J. Appl. Math. & Informatics Vol. 26(2008), No. 3 - 4, pp. 531 - 540
Website: http://www.kcam.biz

# SOME PROBLEMS AND RESULTS ON CIRCUIT GRAPHS AND TRIANGULAR GRAPHS

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ABSTRACT. We discuss the decomposition problems on circuit graphs and triangular graphs, and show how they can be applied to obtain results on spanning trees or hamiltonian cycles. We also prove that every circuit graph containing no separating 3-cycles can be extended by adding new edges to a triangular graph containing no separating 3-cycles.

AMS Mathematics Subject Classification : 05C38; 05C75
*Key words and phrases* : Circuit graphs; triangular graphs; decompositions; spanning subgraphs.

## 1. Introduction

Any notation not explained below is taken from [4]. We consider only finite simple graphs. Let $G$ be a graph. For an edge $e$ of $G$ with incident vertices $x$ and $y$, we use $xy$ or $yx$ to denote $e$. For a vertex-set $X$ of $G$, the subgraph of $G$ *induced by* $X$, denoted by $G[X]$, is the graph with the vertex set $X$ and the edge set $\Big\{xy \in G \mid x, y \in X\Big\}$. A vertex set $X$ of $G$ is a *vertex cut* if $G - X$ is disconnected and $G - X'$ is connected for any subset $X'$ of $X$. If $\{x\}$ is a vertex cut, then $x$ is a *cut vertex* of $G$.

For any vertex $x \in V(G)$, let $N_G(x) := \Big\{u \in V(G) \mid u \text{ is adjacent to } x\Big\}$. With $d_G(x) = |N_G(x)|$ we call the *degree* of $x$ in $G$. If $H$ is a subgraph of $G$, then an $H$*-bridge* of $G$ is either an edge together its endvertices in $E(G) \setminus E(H)$ joining two vertices of $H$ which is called a *trivial* $H$-bridge, or it is a connected component of $G - H$ together with all edges (and their endvertices) of $G$ joining this component to $H$ which is called *nontrivial*. A cycle $Z$ in a connected graph

Received September 3, 2007.
This Work was Supported by Hanshin University Research Grants.

$G$ is called a *separating cycle* if $G - Z$ is disconnected. For a positive number $k$, a *separating $k$-cycle* is similarly defined. We define further a graph is called *$Sk$-free* if it contains no separating $k$-cycles.

Let $G$ be a plane graph. The *faces* of $G$ are the connected components (in topological sense) of the complement of $G$ in the plane. The boundary of the unbounded face of $G$ is the *outer walk* of the graph, or the *outer cycle* if it is a cycle. A cycle is a *facial cycle* in a plane graph if it bounds a face of the graph. Let $C$ be a cycle in $G$. We denote by $\overline{C}$ the subgraph of $G$ consisted of the vertices and edges lying on $C$ and in the interior of $C$. A plane graph is a *circuit graph* if it is isomorphic to $\overline{C}$, where $C$ is a cycle in a 3-connected plane graph. A 2-connected plane graph is *triangular* if every bounded facial cycle is a 3-cycle. We can easily verify that every triangular graph is a circuit graph. It is well known that if $G$ is a 2-connected plane graph then every face of $G$ is bounded by a cycle. Also note that if $G$ is plane graph and $x, y, z, v$ occur on a facial cycle in this cyclic order, then $G$ contains no vertex disjoint paths from $x$ to $z$ and from $y$ to $v$, respectively.

The concept of circuit graphs which are frequently used in a proof process using the mathematical induction was introduced by D. Barnette [1]. He gave some results concerning the existence of spanning trees or hamiltonian paths in a circuit graph by investigating the properties and the characteristic of such graphs (see [1] and [2]). Circuit graphs have since been constructed for some classes of 3-connected planar graphs or infinite graphs as well (see [3] or [6]), in particular very recent result due to Jung [7] suggests that the characteristic of circuit graphs can be applied to solve the cyclability problems for finite or infinite planar graphs.

On the other hand, Thomassen [9] showed that every triangulation of the orientable surface with no noncontractable cycle of a limited length has a spanning tree of maximum degree at most 4, and Ellingham and Gao [5] improved the result for 4-connected triangulations. In the proofs of their results we see that the properties of the triangular graphs play a crucial role for proving these results. Moreover, triangular graphs are useful for searching spanning trees or paths in certain classes of maximal planar graphs or triangulations (see [8]).

Motivated by the above statements, we study the decomposition problems on circuit graphs and triangular graphs, and further it will be shown show how they can be applied to obtain results on spanning trees or hamiltonian cycles. In addition, by drawing attention to natural characterizations of the circuit graphs in terms of their subgraphs and investigating some properties concerning the triangular graphs, we prove that every circuit graph containing no separating 3-cycles can be extended by adding new edges to a triangular graph containing no separating 3-cycles.

This paper organized as follows. Following this section on introduction, Section 2 gives one of our main theorems which characterizes the circuit graphs by presenting 5 equivalent forms, and in addition some corollaries of the result are included. In Section 3, we treat the 1-decomposition of a circuit graph and 2-decomposition of a triangular graph, and moreover we show how they can be applied to obtain results on spanning trees or hamiltonian cycles. As an application of these results, we show the existence of a spanning subgraph in a triangular graph which contains the outer cycle of the graph and satisfies some additional properties, which generalizes the theorem of Barnette. Finally in Section 4, we study S3-free plane graphs (=graphs which contain no separating 3-cycles) and prove that every S3-free circuit graph can be extended by adding new edges to a S3-free triangular graph.

## 2. Characterization theorem for circuit graphs

In [7] three equivalent forms concerning circuit graphs (due to Barnette [1]) were given. In this section we present an extended theorem including two additional properties equivalent to them, and using the result we prove some corollaries.

**Theorem 2.1.** *Let $G$ be a 2-connected plane graph with outer cycle $C$. Then following statements are equivalent.*

(1) *$G$ is a circuit graph.*
(2) *For every vertex cut $S$ of $G$ with $|S| = 2$, every component of $G - S$ contains a vertex of $C$.*
(3) *If $S$ is a vertex cut with $|S| = 2$ and $G_1, G_2$ are subgraphs of $G$ with $G_1 \cup G_2 = G$ and $V(G_1 \cap G_2) = S$, then $G_i - S$ is connected and $C \not\subseteq G_i$, $(i = 1, 2)$.*
(4) *$G \cup (C \times \{v\})$ is 3-connected, for a further vertex $v$.*
(5) *There is a 3-connected planar graph $\widetilde{G}$ and a vertex $x \in V(\widetilde{G})$ such that $G$ is isomorphic to $\widetilde{G} - x$.*

*Proof.* Since the equivalence of the properties (1), (2) and (4) is already verified in [7], we have to show that the assertions (3) and (5) are also equivalent to them.

(2) $\Rightarrow$ (3). We first show that $G_i - S$ is connected. To see this, consider the number of the components of $G - S$. Since $S$ is a vertex cut of $G$, $G - S$ has at least 2 components. On the other hand, since $C - S$ has at least 2 paths and each component of $G - S$ contains a vertex of $C$, it follows that $G - S$ has at most 2 components. Therefore $G - S$ has exactly 2 components, namely $G_1 - S$ and $G_2 - S$, which means that both $G_i - S$ $(i = 1, 2)$ are connected.

To show $C \not\subseteq G_i$, $(i = 1, 2)$, suppose to the contrary that $C \subseteq G_1$. (The case $C \subseteq G_2$ can be similarly verified.) Then, since $C$ is the outer cycle of $G$, $G_2 - S$ must lie in the interior of $C$, which implies that each component of $G_2 - S$ (which is simultaneously a component of $G - S$) cannot contain a vertex of $C$. A contradiction.

(3) $\Rightarrow$ (2). Suppose that a component (say $Q$) of $G - S$ does not contain a vertex of $C$. Note that, since $(G_1 - S) \cap (G_2 - S) = \emptyset$, $Q$ is contained either entirely in $G_1 - S$ or in $G_2 - S$. But, from the hypothesis that both $G_1 - S$ and $G_2 - S$ are connected, we conclude that either $Q = G_1 - S$ or $Q = G_2 - S$. We may without loss of generality assume $Q = G_1 - S$. Then, since $G_1 - S$ contains no vertex of $C$, $G_2$ must contain entirely $C$, which contradicts the assumption.

(4) $\Leftrightarrow$ (5). Since the assertion (4) $\Rightarrow$ (5) is obviously true, we have to prove the converse. By the 3-connectedness of $\widetilde{G}$, there exists only one facial cycle of $\widetilde{G} - x$ containing al neighbors of $x$ in $\widetilde{G}$. Consider

$$\widetilde{G}' = \widetilde{G} \cup \left\{ xy \,|\, y \in N_{\widetilde{G}}(x) \right\}.$$

Since $\widetilde{G}'$ is obviously is 3-connected, $G$ (with the outer cycle $C$) satisfies the condition (4). □

**Corollary 2.2.** *Let $G$ be a circuit graph with outer cycle $C$, and let $G_1, G_2$ are subgraphs of $G$ with $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{x, y\}$. Then $G_i \cup \{xy\}$ is a circuit graph with outer cycle $(G_i \cap C) \cup \{xy\}$, $(i = 1, 2)$.*

*Proof.* Set $G_1' := G_1 \cup \{xy\}$. It is clear that $G_1'$ is 2-connected. We can also see that $xy$ lies on the outer cycle of $G_1'$, for otherwise, $G_1' - \{x, y\}$ must have at least 2 components. But, since $x, y \in V(C)$, it follows that $C \subseteq G_1$, which contradicts to (4) in Theorem 2.1. Therefore $C_1 := (C \cap G_1) \cup \{xy\}$ is the outer cycle of $G_1'$.

In order to show that $G_1'$ is a circuit graph, let $v$ be a vertex ($\notin V(G_1')$) and set $\widetilde{G}_1 := G_1' \cup \left(C_1 \times \{v\}\right)$. Then, by Theorem 2.1, we have to show that $\widetilde{G}_1$ is 3-connected. To see this, suppose that there would exist vertex cut $T \subseteq V(\widetilde{G}_1)$ with $|T| = 2$, such that $\widetilde{G}_1 - T$ is disconnected.

First it is $v \notin T$, for otherwise $\widetilde{G}_1$ is not 2-connected. Further we see that $T \nsubseteq V(C_1)$, because of $N_{\widetilde{G}_1}(v) = V(C_1)$. Therefore $T$ must contain a vertex of $\widetilde{G}_1 - C_1$. But, in this case, $G$ is not a circuit graph, which is also impossible. □

**Corollary 2.3.** *If $Z$ be a cycle in a circuit graph, then $\overline{Z}$ is a circuit graph.*

*Proof.* Let $G$ be a circuit graph containing $Z$ (with outer cycle $C$). Consider the graph $\widetilde{G} := G \cup (C \times \{v\})$ for a further vertex $v$ ($\notin V(G)$). Since $\widetilde{G}$ is 3-connected by Theorem 2.1 and $Z$ is a cycle in $\widetilde{G}$, we conclude that $\overline{Z}$ is a circuit. □

## 3. Decompositions

It is clear that every connected graph $H$ has a following decomposition:

There exist subgraphs $H_1, \ldots, H_m$ and cut vertices $x_1, \ldots, x_{m-1}$ of $H$ with

$$H = \bigcup_{k=1}^{m} H_k, \quad \text{and} \quad \left[\bigcup_{i=1}^{k-1} H_i\right] \bigcap H_k = \{x_{k-1}\}, \quad k = 2, \ldots, m \qquad (*)$$

such that $H_k$ ($k = 1, \ldots, m$) is either 2-connected or isomorphic to $K_2$. We call such a decomposition by a *1-decomposition* of $H$. If a connected graph $H$ has a form (*), we say that $H_k$ is a *block* of $H$. An *endblock* is a block containing at most one cut vertex. A block is *trivial* if it is isomorphic to $K_2$. We say that a 1-decomposition of $H$ is *linear* if $H$ contains at most 2 endblocks; i.e., there is an alternating sequence of blocks $H_1, \ldots, H_m$ and cut vertices $x_1, \ldots, x_{m-1}$ of $H$ ($m \geq 1$) with

$$H_1,\ x_1,\ H_2, \cdots, x_{m-1},\ H_m$$

such that $H_k$ contains the cut vertices $x_{k-1}$ and $x_k$, ($k = 2, \ldots, m-1$).

**Theorem 3.1.** *Let $G$ be a circuit graph with outer cycle $C$ and let $x \in V(C)$ be arbitrary. Then $G - x$ has a linear 1-decomposition such that every nontrivial block is a circuit graph.*

*Proof.* Set $G' = G - v$. First consider the case that $G'$ is 2-connected. In this case we clearly have to show that $G'$ is a circuit graph. To prove this, let us denote the outer cycle of $G'$ by $C'$, and suppose for contradiction that it is not the case. Then, by Theorem 2.1, there exist a vertex cut $S$ with $|S| = 2$ and a component (say $Q$) of $G' - S$ such that $Q$ contains no vertex of $C'$; i.e., $Q$ lies entirely in the interior of $C'$. But, since $C$ is the outer cycle of $G$ and $C' \subseteq \overline{C} = G$, $Q$ cannot contain a vertex of $C$. Using again Theorem 2.1, we conclude that $G$ cannot be a circuit graph, which contradicts the assumption.

Now consider the case that $G'$ is connected, but not 2-connected. Let

$$N_G(v) \cap V(C) = \{x, y\}$$

and suppose to the contrary that $G'$ doesn't have a linear 1-decomposition. Let $Q_1, \ldots, Q_r$ be the endblocks of $G'$ with the cut vertex $u_1, \ldots, u_r$. First note that $r \geq 3$, since $G'$ is not linear. If we may without loss of generality assume that $x \in Q_1$ and $y \in Q_r$, then each of the remaining endblocks (except for its cut vertex) $Q_i - u_i$ ($i = 2, \ldots, r-1$) must be contained in the interior of $C$. Then, since $S := \{u_2, v\}$ separates $G$, it follows that $Q_i - u_i$ is a component of $G - S$ which does not contain a vertex of $C$. Thus, by Theorem 2.1, $G$ is not a circuit graph, which contradicts the assumption. The fact that each nontrivial block of $G'$ is circuit graph can be verified using similar method above. □

We will now consider the triangular graphs.

**Proposition 3.2.** *Let $G$ be a triangular graph. Then there exist subgraphs $G_1, \dots, G_m$ and edges $e_1, \dots, e_{m-1}$ of $G$ with*

$$G = \bigcup_{k=1}^{m} G_k, \quad \text{and} \quad \left[\bigcup_{i=1}^{k-1} G_i\right] \bigcap G_k = \{e_{k-1}\}, \quad k = 2, \dots, m \qquad (**)$$

*such that $G_k$ $(k = 1, \dots, m)$ is either a 3-cycle or 3-connected triangular graph.*

*Proof.* The assertion can be easily verified by induction on $m$. □

We may say that for a triangular graph $G$ the decomposition in Proposition 3.2 is a *2-decomposition*. If a triangular graph has a form (**), we say that $G_k$ is a *term* of $G$ and the edge $e_k$ an *edge-cut* of $G$, $(k = 1, \dots, m)$. If a term contains at most one edge-cut, it is called a *endterm* of $G$. A term is trivial if it is isomorphic to $K_3$. A 2-decomposition is *linear* if there is an alternating sequence of 3-connected triangular graphs $G_1, \dots, G_m$ and edge-cuts $e_1, \dots, e_{m-1}$ of $G$ $(m \geq 1)$ with

$$G_1,\ e_1,\ G_2, \cdots, e_{m-1},\ G_m$$

such that $G_k$ $(k = 2, \dots, m-1)$ contains the edge-cuts $e_{k-1}$ and $e_k$.

**Lemma 3.3.** *Let $G$ be a triangular graph with outer cycle $C$ containing endterms $T$ and $T'$ $(T \neq T')$ with edge-cuts $xy$ and $x'y'$, respectively. Let further $v \in \big(V(T) \cap V(C)\big) \setminus \{x, y\}$, $v' \in \big(V(T') \cap V(C)\big) \setminus \{x', y'\}$ arbitrary. Then $G$ has a linear 2-decomposition if and only if both the two $v, v'$-paths on $C$ are induced.*

*Proof.* Suppose that a $v, v'$-path (say $J$) on $C$ is not induced, in cause of two vertices $u, w \in V(J)$ with $uw \in E(G)$ but $uw \notin E(J)$. Then $uw$ is an edge-cut of $G$, and it follows that $G$ must contain an endterm (distinct from $T$ and $T'$). Therefore $G$ is not linear. The converse is obvious. □

We will now prove a similar argument as Theorem 3.1 using Lemma 2.3.

**Theorem 3.4.** *Let $G$ be a 3-connected triangular graph with outer cycle $C$ with $|V(C)| \geq 4$, and assume that $G$ contain no separating 3-cycles. Further let $x \in V(C)$ be arbitrary. Then $G - x$ is a linear 2-decomposition such that every nontrivial term is a 3-connected triangular graph.*

*Proof.* From the 3-connectedness and triangularity of $G$, $G - x$ is 2-connected triangular graph. Let

$$N_G(x) \cap V(C) = \{v, v'\}.$$

Let further $J_1$ and $J_2$ be the $v, v'$-paths on the outer cycle of $G - x$ with $J_1 = C - x$. Then clearly $V(J_2) = N_G(x)$, since $G$ is triangular. In order to prove that $G$ has a linear 2-decomposition, we only show that $J_1$ and $J_2$ are induced paths, by Lemma 3.3.

First we see that, from the facts that $J_1 \subseteq C$ and $G$ is 3-connected, $J_1$ is an induced path. But $J_2$ is also induced, for otherwise, as in the proof of Lemma 3.3, there exist two vertices (say $u, w$) on $J_2$ with $uw \in E(G) \setminus E(J_2)$. Then it can be easily seen that $\{x, u, w\}$ separates $G$. Since $xu, xw \in E(G)$ (from the triangularity of $G$) and $uw \in E(G)$, $\{x, u, w\}$ is a separating 3-cycle, which contradicts the hypothesis. The fact that every nontrivial term of the 2-decomposition of $G$ is a 3-connected triangular graph follows from the definition. □

Below we derive one of our main results as a consequence of Theorem 3.4, which provides the existence of spanning trees possessing some particular properties concerning degree conditions.

**Theorem 3.5.** *Let $G$ be a triangular graph with outer cycle $C$, and let $u, v \in V(C)$. Then there exists a spanning subgraph $H$ of $G$ with $C \subseteq H$ such that $d_H(x) \leq 3$ for all $x \in V(H)$ and $d_H(u) = d_H(v) = 2$.*

*Proof.* We proceed by induction on the number of vertices $|V(G)|$, the result being trivial when $|V(G)| = 3$. We distinguish two cases.

*Case 1 :* $G$ contains no trivial $C$-bridge.

Choose an arbitrary vertex $w \in V(C) \setminus \{u, v\}$, and set $N_G(w) = \{w_1, \ldots, w_k\}$ in the natural order. Then we first see that $k \geq 3$, for otherwise, i.e., $N_G(w) = \{w_1, w_2\}$, we have $w_1 w_2 \in E(G)$ by the triangularity of $G$, and thus this edge is a trivial $C$-bridge, a contradiction. Further, by the fact that $G$ is 3-connected from the assumption, we have $w_2, \ldots, w_{k-1} \notin V(C)$, and therefore

$$C' := (C - w) \cup \{w_1, w_2, \ldots, w_{k-1}, w_k\}$$

is the outer cycle of $G - w$.

Apply the theorem inductively to $G - w$ with $C'$ playing the role $C$, to obtain a spanning subgraph $H'$ in $G - w$ with $C' \subseteq H'$ such that $d_{H'}(u) = d_{H'}(v) = 2$ and $d_{H'}(x) \leq 3$ for all $x \in V(H')$. Then by setting

$$H := \left[H' \cup \{ww_1, ww_2, ww_k\}\right] \setminus \{w_1 w_2, w_k w_{k-1}\}$$

we get a spanning subgraph of $G$ as desired.

*Case 2 :* $G$ contains a trivial $C$-bridge.

Let $yz \in E(G)$ be a trivial $C$-bridge in $G$, and set $G_1$ and $G_2$ be the subgraphs of $G$ with

$$G_1 \cup G_2 = G \qquad \text{and} \qquad G_1 \cap G_2 = \{yz\}.$$

Let us denote the $y, z$-path on $C$ contained in $G_i$ by $P_i$ $(i = 1, 2)$. Then we can without difficulty see that both $G_1$ and $G_2$ are triangular graph with the outer cycles

$$C_1 := P_1 \cup \{yz\} \qquad \text{and} \qquad C_2 := P_2 \cup \{yz\}, \quad \text{respectively.}$$

*Subcase 2.1* : $u \in V(P_1) \setminus \{y, z\}$ and $v \in V(P_2) \setminus \{y, z\}$ $\Big($or $u \in V(P_2) \setminus \{y, z\}$ and $v \in V(P_1) \setminus \{y, z\}\Big)$.

We will prove the former case. (The latter case can be analogously verified.) Since $G_i$ (with the outer cycle $C_i$) satisfies the hypothesis in the theorem, we can apply this theorem inductively to each $G_i$ with $C_i \subseteq H_i$ $(i = 1, 2)$ satisfying the conditions

$$d_{H_1}(u) = d_{H_1}(y) = d_{H_2}(v) = d_{H_2}(z) = 2.$$

Then, by putting $H = (H_1 \cup H_2) - \{yz\}$, we obtain a subgraph of $G$ satisfying the desired properties.

*Subcase 2.2* : $u, v \in V(P_1)$. (In the case $u, v \in V(P_2)$ it can be similarly verified.)

As in the proof above, we can also obtain subgraphs $H_1$ of $G_1$ and $H_2$ of $G_2$ with

$$d_{H_1}(u) = d_{H_1}(v) = d_{H_2}(y) = d_{H_2}(z) = 2.$$

Then $H = (H_1 \cup H_2) - \{yz\}$ is a subgraph of $G$ as desired. □

We note that the following consequence of Theorem 3.5 is a slight variation of Barnette's Theorem [1] on spanning trees in circuit graphs.

**Corollary 3.6.** *Let $G$, $C$, $u$ and $v$ as in Theorem *** be given. Then there exists a spanning 3-tree in $G$ with $d_T(u) = 1$ and $d_T(v) \leq 2$.*

*Proof.* If $H$ is a subgraph of $G$ obtained from Theorem 3.5, we delete from $H$ one of the edges incident to each facial cycle to get a spanning 3-tree with the required degree constraints. □

## 4. S3-free plane graphs

Clearly every planar graph can be extended to a maximal planar graph by adding some new edges. We now prove that every 2-connected S3-free planar graph can be extended to a S3-free maximal planar graph, and as a corollary we show that every S3-free circuit graph is a spanning subgraph of a S3-free triangular graph.

**Theorem 4.1.** *Let $H$ be a 2-connected S3-free plane graph. Then there exists a S3-free maximal planar graph $G$ satisfying the properties*

$$H \subseteq G \qquad \text{and} \qquad V(H) = V(G)$$

*Proof.* If every facial cycle of $H$ is a 3-cycle, then $H$ already is a maximal planar graph, and therefore we are done. Otherwise, choose a facial cycle (say $Z$) with

$$V(Z) = \{v_1, v_2, \ldots, v_k\}, \qquad k \geq 4$$

*Case 1 :* Either $v_1v_3 \in E(H)$ or $v_2v_4 \in E(H)$.

We will only consider the former case. In this case we see that $v_2v_4 \notin E(H)$. We now add a new edge $v_2v_4$ in the interior of $Z$. Then the constructed graph may be denoted by $H'$.

*Case 2 :* Both $v_1v_3 \notin E(H)$ and $v_2v_4 \notin E(H)$.

If there does not exist a vertex $w \in V(H)$ with $wv_1, wv_3 \in E(H)$, then we can simply add an edge $v_2v_4$. Now let $w \in V(H)$ with $wv_1, wv_3 \in E(H)$. Then we first verify that $v_2v_4 \notin E(H)$, by planarity of $H$. To obtain a new graph, denoted by $H'$, adding a new edge $v_2v_4$ in $H$ without creating a separating 3-cycle, we have only to show that there exists no vertex of $H$ adjacent to both $v_2$ and $v_4$.

To see this, it must be noted that each vertex of $H$ must lie either in the interior of 4-cycle $v_1v_2v_3wv_1$ or in the exterior of the cycle $v_1wv_3v_4\cdots v_1$. If a vertex $u$ lies in the interior of the cycle $v_1v_2v_3wv_1$ $u$ cannot be adjacent to $v_4$. On the other hand, if a vertex $u$ lies in the exterior of the cycle $v_1wv_3v_4\cdots v_1$, it cannot be adjacent to $v_2$.

In any case, the constructed graph $H'$ remains 2-connected S3-free plane graph with $V(H') = V(H)$ and further $E(H') = E(H) + 1$. By noting that the number of edges of a maximal planar graph is $3|G| - 6$, our construction eventually terminates since the number of edges is increased by one at each stage. □

As an immediate consequence of Theorem 4.1, we have:

**Corollary 4.2.** *Every S3-free circuit graph can be extended by adding edges to a S3-free triangular graph.*

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