the equation is not energy conservation, we have given its the energy conservative finite difference scheme with condition. Convergence and stability of the difference solution were proved. Numerical results demonstrate that the method is efficient and reliable.

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1. Introduction

The RLW-Burgers equation in general has the following form [1]:

$$u_t + \gamma u_x + \delta u u_x - \alpha u_{xx} - \beta u_{xxt} = 0, \tag{1}$$

which is a model equation of describing the propagation of surface water waves in a channel. In equation (1), the variables are all scaled, with x proportional to the horizontal coordinate along the channel, t proportional to the elapsed time, and $u = u(x,t)$ proportional to the vertical displacement of the surface of the water from its equilibrium position, the coefficients $\gamma, \delta, \alpha, \beta$ are all constants, here α and β are called dissipative and dispersive coefficient, respectively. Equation (1) represents a balance relation among the dispersion, dissipation and nonlinearity, and equation (1) is called the RLW-Burgers equation, since it is the so-called regularized long-wave (RLW) equation

$$u_t + \gamma u_x + \delta u u_x - \beta u_{xxt} = 0, \tag{2}$$

with a Burgers-type dissipative term " $-\alpha u_{xx}$"$(\alpha > 0)$ appended. Mathematical theory and numerical methods for (2) was considered in [3-7].

$$u_t + u_x + \alpha(u^p)_x - \beta u_{xxt} = 0, \tag{3}$$

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which is usually called the generalized regularized long-wave (GRLW) equation, has been studied by many authors too. Doğan Kaya [8] gives the exact solitary-solutions of GRLW equation (3), and construct numerical solutions using ADM (Adomian decomposition method) without using any discretization technique. Recently, Luming Zhang gives a finite difference scheme for generalized regularized long-wave equation in [9].

Without loss generality, here we will take $\gamma = 1, \delta = 12$ in the equation (1), and therefore the RLW-Burgers equation to be discussed in the present paper is of the following form

$$u_t + u_x + 12uu_x - \alpha u_{xx} - \beta u_{xxt} = 0, \tag{4}$$

$$u|_{t=0} = u_0(x). \tag{5}$$

Mingliang Wang gives the exact solutions of this equation in [2]. Motivated by Zhang's work, we study equation (4). Although the equation is not energy conservation, we have given its the energy conservative finite difference scheme with condition. Convergence and stability of the difference solution were proved. Numerical results demonstrate that the method is efficient and reliable.

The propose of this paper is to present a conservative finite difference scheme for the initial value problem (4), (5), and proof convergence and stability of the scheme. An outline of the paper is as follows. In section 2, a conservative finite difference scheme for the initial value problem (4), (5) is proposed. In section 3, convergence and stability of the scheme are proved. Numerical experiments are reported in section 4.

2. Finite difference scheme and conservation law

As usual, the following notations will be used

$$(u_j^n)_x = \frac{u_{j+1}^n - u_j^n}{h}, \qquad (u_j^n)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h},$$

$$(u_j^n)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \qquad (u_j^n)_{\hat{t}} = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau},$$

$$(u^n, v^n) = h\sum_j u_j^n v_j^n, \qquad \|u^n\|^2 = (u^n, u^n), \qquad \|u^n\|_\infty = \sup_j |u_j^n|$$

where h and τ are the spatial and temporal step sizes, respectively, and $x_j = jh, t_n = n\tau$. Superscript n denotes a quantity associated with the time-level t_n and subscript j denotes a quantity associated with space mesh point x_j. In this paper, C denote general constant, which may have different value in different place.

Now we consider the finite difference method for the problem (4), (5). Since

$$(u^2)_x = \frac{2}{3}[uu_x + (u^2)_x],$$

then the following finite difference scheme is considered.

$$(u_j^n)_{\hat{t}} + \frac{1-\theta}{2}\Big[(u_j^{n+1})_{\hat{x}} + (u_j^{n-1})_{\hat{x}}\Big] + \theta(u_j^n)_{\hat{x}} - \beta(u_j^n)_{x\bar{x}\hat{t}} - \alpha(u_j^n)_{x\bar{x}}$$
$$+2\Big[u_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}} + [u_j^n(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}\Big] = 0, \tag{6}$$

$$u_j^0 = u_0(jh), \tag{7}$$

where $0 \leq \theta \leq 1$ is a real constant.

Lemma 1. *The finite difference scheme (6), (7) is conservative for discrete energy, i.e.,*

$$E^n = \frac{1}{2}(\|u^n\|^2 + \|u^{n+1}\|^2) + \frac{\beta}{2}(\|u_x^{n+1}\|^2 + \|u_x^n\|^2) + \theta\tau h\sum_j (u_j^n)_{\hat{x}} u_j^{n+1} +$$
$$(-1)^{n+1}\alpha\tau\sum_j [u_j^{n+1}(u_j^n)_x + u_j^n(u_j^{n+1})_x] = \ldots = E_0, \tag{8}$$

Proof. Computing the inner product of (6) with $u^{n+1}+u^{n-1}$, we obtain

$$\frac{1}{2\tau}(\|u^{n+1}\|^2 - \|u^{n-1}\|^2) + \frac{\beta}{2\tau}(\|u_x^{n+1}\|^2 - \|u_x^{n-1}\|^2) + \theta h\sum_j (u_j^n)_{\hat{x}}(u_j^{n+1}+u_j^{n-1})$$
$$+(p, u^{n+1}+u^{n-1}) - \alpha\sum_j (u_j^n)_{x\bar{x}}(u_j^{n+1}+u_j^{n-1}) = 0, \tag{9}$$

where $p = 2[u_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}} + [u_j^n(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}]$. Now, we compute last three terms of the left-hand side in (9).

$$\begin{aligned}
&\theta h\sum_j (u_j^n)_{\hat{x}}(u_j^{n+1}+u_j^{n-1})\\
&= h\theta\sum_j (u_j^n)_{\hat{x}} u_j^{n+1} + \frac{\theta}{2}\sum_j (u_{j+1}^n - u_{j-1}^n)u_j^{n-1}\\
&= h\theta\sum_j (u_j^n)_{\hat{x}} u_j^{n+1} + \frac{\theta}{2}\sum_j (u_{j+1}^n u_j^{n-1} - u_{j-1}^n u_j^{n-1})\\
&= h\theta\sum_j (u_j^n)_{\hat{x}} u_j^{n+1} + \frac{\theta}{2}\sum_j (u_j^n u_{j-1}^{n-1} - u_j^n u_{j+1}^{n-1})\\
&= \theta h\sum_j [(u_j^n)_{\hat{x}} u_j^{n+1} - (u_j^{n-1})_{\hat{x}} u_j^n],
\end{aligned} \tag{10}$$

$$\begin{aligned}
&(p, u^{n+1}+u^{n-1})\\
&= 2h\sum_j [u_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}} + [u_j^n(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}](u_j^{n+1}+u_j^{n-1})\\
&= \sum_j [u_j^n(u_{j+1}^{n+1} - u_{j-1}^{n+1} + u_{j+1}^{n-1} - u_{j-1}^{n-1})\\
&\quad + u_{j+1}^n(u_{j+1}^{n+1}+u_{j+1}^{n-1}) - u_{j-1}^n(u_{j-1}^{n+1}+u_{j-1}^{n-1})](u_j^{n+1}+u_j^{n-1})
\end{aligned} \tag{11}$$

$$
\begin{aligned}
&= \sum_j [u_j^n(u_{j+1}^{n+1} + u_{j+1}^{n-1}) + u_{j+1}^n(u_{j+1}^{n+1} + u_{j+1}^{n-1})](u_j^{n+1} + u_j^{n-1}) \\
&\quad - \sum_j [u_{j+1}^n(u_j^{n+1} + u_j^{n-1}) + u_j^n(u_j^{n+1} + u_j^{n-1})](u_{j+1}^{n+1} + u_{j+1}^{n-1}) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
&-\alpha h \sum_j (u_j^n)_{x\bar{x}}(u_j^{n+1} + u_j^{n-1}) \\
&= -\frac{\alpha}{h}\sum_j (u_{j+1}^n - 2u_j^n + u_{j-1}^n)(u_j^{n+1} + u_j^{n-1}) \\
&= -\alpha \sum_j [u_j^{n+1}(u_j^n)_x + u_j^n(u_j^{n+1})_x + u_j^{n-1}(u_j^n)_x + u_j^n(u_j^{n-1})_x] \qquad (12) \\
&= -\alpha \sum_j [u_j^{n+1}(u_j^n)_x + u_j^n(u_j^{n+1})_x] - \alpha \sum_j [u_j^{n-1}(u_j^n)_x + u_j^n(u_j^{n-1})_x].
\end{aligned}
$$

Substitute (10), (11) and (12) into (9), and let

$$
\begin{aligned}
E^n = &\frac{1}{2}(\|u^n\|^2 + \|u^{n+1}\|^2) + \frac{\beta}{2}(\|u_x^{n+1}\|^2 + \|u_x^n\|^2) \\
&+\theta\tau h \sum_j (u_j^n)_{\hat{x}} u_j^{n+1} + (-1)^{n+1}\alpha\tau \sum_j [u_j^{n+1}(u_j^n)_x + u_j^n(u_j^{n+1})_x], \qquad (13)
\end{aligned}
$$

we obtain (8).

3. Convergence and stability of finite difference scheme

First, we consider the truncation error of the finite difference scheme (6), (7). Suppose $v_j^n = u(x_j, t_n)$, then we have

$$
\begin{aligned}
Er_j^n = &(v_j^n)_{\hat{t}} + \frac{1-\theta}{2}[(v_j^{n+1})_{\hat{x}} + (v_j^{n-1})_{\hat{x}}] + \theta(v_j^n)_{\hat{x}} - \beta(v_j^n)_{x\bar{x}\hat{t}} \\
&+2[v_j^n(v_j^{n+1} + v_j^{n-1})_{\hat{x}} + [v_j^n(v_j^{n+1} + v_j^{n-1})]_{\hat{x}}] - \alpha(v_j^n)_{x\bar{x}}.
\end{aligned} \qquad (14)
$$

According to Taylor's expansion, it can be easily obtained that linear part of (14) at point (x_j, t_n) satisfies

$$
\begin{aligned}
&(v_j^n)_{\hat{t}} + \frac{1-\theta}{2}[(v_j^{n+1})_{\hat{x}} + (v_j^{n-1})_{\hat{x}}] + \theta(v_j^n)_{\hat{x}} - \beta(v_j^n)_{x\bar{x}\hat{t}} - \alpha(v_j^n)_{x\bar{x}} \\
&= (v_t + v_x - \beta v_{xxt} - \alpha v_{xx})|_{(x_j,t_n)} + O(h^2 + \tau^2). \qquad (15)
\end{aligned}
$$

The last 2th term of (14) can be written as

$$
\begin{aligned}
Q &= 2[v_j^n(u_j^{n+1} + v_j^{n-1})_{\hat{x}} + [v_j^n(u_j^{n+1} + v_j^{n-1})]_{\hat{x}}] \\
&= 2[v_j^n(v_j^{n+1} + v_j^{n-1})_{\hat{x}}] + 2[v_j^n(u_j^{n+1} + v_j^{n-1})]_{\hat{x}} \\
&= 2[v_j^n(v_j^{n+1} + v_j^{n-1})_{\hat{x}}] + \tfrac{1}{h}[v_{j+1}^n(v_{j+1}^{n+1} + v_{j+1}^{n-1}) - v_{j-1}^n(v_{j-1}^{n+1} + v_{j-1}^{n-1})]. \qquad (16)
\end{aligned}
$$

By making Taylor's expansion and miscellaneous computation for above equation, we get

$$
Q = (6(u^2)_x)|_{(x_j,t_n)} + O(h^2 + \tau^2).
$$

Therefore $Er_j^n = O(h^2 + \tau^2)$.

Lemma 2. *Assume $u(x,t)$ is enough smooth, then the local truncation error of finite difference scheme (6), (7) is $O(h^2+\tau^2)$.*

Next, we consider convergence and stability of the finite scheme (6), (7).

Lemma 3. (Discrete Sobolev's inequality [10]). *For any discrete function u_h and for any given $\varepsilon > 0$, there exists a constant $K(\varepsilon, n)$, depending only ε and n, such that*

$$\|u^n\|_\infty \leq \epsilon\|u_x^n\| + K(\varepsilon,n)\|u^n\|.$$

Lemma 4. [10]. *Suppose that the discrete function w_h satisfies recurrence formula*

$$w_n - w_{n-1} \leq A\tau w_n + B\tau w_{n-1} + C_n\tau,$$

where A,B and C_n $(n = 1, ..., N)$ are nonnegative constants. then

$$\|w_h\|_\infty \leq (w_0 + \tau\sum_{k=1}^{N} C_k)e^{2(A+B)\tau},$$

where τ is small, such that $(A+B)\tau \leq \frac{N-1}{2N}(N>1)$.

Lemma 5. *Assume $u_0 \in H_0^1$, then there is the estimation for the solution of the difference scheme (6), (7),*

$$\|u^n\| \leq C, \|u_x^n\| \leq C, \|u^n\|_\infty \leq C.$$

Proof. We obtain from (8)

$$\begin{aligned}
&\frac{1}{2}(\|u^n\|^2 + \|u^{n+1}\|^2) + \frac{\beta}{2}(\|u_x^{n+1}\|^2 + \|u_x^n\|^2) \\
&\leq C + \theta\tau h\sum_j (u_j^n)_{\hat{x}} u_j^{n+1} + |\ \alpha\ |\tau\sum_j |\ u_j^{n+1}(u_j^n)_x + u_j^n(u_j^{n+1})_x\ | \\
&\leq C + \frac{\theta\tau}{2}(\|u^{n+1}\|^2 + \|u_x^n\|^2) + \frac{|\alpha|\tau}{2h}(\|u^{n+1}\|^2 + \|u_x^n\|^2 + \|u^n\|^2 \\
&+\|u_x^{n+1}\|^2)
\end{aligned} \tag{17}$$

It is

$$\begin{aligned}
&(1-\gamma|\alpha| - \theta\tau)\|u^{n+1}\|^2 + (1-\gamma|\alpha|)\|u^n\|^2 + (\beta-\gamma|\alpha|)\|u_x^{n+1}\|^2 \\
&+(\beta - \theta\tau - \gamma|\alpha|)\|u_x^n\|^2 \leq C,
\end{aligned} \tag{18}$$

where $\gamma = \frac{\tau}{h}$. Let τ be small, such that $\gamma < \min(\frac{1}{|\ \alpha\ |}, \frac{1}{\beta})$, we obtain from (18) $\|u^n\| \leq C, \|u_x^n\| \leq C$. It follows from Lemma 3 that $\|u^n\|_\infty \leq C$.

Theorem 1. *Assume $u_0(x) \in H_0^1$, and $u \in C^{(4,3)}$, then the solution of the conservative difference scheme (6), (7) converges to the solution of the problem (4), (5) with order $O(h^2+\tau^2)$ by l_∞ norm.*

Proof. Let $e_j^n = v_j^n - u_j^n$. Subtracting (6) from (14), we have

$$\begin{aligned} Er_j^n =& (e_j^n)_{\hat{t}} + \frac{1-\theta}{2}[(e_j^{n+1})_{\hat{x}} + (e_j^{n-1})_{\hat{x}}] + \theta(e_j^n)_{\hat{x}} + 2[v_j^n(v_j^{n+1}+v_j^{n-1})_{\hat{x}} \\ &+[v_j^n(v_j^{n+1}+v_j^{n-1})]_{\hat{x}} - u_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}} - [u_j^n(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}] \\ &-\beta(e_j^n)_{x\bar{x}\hat{t}} - \alpha(e_j^n)_{x\bar{x}}, \end{aligned} \tag{19}$$

Computing the inner product of above equation with $e^{n+1}+e^{n-1}$, we get

$$\begin{aligned} &(Er^n, e^{n+1}+e^{n-1}) \\ &= \frac{1}{2\tau}(\|e^{n+1}\|^2 - \|e^{n-1}\|^2) + \frac{\beta}{2\tau}(\|e_x^{n+1}\|^2 - \|e_x^{n-1}\|^2) + \theta h\sum_j (e_j^n)_{\hat{x}}(e_j^{n+1} \\ &+e_j^{n-1}) + (I+II, e^{n+1}+e^{n-1}) - \alpha h\sum_j (e_j^n)_{x\bar{x}}(e_j^{n+1}+e_j^{n-1}), \end{aligned} \tag{20}$$

where

$$I = 2[v_j^n(v_j^{n+1}+v_j^{n-1})_{\hat{x}} - u_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}}],$$

$$II = 2[[v_j^n(v_j^{n+1}+v_j^{n-1})]_{\hat{x}} - [u_j^n(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}].$$

According to Lemma 5, the 4th term of right-hand side of (20) is follows:

$$\begin{aligned} &(I, e^{n+1}+e^{n-1}) \\ &= 2h\sum_j [v_j^n(v_j^{n+1}+v_j^{n-1})_{\hat{x}} - u_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}}](e_j^{n+1}+e_j^{n-1}) \\ &= 2h\sum_j [v_j^n(e_j^{n+1}+e_j^{n-1})_{\hat{x}} + (v_j^n-u_j^n)(u_j^{n+1}+u_j^{n-1})_{\hat{x}}](e_j^{n+1}+e_j^{n-1}) \\ &= 2h\sum_j [v_j^n(e_j^{n+1}+e_j^{n-1})_{\hat{x}} + e_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}}](e_j^{n+1}+e_j^{n-1}) \\ &\le Ch\sum_j [|(e_j^{n+1}+e_j^{n-1})_{\hat{x}}||e_j^{n+1}+e_j^{n-1}| + |e_j^n||e_j^{n+1}+e_j^{n-1}|] \\ &\le C(\|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2), \end{aligned} \tag{21}$$

$$\begin{aligned} &(II, e^{n+1}+e^{n-1}) \\ &= 2h\sum_j [[v_j^n(v_j^{n+1}+v_j^{n-1})]_{\hat{x}} - [u_j^n(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}](e_j^{n+1}+e_j^{n-1}) \\ &= \sum_j [v_{j+1}^n(v_{j+1}^{n+1}+v_{j+1}^{n-1}) - v_{j-1}^n(v_{j-1}^{n+1}+v_{j-1}^{n-1}) \\ &\quad -u_{j+1}^n(u_{j+1}^{n+1}+u_{j+1}^{n-1}) + u_{j-1}^n(u_{j-1}^{n+1}+u_{j-1}^{n-1})](e_j^{n+1}+e_j^{n-1}) \\ &= 2h\sum_j [[e_j^n(u_j^{n+1}+u_j^{n-1})]_{\hat{x}} + [v_j^n(e_j^{n+1}+e_j^{n-1})]_{\hat{x}}](e_j^{n+1}+e_j^{n-1}) \\ &\le C(\|e_x^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^{n-1}\|^2) \end{aligned} \tag{22}$$

The last term of (20) can be written as

$$
\begin{aligned}
&-\alpha h \sum_j (e_j^n)_{x\bar{x}}(e_j^{n+1}+e_j^{n-1}) \\
&= -\alpha \sum_j [e_j^{n+1}(e_j^n)_x + e_j^n(e_j^{n+1})_x + e_j^n(e_j^{n-1})_x + e_j^{n-1}(e_j^n)_x] \\
&\le C \mid \alpha \sum_j [e_j^{n+1}(e_j^n)_x - e_{j+1}^{n+1}(e_j^n)_x \quad + e_j^n(e_j^{n+1})_x - e_{j+1}^n(e_j^{n+1})_x \\
&\quad + e_j^n(e_j^{n-1})_x - e_{j+1}^n(e_j^{n-1})_x + e_j^{n-1}(e_j^n)_x - e_{j+1}^{n-1}(e_j^n)_x] \mid \\
&\le 2C \mid \alpha \mid h |(e_j^{n+1})_x(e_j^n)_x + (e_j^n)_x(e_j^{n-1})_x| \\
&\le C[\|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|e_x^{n-1}\|^2]
\end{aligned} \tag{23}
$$

In addition, there exist obviously that

$$
(Er^n, e^{n+1}+e^{n-1}) \le \|Er^n\|^2 + \tfrac{1}{2}(\|e^{n+1}\|^2 + \|e^{n-1}\|^2) \tag{24}
$$

Substituting (21)-(24) into (20), we get

$$
\begin{aligned}
&\frac{1}{2\tau}(\|e^{n+1}\|^2 - \|e^{n-1}\|^2) + \frac{\beta}{2\tau}(\|e_x^{n+1}\|^2 - \|e_x^{n-1}\|^2) \\
&\le \|Er^n\|^2 + \tfrac{1}{2}(\|e^{n+1}\|^2 + \|e^{n-1}\|^2) + \frac{\theta}{2}(\|e_x^n\|^2 + \|e^{n+1}\|^2 + \|e^{n-1}\|^2) \\
&+C(\|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|e_x^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2).
\end{aligned} \tag{25}
$$

Let $B^n = \frac{1}{2}(\|e^{n+1}\|^2 + \|e^n\|^2) + \frac{\beta}{2}(\|e_x^{n+1}\|^2 + \|e_x^n\|^2)$. Then (25) can be rewritten as

$$
B^n - B^{n-1} \le \tau\|Er^n\|^2 + C\tau(B^n + B^{n-1}). \tag{26}
$$

By lemma 4 it can immediately be obtained that

$$
B^N \le (B^0 + T \sup_{1\le n\le N} \|Er^n\|^2)e^{CT}. \tag{27}
$$

Thus we can choose a two order method to compute u^1 such that $B^0 \le [O(h^2+\tau^2)]^2$ It follows from (27) that $\|e^n\| \le O(h^2+\tau^2)$, $\|e_x^n\| \le O(h^2+\tau^2)$. According to Lemma 3 there exist that $\|e^n\|_\infty \le O(h^2+\tau^2)$. Similarly, it can be proved that

Theorem 2. *Under the conditions of the Theorem 1, the solution of conservative finite difference scheme (6), (7) is stable by l_∞ norm.*

4. Algorithm and numerical experiments

Let in (4), $\alpha = 1, \beta = 1$. Then, the solitary wave solution of (4) is $u(x,t) = -\frac{23}{120} - \frac{1}{5}tanh\left(x + \frac{t}{10}\right) + \frac{1}{10}\left[tanh(x + \frac{t}{10})\right]^2$. In the numerical experiments, we solve the problem (4), (5) in [-50,50]. According to solitary wave solution, we take $u_0^n = u(-50, t_n), u_J^n = u(50, t_n), u_0(x) = -\frac{23}{120} - \frac{1}{5}tanh(x) + \frac{1}{10}[tanh(x)]^2$. Thus, the system(6), (7) can be written as

$$
A_j^n u_{j-1}^{n+1} + B_j^n u_j^{n+1} + C_j^n u_{j+1}^{n+1} = D_j^n (j = 1, ..., J-1, n = 1, ..., N-1), \tag{28}
$$

$$u_j^0 = u_0(jh)(j = 1, ..., J), \tag{29}$$

where

$$A_j^n = -\frac{\gamma(1-\theta)}{2} - 2\gamma(u_j^n + u_{j-1}^n) - \frac{1}{h^2}, (j = 1, ..., J-1) \tag{30}$$

$$B_j^n = 1 + \frac{2}{h^2}, (j = 1, ..., J-1, n = 1, ..., N-1), \tag{31}$$

$$C_j^n = \frac{\gamma(1-\theta)}{2} + 2\gamma(u_j^n + u_{j+1}^n) - \frac{1}{h^2}, (j = 1, ..., J-1) \tag{32}$$

$$D_j^n = B_j^n u_j^{n-1} - \Big(C_j^n + \frac{2}{h^2}\Big)u_{j+1}^{n-1} - \Big(A_j^n + \frac{2}{h^2}\Big)u_{j-1}^{n-1} - \gamma\theta(u_{j+1}^n - u_{j-1}^n) + \frac{2\gamma}{h}(u_{j+1}^n - 2u_j^n + u_{j-1}^n), (j = 1, ..., J-1, n = 1, ..., N-1) \tag{33}$$

Then (30)-(33) is a linear tri-diagonal system for u_j^{n+1}. Hence it can be solved by the formulas

$$u_j^{n+1} = p_j u_{j+1}^{n+1} + q_j, j = J-1, J-2, ..., 1, \tag{34}$$

where

$$p_j = -\frac{C_j^n}{B_j^n + A_j^n p_{j-1}}, q_j = \frac{D_j^n - A_j^n q_{j-1}}{B_j^n + A_j^n p_{j-1}}, j = 1, 2, \cdots, J-1 \tag{35}$$

$$p_0 = 0, q_0 = 0. \tag{36}$$

We take $h = 0.2, \tau = 0.1$, and θ equal to 0, 0.25, 0.5, 0.75 and 1, respectively . The maximal error $\|e^n\|_\infty$ are listed in Table. According to the data of the table, we can draw a conclusion that our scheme (6), (7) is efficient and reliable.

t	$\|e^n\|_\infty$				
	$\theta = 0.0$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 1.0$
0.2	7.650×10^{-5}	7.697×10^{-5}	7.741×10^{-5}	7.785×10^{-5}	7.827×10^{-5}
0.3	6.954×10^{-5}	6.913×10^{-5}	6.876×10^{-5}	6.845×10^{-5}	6.819×10^{-5}
0.4	1.490×10^{-4}	1.499×10^{-4}	1.507×10^{-4}	1.514×10^{-4}	1.520×10^{-4}
0.5	1.334×10^{-4}	1.327×10^{-4}	1.321×10^{-4}	1.317×10^{-4}	1.315×10^{-4}
0.6	2.160×10^{-4}	2.173×10^{-4}	2.183×10^{-4}	2.190×10^{-4}	2.193×10^{-4}
0.7	1.918×10^{-4}	1.900×10^{-4}	1.885×10^{-4}	1.883×10^{-4}	1.887×10^{-4}
0.8	2.774×10^{-4}	2.805×10^{-4}	2.830×10^{-4}	2.850×10^{-4}	2.862×10^{-4}
0.9	2.474×10^{-4}	2.443×10^{-4}	2.420×10^{-4}	2.406×10^{-4}	2.401×10^{-4}
1.0	3.385×10^{-4}	3.426×10^{-4}	3.456×10^{-4}	3.475×10^{-4}	3.481×10^{-4}

5. Conclusions

We have illustrated how an energy conservative finite difference scheme can be used to solve RLW-Burgers equation. The equation is not energy conservation, and perhaps the first time we have given the energy conservative finite difference scheme for it. The accuracy of the numerical solutions indicates that the method is well suited for the solution of the RLW-Burgers equation. It is different from the opinion in [9] that numerical solutions for the equation obtained by this scheme seem to have little effect with the variation of the parameter θ in this paper.

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