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NONDIFFERENTIABLE SECOND ORDER SELF AND SYMMETRIC DUAL MULTIOBJECTIVE PROGRAMS

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ABSTRACT. In this paper, we construct a pair of Wolfe type second order symmetric dual problems, in which each component of the objective function contains support function and is, therefore, nondifferentiable. For this problem, we validate weak, strong and converse duality theorems under bonvexity - boncavity assumptions. A second order self duality theorem is also proved under additional appropriate conditions.

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1. Introduction

Following Dorn [7], first order symmetric and self duality results in mathematical programming have been derived by a number of authors, notably, Dantzig et al [5] Mond [11], Bazaraa and Goode [1], Mond and Weir [13]. Later Weir and Mond [16] discussed symmetric duality in multiobjective programming by using the concept proper efficiency. Chandra and Prasad [3] presented a pair of multiobjective programming problem by associating a vector valued infinite game to this pair. Gulati, Husain and Ahmed [8] also established duality results for multiobjective symmetric dual problem without non-negativity constraints.

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bound for the value of the objective function when approximations are used [9]. Motivated by Mangasarian [9], Mond [12] was the first to study Wolfe type second order symmetric duality bonvexity-boncavity. Subsequently, Bector and Chandra [2] established second

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order symmetric and self duality results for a pair of non-linear programs under pseudobonvexity-pseudoboncavity condition. Devi [6] formulated a pair of second order symmetric dual programs and established corresponding duality results involving η-bonvex functions and Mishra [10] extended the results of [6] to multiobjective nonlinear programming.Recently, Suneja et. al [15] presented a pair of Mond-Weir type multiobjective second order symmetric and self dual program without non negativity constraint and proved vairous duality results under bonvexity and pseudobonvexity. In this paper, we construct in the spirit of Mond and Schechtor [14] a pair of Wolfe type multiobjective second order symmetric dual programs in which a support function occurs in each component of in the objective function and hence non-differentiable. We validate various duality results under pseudobonvexity-pseudoboncavity assumption. A self duality theorem is also proved. Some special cases are also derived form our results. The importance of this kind of programs containing $\sqrt{x^TBx}$ or a support function lies in the fact that even though objective function and/or constraint functions are nonsmooth, a simple representation for the dual may be found.

2. Notations and Pre-requisites

The following conventions for vectors x and y in n-dimensional Euclidian space R^n will be used:

$$x < y \Leftrightarrow x_i < y_i, \quad i = 1, 2, \ldots, n,$$
$$x \leqq y \Leftrightarrow x_i \leqq y_i, \quad i = 1, 2, \ldots, n,$$
$$x \leq y \Leftrightarrow x_i \leq y_i, \quad i = 1, 2, \ldots, n, \text{ but } x \neq y$$
$$x \nleq y \text{ is the negation of } x \leq y.$$

For $x, y \in R$, $x \leq y$ and $x < y$ have the usual meaning. Let ϕ be a twice differentiable from $R^n \times R^m \to R$. Then $\nabla_1\phi$ and $\nabla_2\phi$ denote gradient vectors with respect to x and y, respectively; $\nabla_1^2\phi$ and $\nabla_2^2\phi$ are respectively, the $n \times n$ and $m \times m$ symmetric Hessian matrices. $\frac{\partial}{\partial y_i}(\nabla_2^2\phi)$ is the $m \times m$ matrix obtained by differentiating the elements of $\nabla_2^2\phi$ with respect to y_i and $\nabla_2(\nabla_1^2\phi(x,y)q)$ denotes the matrix whose (i,j)th the element is $\frac{\partial}{\partial y_i}(\nabla_1^2\phi(x,y)q)_j$, where $q \in R^n$.

Definition 1. Let C be compact convex set in R^n. The support function of C is defined by $s(x|C) = \max\{x^Ty : y \in C\}$.

Definition 2. Let Q be a nonempty convex set in R^n, and let $\psi : Q \to R$ be convex. Then z is called a subgradient of ψ at $\bar{x} \in Q$ if

$$\psi(x) \geq \psi'(\bar{x}) + z^T(x - \bar{x}), \quad \text{for all } x \in Q.$$

A support function, being convex and every where finite, has a subdifferential, that is, there exist z such that $s(y|C) \geq s(x|C) + z^T(y - x)$, for all $x \in C$. The set of all subdifferential of $s(y|C)$ is given by $\partial s(x|C) = \{z \in C : z^Tx = s(x|C)\}$.

For a set Γ, the normal cone to Γ at a point $x \in \Gamma$ is defined by

$$N_\Gamma(x) = \{y | y^T(z - x) \leqq 0, \quad \text{for all} \quad z \in \Gamma\}.$$

Where C is a compact convex set, y is in $N_C(x)$ if and only if $s(y|c) = x^T y$, i.e., x is the subdifferentiable of s at y.

Consider the following multiobjective program:

(VP) Minimize $\phi(x) = (\phi_1(x), \phi_2(x), ...\phi_k(x))$

subject to $x \in X_0$

where $f : R^n \to R^n$ and $X_0 \subseteq R^n$.

Definition 3. A feasible point $\bar{x}$ is said to be a weak minimum of (VP),if there does not exist any $x \in X_0$ such that $\phi(x) < \phi(\bar{x})$.

Definition 4. A feasible point $\bar{x}$ is said to be efficient solution of (VP), if there does not exist any feasible x such that $\phi(x) \leq \phi(\bar{x})$.

An efficient solution of (VP) is obviously a weak minimum to (VP).

Definition 5. A feasible point $\bar{x}$ is said to be properly efficient solution of (VP), if it is an efficient solution of (VP) and if there exists a scalar $M > 0$ such that for each i and $x \in X_0$ satisfying $\phi_i(x) < \phi_i(\bar{x})$, we have

$$\phi_i(\bar{x}) - \phi_i(x) \leq M(\phi_j(x) - \phi_j(\bar{x})),$$

for some j, satisfying $\phi_j(x) > \phi_j(\bar{x})$.

Definition 6. A twice differentiable functions $f : R^n \times R^m \to R$ is said to be

(i) Bonvex in x, if for all $x, q, v \in R^n$ at $u \in R^m$ and fixed y

$$f(x,v) - f(u,v) \geq (x-u)^T[\nabla_x f(u,v) + \nabla_x^2 f(u,v)q] - \frac{1}{2}q^T \nabla_x^2 f(u,v)q$$

(ii) Boncave in y, if for for all $y, p, u \in R^m$ at $v \in R^n$ and fixed x

$$f(x,v) - f(x,y) \leq (v-y)^T[\nabla_y f(x,y) + \nabla_y^2 f(x,y)p] - \frac{1}{2}p^T \nabla_y^2 f(x,y)p$$

(iii) Skew-symmetric, when both x and y are in R^n and $f(x,y) = -f(y,x)$, for all in the domain of f.

3. Second order symmetric multiobjective duality

We have taken the auxiliary vectors p and q same throughout the formulations of two problems because it seems more natural than different p's and q's in [15].

In this section, we present a pair of Wolfe type non-differentiable multiobjective dual programs and validate weak, strong and converse duality theorems:

Consider the following two programs:

Primal Program:

(SWP) : Minimize $F(x,y,z,p) = F_i(x,y,z_1,p), \ldots F_k(x,y,z_k,p)$

subject to

$$\sum_{i=1}^{k} \lambda_i(\nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y)p) \leqq 0 \tag{1}$$

$$z_i \in D_i, i = 1,2,\dots,k \tag{2}$$

$$x \geqq 0 \tag{3}$$

$$\lambda = \in \wedge^+ \tag{4}$$

and

Dual Program:

(SWD) : Minimize $G(u,v,w,q) = G_1(u,v,w_1,q),\dots G_k(u,v,w_k,q)$
subject to

$$\sum_{i=1}^{k} \lambda_i(\nabla_1 f_i(u,v) - w_i + \nabla_1^2 f_i(u,v)q) \geqq 0 \tag{5}$$

$$w_i \in C_i, i = 1,2,\dots,k \tag{6}$$

$$v \leqq 0 \tag{7}$$

$$\lambda = \in \wedge^+ \tag{8}$$

where

i. $F_i(x,y,z_i,p) = f_i(x,y) + s(x \mid C_i) - y^T z_i - \frac{1}{2}p^T \nabla_2^2 f_i(x,y)p.$
$-y^T \sum_{i=1}^{k} \lambda_i(\nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y)p).$

ii. $G_i(u,v,w_i,q) = f_i(u,v) - s(v \mid D_i) + u^T w_i - \frac{1}{2}q^T \nabla_1^2 f_i(u,v)q.$
$-u^T \sum_{i=1}^{k} \lambda_i(\nabla_1 f_i(u,v) - w_i + \nabla_1^2 f_i(u,v)q)$, and

iii. for each i, $s(x \mid C_i)$ and $s(v \mid D_i)$ represent support functions of compact convex sets C_i and D_i in R^n and R^m, respectively.

iv. $w = (w_1,\dots,w_k)$ with $w_i \in C_i$ and $z = (z_1,\dots,z_k)$ for each $\{i = 1,2,\dots,k\}$.

v. $\wedge^+ = \{\lambda \in R^k \mid \lambda = (\lambda_i,\dots\lambda_k), \lambda > 0, \sum_{i=1}^{k} \lambda_i = 1\}$.

Theorem 1 (Weak Duality). *Let (x,y,λ,z,p) satisfies the constraints of (SWD) of (u,v,λ,w,q) satisfies the constraints of (SWD). If for each $i \in \{1,2,\dots,k\}$, $f_i(\cdot,y)$ is bonvex at x for fixed y and $g_i(x,\cdot)$ be boncave at y for fixed x for feasible $(x,y,u,v,\lambda,p,q,z,w,)$ then $F(x,y,z,p) \leq G(u,v,w,q)$.*

Proof. By bonvexity of $f_i(.,y)$ for fixed y at u, we have.

$$f_i(x,v) - f_i(u,v)$$
$$\leqq (x-u)^T[\nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v)q] - \frac{1}{2}q^T \nabla_1^2 f_i(u,v)q \tag{9}$$

and by boncavity of $f_i(x,\cdot)$ for fixed x at v, we have

$$f_i(x,v) - f_i(x,y)$$

$$\leqq \quad (v-y)^T[\nabla_2 f_i(x,y) + \nabla_2^2 f_i(x,y)p] - \frac{1}{2}p^T\nabla_2^2 f_i(x,y)p. \tag{10}$$

Multiplying (10) by (-1) and adding the resulting inequality to (9) we obtain

$$\begin{aligned}&[f_i(x,y) - \frac{1}{2}p^T\nabla_2^2 f_i(x,y)p - y^T\{\nabla_2 f_i(x,y) + \nabla_2^2 f_i(x,y)p\}]\\ &-[f_i(u,v) - \frac{1}{2}q^T\nabla_2^2 f_i(u,v)q - u^T\{\nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v)q\}]\\ &\quad\geq \quad x^T\{\nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v)q\} - v^T\{\nabla_2 f_i(x,y) + \nabla_2^2 f_i(x,y)p\}\end{aligned}$$

or

$$\begin{aligned}&[f_i(x,y) - y^Tz_i - \frac{1}{2}p^T\nabla_2^2 f_i(x,y)p - y^T\{\nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y)p\}]\\ &-[f_i(u,v) - u^Tw_i - \frac{1}{2}q^T\nabla_2^2 f_i(u,v)q - u^T\{\nabla_1 f_i(u,v) + w_i + \nabla_1^2 f_i(u,v)q\}]\\ &\geq x^T\{\nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v)q\} - v^T\{\nabla_2 f_i(x,y) + \nabla_2^2 f_i(x,y)p\}.\end{aligned}$$

Multiplying this by $\lambda_i > 0$, $i \in \{1,2\ldots,k\}$ and summing and using $\sum_{i=1}^{k}\lambda_i = 1$ we have

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\Big[f_i(x,y) - y^Tz_i - \frac{1}{2}p^T\nabla_2^2 f_i(x,y)p - y^T\\ &\qquad\sum_{i=1}^{k}\lambda_i\{\nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y)p\}\Big]\\ &-\sum_{i=1}^{k}\lambda_i\Big[f_i(u,v) - u^Tw_i - \frac{1}{2}q^T\nabla_1^2 f_i(u,v)q - u^T\\ &\sum_{i=1}^{k}\lambda_i\{\nabla_1 f_i(u,v) + w_i + \nabla_1^2 f_i(u,v)q\}\Big]\\ &\quad\geq\quad x^T\sum_{i=1}^{k}\lambda_i\{\nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v)q\}\\ &\qquad\quad -v^T\sum_{i=1}^{k}\lambda_i\{\nabla_2 f_i(x,y) + \nabla_2^2 f_i(x,y)q\}.\end{aligned}$$

Using (1) with (7) and (5) with (3), this inequality becomes

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i[f_i(x,y) - y^Tz_i - \frac{1}{2}p^T\nabla_2^2 f_i(x,y)p\\ &-y^T\sum_{i=1}^{k}\lambda_i\{\nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y)p\}]\end{aligned}$$

$$-\sum_{i=1}^{k}\lambda_i[f_i(u,v)-u^Tw_i-\frac{1}{2}q^T\nabla_1^2 f_i(u,v)q$$
$$-u^T\sum_{i=1}^{k}\lambda_i\{\nabla_1 f_i(u,v)+w_i+\nabla_1^2 f_i(u,v)q\}]$$
$$\geqq \ -\sum_{i=1}^{k}\lambda_i(x^Tw_i)-\sum_{i=1}^{k}\lambda_i(v^Tz_i).$$

Since $-s(x \mid C_i) \leqq -x^Tw_i$ for $w_i \in C_i$ and $-s(v \mid D_i) \leqq -v^Tz_i$, $i=1,2,\dots,k$, therefore, this inequality reduces to

$$\sum_{i=1}^{k}\lambda_i[f_i(x,y)+s(x+C_i)-y^Tz_i-\frac{1}{2}p^T\nabla_2^2 f_i(x,y)p$$
$$-y^T\sum_{i=1}^{k}\lambda_i\{\nabla_2 f_i(x,y)-z_i+\nabla_2^2 f_i(x,y)p\}]$$
$$\geqq \ \sum_{i=1}^{k}\lambda_i[f_i(u,v)-s(v\mid D_i)+u^Tw_i-\frac{1}{2}q^T\nabla_1^2 f_i(u,v)q$$
$$-u^T\sum_{i=1}^{k}\lambda_i\{\nabla_1 f_i(u,v)+w_i+\nabla_1^2 f_i(u,v)q\}],$$

i.e., $\sum_{i=1}^{k}\lambda_i F_i(x,y,z_i,p) \geqq \sum_{i=1}^{k}\lambda_i G_i(u,v,w_i,q)$ or $\lambda^T F(x,y,z,p) \geqq \lambda^T G(u,v,w,q)$. Thus, $F(x,y,z,p) \leq G(u,v,w,q)$, as we wished. □

Theorem 2 (Strong duality). *Let for each $i \in \{1,2,\dots k\}$, f_i be thrice differentiable on $R^n \times R^n$. Let $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ be properly efficient solution of (SWP); for $\lambda \leq \bar{\lambda}$ in (SWP) and assume that*

(A_1)*: the set $\{\nabla_2^2 f_1(\bar{x},\bar{y}),\nabla_2^2 f_2(\bar{x},\bar{y}),\dots\nabla_2^2 f_k(\bar{x},\bar{y}),\}$ is linearly independent.*

(A_2)*: the set $\nabla_2(\nabla_2^2(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p})$ is positive or negative definite.*

(A_3)*: the set $\{\nabla_2 f_1(\bar{x},\bar{y})+\bar{w}_1+\nabla_2^2 f_1(\bar{x},\bar{y})\bar{p},\dots,\nabla_2 f_k(\bar{x},\bar{y})+\bar{w}_k+\nabla_2^2 f_k(\bar{x},\bar{y})\bar{p}\}$ is linearly independent.*

Then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q}=0)$ is feasible solution of (SWD) and $F(\bar{x},\bar{y},\bar{z},\bar{p}) = G(\bar{x},\bar{y},\bar{w},\bar{q})$

Moreover, if the hypotheses of Theorem 1 are satisfied for all feasible solution of (SWP) and (SWD), then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q})$ is properly efficient solution for (SWD).

Proof. Since $(\bar{x},\bar{y},\bar{z},\bar{p})$ is a properly efficient solution of (SWP), then it is also weak minimum. Hence there exist $\alpha \in R^n$ with $\alpha=(\alpha_1,\dots,\alpha)$, $\beta \in R^m$, $\eta \in R^k$ and $\mu \in R^k$ with $\mu=(\mu_1,\dots\mu_k)$ and $\theta \in C_i$, $i=1,2,\dots,k$ such that the following Fritz John optimality conditions [4] are satisfied at $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$:

$$\sum_{i=1}^{k} \alpha_i(\nabla_1 f_i(\bar{x},\bar{y}) + \theta_i) + \sum_{i=1}^{k}(\beta - (\alpha^t e)\bar{y})^T \bar{\lambda}_i \nabla_{21} f_i(\bar{x},\bar{y})$$

$$+ \sum_{\lambda=1}^{k}\{(\beta - (\alpha^T e)\bar{y})^T \bar{\lambda}_i - \frac{\alpha_i \bar{p}}{2}\}^T \nabla_1(\nabla_2^2 f_i(\bar{x},\bar{y})p) = \eta, \tag{11}$$

$$\sum_{i=1}^{k}(\alpha_i - (\alpha^T e)\lambda_i)^T(\nabla_2 f_i(\bar{x},\bar{y}) - \bar{z}_i)$$

$$+ \sum_{i=1}^{k}\{(\beta - (\alpha^T e)(\bar{y} + \bar{p})^T)\lambda_i \nabla_2^2 f_i(\bar{x},\bar{y})\}$$

$$+ \sum_{\lambda=1}^{k}\{(\beta - (\alpha^T e)\bar{y})\bar{\lambda}_i - \frac{\alpha_i \bar{p}}{2}\}^T \nabla_2[\nabla_2^2 f_i((\bar{x},\bar{y})\bar{p}) = 0], \tag{12}$$

$$\{(\beta - (\alpha^T e)\bar{y})\bar{\lambda}_i - \alpha_i \bar{p}\}^T \nabla_2^2 f_i(\bar{x},\bar{y}) = 0, \tag{13}$$

$$(\beta - (\alpha^T e)\bar{y})^T(\nabla_2 f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_2^2 f_i(\bar{x},\bar{y})\bar{p}) - \mu_i = 0,$$
$$i = 1, 2, \ldots, k, \tag{14}$$

$$-\alpha_i \bar{y} + (\beta - (\alpha^T e)\bar{y})^T \lambda_i \in N_{D_i}(\bar{z}_i),\ i = 1, 2, \ldots, k, \tag{15}$$

$$\beta^T \sum_{i=1}^{k} \bar{\lambda}_i\{\nabla_2 f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_2^2 f_i(\bar{x},\bar{y})\bar{p}\} = 0, \tag{16}$$

$$\eta^T \bar{x} = 0, \tag{17}$$

$$\bar{\lambda}^T \mu = 0, \tag{18}$$

$$(\alpha, \beta, \eta, \mu) \geqq 0, \tag{19}$$

$$(\alpha, \beta, \eta, \mu) \neq 0. \tag{20}$$

Since $\lambda > 0$ and $\mu \geqq 0$, (18) implies $\mu = 0$. In view of the assumption (A_1), (13) yields

$$(\beta - (\alpha^T e)\bar{y})\lambda_i = \alpha_i \bar{p}, \quad i = 1, 2, \ldots, k. \tag{21}$$

Using (21) in (12) we have

$$\sum_{i=1}^{k}(\alpha_i - (\alpha^T e)\bar{\lambda}_i)\{(\nabla_2 f_i(\bar{x},\bar{y}) - \bar{z}_i) + \nabla_2^2 f_i((\bar{x},\bar{y})\bar{p})\}$$

$$+\frac{1}{2}(\beta - (\alpha^T e)\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_2(\nabla_2^2 f(\bar{x},\bar{y})\bar{p}) = 0. \tag{22}$$

Post multiplying (22) by $(\beta - (\alpha^T e)\bar{y})$ and the using (14) with $\mu_i = 0$, we obtain

$$(\beta - (\alpha^T e)\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_2 (\nabla_2^2 \bar{\lambda}_i f(\bar{x}, \bar{y})\bar{p})(\beta - (\alpha^T e)\bar{y}) = 0$$

which because of the condition (A_2) implies

$$(\beta - (\alpha^T e)\bar{y}) = 0. \tag{23}$$

Using (23) in (22), we have

$$\sum_{i=1}^{k} (\alpha_i - (\alpha^T e)\bar{\lambda}_i)(\nabla_2 f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_2^2 f_i(\bar{x}, \bar{y})\bar{p}) = 0.$$

This, in view of (A_3), gives

$$\alpha_i - (\alpha^T e)\bar{\lambda}_i = 0 \quad i = 1, 2, \ldots, k. \tag{24}$$

If $\alpha_i = 0$, $i = 1, 2, \ldots k$, then from (23) and (11) $\beta = 0$ and $\eta = 0$, respectively. Consequently, we get $(\alpha, \beta, \mu, \eta) = 0$, contradicting to (20). Hence $\alpha_i > 0$. Then from (21) together with (23), we have

$$\bar{p} = 0. \tag{25}$$

Using (23) and (25) in (11), we have $\sum_{i=1}^{k} \alpha_i (\nabla_1 f_i(\bar{x}, \bar{y}) + \theta_i) = \eta$, which by (24) implies $(\alpha^T e) \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_1 f_i(\bar{x}, \bar{y}) + \theta_i) = \eta$.

This with (17) and (19) respectively gives $\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_1 f_i(\bar{x}, \bar{y}) + \theta_i) = 0$, which, because of (19) and (17) along respectively, yields

$$\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_1 f_i(\bar{x}, \bar{y}) + \theta_i) \geqq 0, \tag{26}$$

and

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_1 f_i(\bar{x}, \bar{y}) + \theta_i) = 0. \tag{27}$$

From (23), we have

$$\bar{y} \geqq 0. \tag{28}$$

From (16),(27) and (28), we obtain $(\bar{x}, \bar{y}, \bar{\lambda}, w, \bar{q} = 0) = (\bar{x}, \bar{y}, \bar{\lambda}, \theta, \bar{q} = 0)$ where $\theta = (\theta_i, \ldots, \theta_k)$ is feasible for (SWD). From (16) together with (23)

$$\bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_2 f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_2^2 f_i(\bar{x}, \bar{y})\bar{p}) = 0. \tag{29}$$

From (15) along with (23) and $\alpha_i > 0$, it implies for each $i \in \{1, 2, \ldots, k\}$

$$\bar{y} \in N_{Di}(\bar{z}_i) \text{ giving } \bar{y}^T \bar{z}_i \leqq s(y \mid D_i). \tag{30}$$

From (16), (27), (29) and (30) along with $\bar{p} = \bar{w} = \bar{q}$, it implies, for each $i \in \{1, 2, \ldots k\}$,

$$\begin{aligned} & f_i(\bar{x}, \bar{y}) + s(\bar{x}|C_i) - y^T \bar{z}_i - \frac{1}{2} p^T \nabla_2^2 f_i(\bar{x}, \bar{y})\bar{p} \\ & \qquad -\bar{y}^T \sum_{i=1}^{k} \lambda_i(\nabla_2 f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_2^2 f_i(\bar{x}, \bar{y})\bar{p}) \\ & = f_i(\bar{x}, \bar{y}) + s(\bar{y}|C_i) - x^T \bar{w}_i - \frac{1}{2} q^T \nabla_1^2 f_i(\bar{x}, \bar{y})\bar{q} \\ & \qquad -\bar{x}^T \sum_{i=1}^{k} \lambda_i(\nabla_2 f_i(\bar{x}, \bar{y}) - \bar{w}_i + \nabla_1^2 f_i(\bar{x}, \bar{y})\bar{q}) \end{aligned}$$

for each $i \in \{1, 2, \ldots, k\}$,

$$F_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}) = G_i(\bar{x}, \bar{y}, \bar{w}_i, \bar{q}). \tag{31}$$

This implies $F(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}) = G(\bar{x}, \bar{y}, \bar{w}_i, \bar{q})$. That is, the objective values of (SWP) and (SWD) and equal.

Now, we shall show the proper efficiency of $(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda}, \bar{q})$ for (SWD) by exhibiting a contradiction. If is not efficient for (SWD) such that $G(\bar{x}, \bar{y}, \bar{z}, \bar{q}) \leq G_1(\bar{u}, \bar{v}, \bar{w}, \bar{q})$, which because of (31) yields. $G_i(\bar{u}, \bar{v}, \bar{w}, \bar{q}) \geqq F_i(\bar{x}, \bar{y}, \bar{z}, \bar{q})$. This contradicts to Theorem 1.

If $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ were improperly efficient solution of (SWD) $_{\bar{\lambda}}$, then for some feasible $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{q}) \in Z$ and some i $G_i(\bar{u}, \bar{v}, \bar{w}_i, \bar{q}) > G_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{q})$ and

$$G_i(\bar{u}, \bar{v}, \bar{w}_i, \bar{q}) - G_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{q}) > M(G_j(\bar{u}, \bar{v}, \bar{w}_j, \bar{q}) - G_j(\bar{x}, \bar{y}, \bar{z}_i, \bar{q})).$$

For any $M > 0$ and all j satisfying $G_j(\bar{x}, \bar{y}, \bar{z}_j, \bar{q}) > G_j(\bar{u}, \bar{v}, \bar{w}_j, \bar{q})$.

This means $G_i(\bar{u}, \bar{v}, \bar{w}_i, \bar{q}) - G_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{q})$ is finite for all $j \neq i$. Since $\bar{\lambda}_i > 0$, for all $i \in \{1, 2, \ldots, k\}$

$$\begin{aligned} & \bar{\lambda}_i G_i(\bar{u}, \bar{v}, \bar{w}_i, \bar{q}) - \bar{\lambda}_i G_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{q}) \\ & \quad > \sum_{j \neq i=1}^{k} \bar{\lambda}_j G_j(\bar{u}, \bar{v}, \bar{w}_j, \bar{q}) - \sum_{j \neq i=1}^{k} \bar{\lambda}_j G_j(\bar{x}, \bar{y}, \bar{z}_i, \bar{q}) \end{aligned}$$

i.e., $\sum_{i=1}^{k} \bar{\lambda}_i G_i(\bar{u}, \bar{v}, \bar{w}_i, \bar{q}) > \sum_{i=1}^{k} \bar{\lambda}_i G_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{q})$. This along with (31) implies

$$\sum_{i=1}^{k} \bar{\lambda}_i G_i(\bar{u}, \bar{v}, \bar{w}_i, \bar{q}) > \sum_{i=1}^{k} \bar{\lambda}_i F_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{p})$$

i.e.,

$$\bar{\lambda}^T G(\bar{u}, \bar{v}, \bar{w}, \bar{q}) > \bar{\lambda}^T F(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}).$$

This again leads to a contradiction to Theorem 1. Then the theorem fully validates. □

Theorem 3 (Converse duality). *Let for each* $i \in \{1, 2, \ldots, k\}$, f_i *be thrice differentiable on* $R^n \times R^n$. *Let* $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{q})$ *be a proper efficient solution of (SWD); fix* $\lambda = \bar{\lambda}$ *in (SWP) and assume that*

(C_1): *the set* $\{\nabla_1^2 f_i(\bar{x}, \bar{y}), \ldots \nabla_1^2 f_k(\bar{x}, \bar{y}),\}$ *is linearly independent.*

(C_2): *the matrix* $\nabla_1(\nabla_1^2(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{q})$ *is positive or negative definite, and*

(C_3): *the set* $\{\nabla_1 f_1(\bar{x}, \bar{y}) + \bar{w}_i + \nabla_1^2 f_1(\bar{x}, \bar{y})\bar{q}, \ldots, \nabla_1 f_k(\bar{x}, \bar{y}) + \bar{w}_k + \nabla_1^2 f_k(\bar{x}, \bar{y})\bar{q}\}$ *is linearly independent.*

Then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ *is feasible solution of (SWP) and*

$$F(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p}) = G(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{q}).$$

Moreover, if the hypotheses of theorem are satisfied for all feasible of (SWP) and (SWD).Then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ *is a properly efficient solution of (SWP).*

Proof. It follows exactly on the lines of Theorem 2. □

4. Second order multiobjective self duality

A mathematical program is said to be self dual, if it is formally identical with its dual, that is, if the dual is recast in the form of the primal. The new program so retained is the same as the primal. In general the programs (SWP) and (SWD) are not self dual without an added restriction on $f_i(x, y)$ with $x \in R^n$ and $y \in R^n$ for $i \in \{1, 2, \ldots, k\}$.

We describe (SWP) and (SWD) as the dual programs if the conclusions of Theorem 2 holds.

Theorem 4 (Self duality). *If the kernel* $f_i(x, y)$ *with* $f_i : R^n \times R^n \to R$ *for* $i = 1, 2, \ldots, k$ *is skew symmetric and* $C_i = D_i$ *for all* $i \in \{1, 2, \ldots, k\}$, *then (SWP) is self dual and* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ *is a joint properly efficient solution then so is* $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ *and*

$$F(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{w}, \bar{q})$$

Proof. Rewriting the dual program in primal form, we have

$$\text{(SWP-1)}: \quad \text{Minimize } -G(x,y,w,q) = (-G_i(x,y,w,q),\dots -G_k(x,y,w_k,q))$$

Subject to

$$-\sum_{i=1}^{k}\lambda_i(\nabla_1 f_i(\bar{x},\bar{y})+w_i+\nabla_1^2 f_i(\bar{x},\bar{y})\bar{q}) \leqq 0$$

$$y \geq 0$$

$$\lambda \in \Lambda^+$$

$$w_i \in C_i, \quad i=1,2,\dots k$$

where

$$-G(\bar{x},\bar{y},\bar{w},\bar{q}) = -f_i(x,y)+x^T w_i + s(y \mid D_i)+\frac{1}{2}q^T\nabla_1 f_i(x,y)q$$
$$+x^T\sum_{i=1}^{k}\lambda_i(\nabla_1 f_i(\bar{x},\bar{y})+w_i+\nabla_1^2 f_i(\bar{x},\bar{y})\bar{q}).$$

Since each f_i is a skew symmetric, $\nabla_1 f_i(x,y) = -\nabla_2 f_i(y,x), \nabla_1^2 f_i(x,y) = -\nabla_2^2 f_i(y,x)$ for all $i \in \{1,2,\dots,k\}$, and $k \in R^n$ and $y \in R^n$. Hence the dual program (SWD-1) can be written as

$$\text{(SWD-1)}: \quad \text{Minimize } G(y,x,w,q) = (G_i(y,x,w,q),\dots G_k(y,x,w,q))$$

Subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_2 f_i(y,x)+z_i+\nabla_2^2 f_i(y,x)q) \leqq 0$$

$$y \geqq 0$$

$$z_i \in C_i$$

$$\lambda \in \Lambda^+$$

where

$$G_i(y,x,w,q) = f_i(y,x)+s(y \mid C_i)+y^T z_i - \frac{1}{2}q^T\nabla_2^2 f_i(y,x)q$$
$$-x^T\sum_{i=1}^{k}\lambda_i(\nabla_2 f_i(y,x)+z_i+\nabla_2^2 f_i(y,x)q).$$

This show that the program (SWP-1) is just the primal program (SWP).

Thus $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q})$ optimal for (SWP) implies $(\bar{y},\bar{x},\bar{\lambda},\bar{w},\bar{q})$ optimal for (SWD). By an analogous argument, $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p})$ optimal for (SWP) implies $(\bar{y},\bar{x},\bar{\lambda},\bar{z},\bar{p})$ optimal for (SWD).

If (SWP) and (SWD) are dual program and $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is jointly optimal.

Then

$$\begin{aligned} 0 &= \bar{x}^T \sum_{i=1}^{k} \lambda_i (\nabla_1 f_i(\bar{x}, \bar{y}) + \bar{w}_i + \nabla_1^2 f_i(\bar{x}, \bar{y}) \bar{q}) \\ &= \bar{y}^T \sum_{i=1}^{k} \lambda_i (\nabla_2 f_i(\bar{x}, \bar{y}) + \bar{z}_i + \nabla_2^2 f_i(\bar{x}, \bar{y}) \bar{p}) \end{aligned}$$

and $\bar{p} = \bar{q} = 0$.

The objective values of the programs (SWP) and (SWD) at $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$,

$$F_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}) = G_i(\bar{x}, \bar{y}, \bar{w}, \bar{q}) = f_i(\bar{x}, \bar{y}), i = 1, 2, \ldots, k. \tag{32}$$

Since $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{z}, \bar{p})$ is also a joint optimal solution, one can similarly show that

$$\begin{aligned} 0 &= \bar{y}^T \sum_{i=1}^{k} \lambda_1^T (\nabla_1 f_i(\bar{y}, \bar{x}) + \bar{z}_i + \nabla_1^2 f_i(\bar{y}, \bar{x}) \bar{p}) \\ &= \bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_2 f_i(\bar{y}, \bar{x}) + w_i + \nabla_2^2 f_i(\bar{y}, \bar{x}) q) \end{aligned}$$

and $\bar{p} = \bar{q} = 0$.

The objective value of (SWP) and (SWD) at $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{z}, \bar{p})$ becomes

$$F_i(\bar{y}, \bar{x}, \bar{z}_i, \bar{p}) = G_i(\bar{y}, \bar{x}, \bar{w}_i, \bar{q}) = f_i(\bar{y}, \bar{x}), i = 1, 2, \ldots, k. \tag{33}$$

From (32) and (33), it implies for each $i \in \{1, 2, \ldots, k\}$,

$$F_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}) = G_i(\bar{y}, \bar{x}, \bar{z}_i, \bar{p}) = f_i(\bar{x}, \bar{y}) = f_i(\bar{y}, \bar{x}) = -f_i(\bar{x}, \bar{y}).$$

Therefore, $F_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}) = f_i(\bar{x}, \bar{y}) = 0, i = 1, 2, \ldots, k$. This implies $F_i(\bar{x}, \bar{y}, \bar{z}, \bar{p}) = 0$. □

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