J. Appl. Math. & Informatics Vol. 26(2008), No. 3 - 4, pp. 511 - 517
Website: http://www.kcam.biz

SOME RESULTS ON TOTAL COLORINGS OF PLANAR GRAPHS

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ABSTRACT. Let G be a planar graph. It is proved that if G does not contain a k-cycle with a chord for some $k \in \{4,5,6\}$, then G is total-$(\Delta(G)+2)$-colorable.

AMS Mathematics Subject Classification : 05C15
Key words and phrases : planar graph, total coloring, cycle

1. Introduction

We consider only simply graphs in this paper unless stated otherwise. A plane graph is a particular drawing of a planar graph in Euclidean plane. For a plane graph G, we denote its vertex set, edge set, face set, maximum degree and minimum degree by $V(G), E(G), F(G), \Delta(G)$ and $\delta(G)$ (or simply V, E, F, Δ and δ), respectively. A *total-k-coloring* of a graph G is a coloring of $V(G) \cup E(G)$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ is the smallest integer k such that G has a total-k-coloring. It is clear that $\chi''(G) \geq \Delta(G)+1$. Behzad [2] and Vizing [13] posed independently the famous conjecture, known as the Total Coloring Conjecture (TCC).

Conjecture A. *For any graph G, $\Delta(G)+1 \leq \chi''(G) \leq \Delta(G)+2$*

This conjecture was verified by Rosenfeld [10] and Vijayaditya [12] for $\Delta(G) = 3$ and by Kostochka [7, 8, 9] for $\Delta(G) \leq 5$.

TCC remains open even for planar graphs, but more is known. Borodin proved it for planar graphs with $\Delta(G) \geq 9$. Yap [14] and Andersen [1] proved the case of $\Delta(G) = 8$. The $\Delta(G) = 7$ case was solved for planar graphs by Sander and Zhao [11]. Recently, Geng and Hou [4] considered planar graphs that do not contain small cycles and got some results. Besides total coloring

Received 07 Mar, 2007. Revised 17 Jan, 2008. *Corresponding author.
This work is supported by a research grant NSFC(60673047)

problems, various extension problems in planar graphs have been extensively studied (e.g. [5], [6]).

In this paper, we consider planar graphs that does not contain a k-cycle with a chord for some $k \in \{4, 5, 6\}$ and get the following result.

Theorem 1. *Let G be a planar graph. If G does not contain a k-cycle with a chord for some $k \in \{4,5,6\}$, then G is total-$(\Delta(G)+2)$-colorable.*

2. Proof of theorem 1

Let us introduce some notations and definitions. Let G be a planar graph. A k- or k^+-*vertex* is a vertex of degree k or at least k, respectively. For $f \in F(G)$, we use $b(f)$ to denote the boundary of f and write $f = [u_1u_2...u_n]$ if $u_1, u_2, ..., u_n$ are the vertices of f in a clockwise order. The degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k- or k^+-*face* is a face of degree k or at least k, respectively. For $v \in V(G)$, let $n_k(v)$ or $n_{k^+}(v)$ denote the number of k-faces or k^+-faces incident with v, respectively. Let $\delta(f)$ denote the minimum degree of vertices incident with f.

In the proof of Theorem 1, we use the technique of discharging. In the beginning, we define the initial charge function for each element in $V(G) \cup F(G)$. By following the rules stated in the proof of the theorem, we will redistribute the charges for the vertices and faces so that the new charges are nonnegative and the sum of the new charges is still the same as before, which leads to a contradiction to Euler's formula.

Proof of Theorem 1. We only consider the case that $\Delta(G) = 6$. Let G be a minimal counterexample to the theorem. Then every proper subgraph of G is totally $(\Delta(G)+2)$-colorable. It is easy to see that G is 2-connected and hence has no vertices of degree 1. Furthermore, G has the following proprieties.

(a) G contains no edge uv with $\min\{d(u), d(v)\} \le \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor$ and $d(u) + d(v) \le \Delta(G) + 2$.

(b) G contains no even cycle $v_1v_2...v_{2t}v_1$ such that $d(v_1) = d(v_3) = ... = d(v_{2t-1}) = 3$.

(c) G contains no $(3,6,6)$-face.

(d) G contains no triangle $f = [v_1v_2v_3]$ such that $d(v_1) = d(v_2) = 4$.

(a) and (b) can be found in [3]. (c) can be found in [4]. We will show (d).

On the contrary, suppose that G does contain a triangle $f = v_1v_2v_3$ such that $d(v_1) = d(v_2) = 4$. Let $G' = G - v_1v_2$. By the minimality of G, G' has a proper total-$(\Delta(G)+2)$-coloring ϕ. We first consider the case that $\phi(v_1) \ne \phi(v_2)$. If there is a color α that does not appear both at v_1 and v_2, then color v_1v_2 with color α. It follows that ϕ is extended to a total-$(\Delta(G)+2)$-coloring of G, contradicting to the choice of G. Thus the colors appear at v_1 do not appear at v_2 and the colors appear at v_2 do not appear at v_1. Since there are $\Delta(G)+2$

colors in total available, there is a color β which does not appear at v_3. If β does not appear at v_1, then recolor v_1v_3 with β and color v_1v_2 with $\phi(v_1v_3)$, Otherwise, recolor v_2v_3 with β and color v_1v_2 with $\phi(v_2v_3)$. In either case, ϕ can be extended to a total-$(\Delta(G)+2)$-coloring of G. This contradicts the minimality of G. Now suppose that $\phi(v_1) = \phi(v_2)$. Then erase the color on v_1. Since there are $\Delta(G) + 2$ colors in total available, there is a color γ such that $\gamma \neq \phi(v_2)$ and γ does not appear on the elements incident or adjacent to v_1. We recolor v_1 with color γ and then the color appearing on v_1 is different from that on v_2. We also get a contradiction by the above arguments. It completes the proof of (d).

It follows from (a) that $\delta(G) \geq 3$. Let G_3 be the subgraph induced by the edges incident with the 3-vertices of G. Since $\Delta(G) = 6$, (a) implies that G does not contain two adjacent 3-vertices. Hence G_3 does not contain any odd cycle. It follows from (b) that G_3 is a forest. In each component T of G_3, choose a maximum vertex u as the root of T, and match each 3-vertex v with the maximum vertices u_1, u_2 adjacent to v that are further from u (Note that the endvertices of T are all $\Delta(G)$-vertices). In this case, u_i is the 3-master of v and v is called dependent of u_i for $i = 1, 2$. Note that each vertex of degree $\Delta(G)$ can be the 3-master of at most one 3-vertex and each 3-vertex is the dependent of two maximum vertices.

Since G is a planar graph, by Euler's formula, we have

$$\sum_{v\in V(G)} (d(v) - 4) + \sum_{f\in F(G)} (d(f) - 4) = -4(|V(G)| - |E(G)| + |F(G)|) = -8 < 0.$$

Now we define the initial charge function $w(x)$ for each $x \in V(G) \cup F(G)$. Let $w(x) = d(x) - 4$ if $x \in V(G) \cup F(G)$. It follows that $\sum_{x\in V(G)\cup F(G)} w(x) < 0$. The discharging method distributes the positive charge to neighbors so as to leave as little positive charge remaining as possible. This leads to $\sum_{x\in V(G)\cup F(G)} w(x) \geq 0$. A contradiction follows.

Case 1. G does not contains a 4-cycle with a chord. By the choice of G, we have the following observation.

$(O_{1.1})$ Every k-vertex is incident with at most $\left\lfloor \frac{k}{2} \right\rfloor$ 3-faces.

To prove the theorem, we are ready to construct a new charge $w^*(x)$ on G as follows:

$(R_{1.1})$ Each 3-vertex receive $\frac{1}{2}$ from each of its 3-master.

$(R_{1.2})$ Each 3-face receive $\frac{1}{2}$ from each of its incident 4^+-vertex.

Clearly, $w^*(f) = w(f) \geq 0$ if $d(f) \geq 4$. If $d(f) = 3$, then $\delta(f) \geq 4$ by (c). Furthermore, the face f is incident with two 5^+-vertices by (d). Thus $w^*(f) \geq w(f) + 2 \times \frac{1}{2} = 0$.

Let v be any vertex of G. If $d(v) = 3$, then v must be the dependent of some two maximum vertices. So $w^*(v) = w(v) + 2 \times \frac{1}{2} = 0$. If $d(v) = 4$, then $w^*(v) = w(v) = 0$. If $d(v) = 5$, then v is incident with at most two 3-faces by $(O_{1.1})$. Thus $w^*(v) \geq w(v) - 2 \times \frac{1}{2} = 0$. If $d(v) = 6$, then v is incident with at most three 3-faces by $(O_{1.1})$ and v may be the 3-master of at most one 3-vertex. So $w^*(v) \geq w(v) - \frac{1}{2} - 3 \times \frac{1}{2} = 0$.

Case 2. G does not contain a 5-cycle with a chord. By the choice of G, we have the following observations.

$(O_{2.1})$ Every k-vertex with $k \geq 4$ is incident with at most $(k-2)$ 3-faces.

$(O_{2.2})$ Let v be a 4^+-vertex. If $n_3(v) \neq 0$, then v is incident with at least two 5^+-faces.

To prove the theorem, we are ready to construct a new charge $w^*(x)$ on G as follows:

$(R_{2.1})$ Each 3-vertex receive $\frac{1}{2}$ from each of its 3-master.

$(R_{2.2})$ From each 5^+-face f to each of its incident vertex v, transfer $\frac{w(f)}{d(f)}$.

$(R_{2.3})$ From each 4^+-vertex to each of its incident 3-faces, transfer $\frac{1}{5}$, if $d(v) = 4$. $\frac{2}{5}$, otherwise.

Clearly, $w^*(f) = w(f) \geq 0$ if $d(f) = 4$ and $w^*(f) = w(f) - d(f)\frac{w(f)}{d(f)} = 0$ if $d(f) \geq 5$. Let f be any 3-face. It follows from (c) that $\delta(f) \geq 4$. By (d), the face f is incident with at least two 5^+-vertices. Thus $w^*(f) \geq w(f) + \frac{1}{5} + 2 \times \frac{2}{5} = 0$.

Let v be any vertex of G. If $d(v) = 3$, then v is the dependent of some two maximum vertices. So $w^*(v) \geq w(v) + 2 \times \frac{1}{2} = 0$. If $d(v) = 4$, then v is incident with at most two 3-faces by $(O_{2.1})$. If v is incident with no 3-face, then $w^*(v) = w(v) = 0$. Otherwise, v is incident with at least two 5^+-faces by $(O_{2.2})$. Thus $w^*(v) = w(v) - 2 \times \frac{1}{5} + 2 \times \frac{1}{5} = 0$. Let $d(v) = 5$. Then v is incident with at most three 3-faces by $(O_{2.1})$. Furthermore, if v is incident with three 3-faces, then the other faces incident with f must be 5^+-faces. So $w^*(v) = w(v) - \max\{2 \times \frac{2}{5}, 3 \times \frac{2}{5} - 2 \times \frac{1}{5}\} = \frac{1}{5} > 0$. If $d(v) = 6$, then v is incident with at most four 3-faces by $(O_{2.1})$ and v may be the 3-master of some 3-vertex. If v is incident with no 3-face, then $w^*(v) \geq w(v) - \frac{1}{2} = \frac{3}{2} > 0$. Otherwise, v is incident with at least two 5^+-faces by $(O_{2.2})$. So $w^*(v) = w(v) - \frac{1}{2} - 4 \times \frac{2}{5} + 2 \times \frac{1}{5} = \frac{3}{10} > 0$.

Case 3. G does not contain a 6-cycle with a chord. By the choice of G, we have the following observation.

$(O_{3.1})$ Every k-vertex with $k \geq 5$ is incident with at most $(k-2)$ 3-faces.

To prove the theorem, we are ready to construct a new charge $w^*(x)$ on G as follows:

($R_{3.1}$) Each 3-vertex receive $\frac{1}{2}$ from each of its 3-master.

($R_{3.2}$) From each 5^+-face f to each of its incident vertex v, transfer $\frac{w(f)}{d(f)}$.

($R_{3.3}$) From each k-vertex v, where $4 \le k \le 5$, to each of its incident 3-face f, transfer $\frac{w(v) + \frac{1}{5}n_5(v) + \frac{1}{3}n_{6^+}(v)}{k}$.

($R_{3.4}$) From each 6-vertex to its incident 3-faces, transfer $\frac{1}{2}$.

Let $\gamma(x \to y)$ denote the amount transfers from element x to element y by the above rules.

Let f be any face of G. Then $w^*(f) = w(f) = 0$ if $d(f) = 4$ and $w^*(f) = w(f) - d(f) \times \frac{w(f)}{d(f)} = 0$ if $d(f) \ge 5$. Assume that $f = [v_1v_2v_3]$ is a 3-face such that $d(v_1) \le d(v_2) \le d(v_3)$. Then $\delta(f) \ge 4$ by (c) and f is incident with at least two 5^+-vertices by (d). If $\delta(f) \ge 5$, then $\gamma(v_i \to f) \ge \min\{\frac{1}{3}, \frac{1}{2}\} = \frac{1}{3}$ by ($O_{3.1}$). So $w^*(f) \ge w(v) + 3 \times \frac{1}{3} = 0$. Otherwise, if f is incident with two 6-vertices, then $w^*(f) \ge w(v) + 2 \times \frac{1}{2} = 0$. So we only consider the following two cases.

Subcase 3.1. f is a $(4,5,5)$-face.

Without loss of generality, let $\gamma(v_2 \to f) \le \gamma(v_3 \to f)$. If $n_3(v_2) \le 2$, then $\gamma(v_2 \to f) \ge \frac{1}{2}$ and $w^*(f) \ge w(f) + 2 \times \frac{1}{2} = 0$. Otherwise, $n_3(v) = 3$ by ($O_{3.1}$). In this case, if $n_{6^+}(v_2) = 2$, then $\gamma(v_2 \to f) \ge \frac{1 + 2 \times \frac{1}{3}}{3} = \frac{5}{9}$. Thus $w^*(f) \ge w(f) + 2 \times \frac{5}{9} = \frac{1}{9} > 0$. Otherwise, it must be the case in Figure 1.

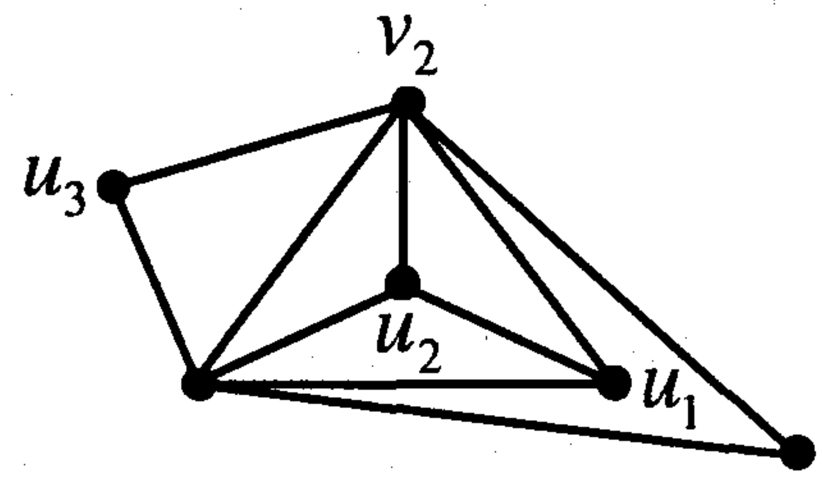

Fig.1 5-vertex incident with three 3-faces and one 4-face

In Figure 1, v_2 is incident with one 6^+-face and so $\gamma(v_2 \to f) = \dfrac{1+\frac{1}{3}}{3} = \dfrac{4}{9}$. Furthermore, u_i is incident with at least one 6^+-face by the choice of G for $i = 1, 2, 3$. So $\gamma(v_1 \to f) \geq \dfrac{\frac{1}{3}}{3} = \dfrac{1}{9}$. Thus $w^*(f) \geq w(f) + \dfrac{1}{9} + 2 \times \dfrac{4}{9} = 0$.

Subcase 3.2. f is a $(4,5,6)$-face.

In this case, $\gamma(v_3 \to f) = \dfrac{1}{2}$. If $n_3(v_2) \leq 2$, then $\gamma(v_2 \to f) \geq \dfrac{1}{2}$ and $w^*(f) \geq w(f) + 2 \times \dfrac{1}{2} = 0$. Otherwise, $n_3(v) = 3$ by $(O_{3.1})$. In this case, if $n_{6^+}(v_2) = 2$, then $\gamma(v_2 \to f) \geq \dfrac{1 + 2 \times \frac{1}{3}}{3} = \dfrac{5}{9}$. Thus $w^*(f) \geq w(f) + \dfrac{5}{9} + \dfrac{1}{2} = \dfrac{1}{18} > 0$. Otherwise, it must be the case in Fig. 1. So $\gamma(v_2 \to f) = \dfrac{1+\frac{1}{3}}{3} = \dfrac{4}{9}$ and $\gamma(v_1 \to f) \geq \dfrac{\frac{1}{3}}{3} = \dfrac{1}{9}$. Thus $w^*(f) \geq w(f) + \dfrac{1}{9} + \dfrac{4}{9} + \dfrac{1}{2} = \dfrac{1}{18} > 0$.

Let v be any vertex of G. If $d(v) = 3$, then $w^*(v) \geq w(v) + 2 \times \dfrac{1}{2} = 0$. If $d(v) = 4$ or 5, then

$$w^*(v) \geq w(v) - d(v) \times \frac{w(v) + \frac{1}{5}n_5(v) + \frac{1}{3}n_{6^+}(v)}{d(v)} + \frac{1}{5}n_5(v) + \frac{1}{3}n_{6^+}(v) = 0.$$

Let v be a 6-vertex. Then v is incident with at most four 3-faces. If v is adjacent to no 3-vertex, then $w^*(v) \geq w(v) - 4 \times \dfrac{1}{2} = 0$. Otherwise, in the evaluation of the lower bound of $w^*(v)$, it suffices to consider the case that v is 3-master of some 3-vertex u. It follows from (c) that the faces incident with u and v are 4^+-faces. Since G does not contain a 6-cycle with a chord, v is not incident with four continuous 3-faces. Thus $n_3(v) \leq 3$ and $w^*(v) \geq w(v) - \dfrac{1}{2} - 3 \times \dfrac{1}{2} = 0$.

This completes the proof of Theorem 1. □

Graph G does not contain a k-cycle with a chord implies that G does not contain a k-cycle. So we have the following corollary.

Corollary. *Let G be a planar graph. If G does not contain a k-cycle for some $k \in \{3, 4, 5, 6\}$, then G is total-$(\Delta(G) + 2)$-colorable.*

Acknowledgement

We would like to thank the referees for providing some very helpful suggestions for revising this paper.

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