J. Appl. Math. & Informatics Vol. **26**(2008), No. 3 - 4, pp. 501 - 510
Website: http://www.kcam.biz

CONVERGENCE OF THE GENERALIZED MULTISPLITTING AND TWO-STAGE MULTISPLITTING METHODS†

SEYOUNG OH, JAE HEON YUN* AND YU DU HAN

ABSTRACT. In this paper, we first provide a convergence result of the generalized two-stage splitting method for solving a linear system whose coefficient matrix is an H-matrix, and then we provide convergence results of the generalized multisplitting and two-stage multisplitting methods for both a monotone matrix and an H-matrix.

AMS Mathematics Subject Classification : 65F10, 65F15.
Key word and phrases : Generalized multisplitting method, generalized two-stage multisplitting method, monotone matrix, H-matrix.

1. Introduction

In this paper, we consider both the generalized multisplitting method and the generalized two-stage multisplitting method for solving a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. For a vector $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) denotes that all components of x are nonnegative (positive). For two vectors x, $y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. These definitions carry immediately over to matrices. A matrix A is called *monotone* if $A^{-1} \geq 0$. A matrix $A = (a_{ij})$ is called an *M-matrix* if $a_{ij} \leq 0$ for $i \neq j$ and A is monotone. The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j \end{cases}.$$

Received October 9, 2007. *Corresponding author.
†This work was supported by the research grant of the Chungbuk National University in 2007

A matrix A is called an *H-matrix* if $\langle A\rangle$ is an M-matrix. Let $\rho(A)$ denote the *spectral radius* of a square matrix A. A representation $A = M - N$ is called a *splitting* of A when M is nonsingular. A splitting $A = M - N$ is called *regular* if $M^{-1} \geq 0$ and $N \geq 0$, *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, and *convergent* if $\rho(M^{-1}N) < 1$. A splitting $A = M - N$ is called an H-splitting of A if $\langle M\rangle - |N|$ is an M-matrix, and an *H-compatible splitting* of A if $\langle A\rangle = \langle M\rangle - |N|$. $A = M - N_1 - N_2$ is a *composite splitting* of A if M is nonsingular, and $A = M - N_1 - N_2$ is a *convergent regular composite splitting* of A if both $A = M_1 - N_1$ and $M_1 = M - N_2$ are convergent regular splittings.

Let $A = M - N_1 - N_2$ be a composite splitting of A and $M = F - G$ be a splitting of M. Then, the *generalized two-stage splitting method* for solving the linear system (1) is as follows.

Algorithm 1: Generalized two-stage splitting method
Given an initial vector x_0
For $i = 1, 2, \ldots,$ until convergence
  $y_0 = x_{i-1}$
  For $j = 1$ to p
    $Fy_j = Gy_{j-1} + (N_1x_{i-1} + N_2x_i + b)$
  $x_i = y_p$

If $N_2 = 0$, then Algorithm 1 reduces to the two-stage splitting method.

A collection of quadruples (M_k, N_1^k, N_2^k, E_k), $k = 1, 2, \ldots, \ell$, is called a *generalized multisplitting* of A if $A = M_k - N_1^k - N_2^k$ is a composite splitting of A for $k = 1, 2, \ldots, \ell$, and E_k's are nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$. The *generalized multisplitting method* associated with this generalized multisplitting for solving the linear system $Ax = b$ is as follows.

Algorithm 2: Generalized multisplitting method
Given an initial vector x_0
For $i = 1, 2, \ldots,$ until convergence
  For $k = 1$ to ℓ
    $M_ky_k = N_1^kx_{i-1} + N_2^kx_i + b$
  $x_i = \sum_{k=1}^{\ell} E_ky_k$

If $N_2^k = 0$ for $k = 1, 2, \ldots, \ell$, then Algorithm 2 reduces to the multisplitting method which was first introduced by O'Leary and White [5] and was further studied by many authors [1, 4, 7].

When the linear systems in Algorithm 2 are also solved iteratively in each processor using the splittings $M_k = F_k - G_k$, one obtains the following *generalized two-stage multisplitting method.*

Algorithm 3: Generalized two-stage multisplitting method
Given an initial vector x_0

For $i = 1, 2, \ldots,$ until convergence
  For $k = 1$ to ℓ
    $y_{k,0} = x_{i-1}$
    For $j = 1$ to p
      $F_k y_{k,j} = G_k y_{k,j-1} + N_1^k x_{i-1} + N_2^k x_i + b$
  $x_i = \sum_{k=1}^{\ell} E_k y_{k,p}$

This paper is organized as follows. In Section 2, we present some preliminary results which we refer to later. In Section 3, we present a convergence result of the generalized two-stage splitting method for solving a linear system whose coefficient matrix is an H-matrix. In Section 4, we present convergence results of the generalized multisplitting and two-stage multisplitting methods for both a monotone matrix and an H-matrix.

2. Preliminaries

Varga [6] showed that for any square matrices A and B, $|A| \leq B$ implies $\rho(A) \leq \rho(B)$. Frommer and Mayer [1] showed that $|A^{-1}| \leq \langle A \rangle^{-1}$ when A is an H-matrix, and Frommer and Szyld [2] showed that if $A = M - N$ is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) \leq \rho(\langle M \rangle^{-1}|N|) < 1$.

Lemma 2.1 ([3]). *Given a nonsingular matrix A and a matrix H such that $I - H$ is nonsingular, there exists a unique pair of matrices B and C such that $A = B - C$ and $H = B^{-1}C$. Moreover, $B = A(I - H)^{-1}$ and $C = B - A$.*

In the context of Lemma 2.1, we say that H induces the unique splitting $A = B - C$.

Theorem 2.2 ([6]). *If $\rho(A) < 1$, then $I - A$ is nonsingular and $(I - A)^{-1} = \sum_{j=1}^{\infty} A^j$.*

3. Convergence of the generalized two-stage splitting method

In this section, we consider convergence of the generalized two-stage splitting method (Algorithm 1) for solving the linear system (1). Algorithm 1 using an outer splitting $A = M - N_1 - N_2$ and an inner splitting $M = F - G$ can be written as

$$x_i = (B - N_2)^{-1}(C + N_1)\,x_{i-1} + (B - N_2)^{-1}\,b, \; i = 1, 2, \ldots, \qquad (2)$$

where $B = M(I - (F^{-1}G)^p)^{-1}$ and $C = M(I - (F^{-1}G)^p)^{-1}(F^{-1}G)^p$. Here $T = (B - N_2)^{-1}(C + N_1)$ is called an iteration matrix for Algorithm 1 with p inner iterations. In equation (2), it is assumed that both $I - (F^{-1}G)^p$ and $B - N_2$ are nonsingular. From Lemma 2.1, it is easy to see that $(F^{-1}G)^p$ induces the unique splitting $M = B - C$ and $A = (B - N_2) - (C + N_1)$. For detail description of (2), see Lanzkron et al [3].

Theorem 3.1 ([3]). *Let $A = M - N_1 - N_2$ be a convergent regular composite splitting of A. If $M = F - G$ is a weak regular splitting of M, then the generalized two-stage splitting method converges for any initial vector x_0. Moreover, $A = (B - N_2) - (C + N_1)$ is a weak regular splitting of A.*

The assumption in Theorem 3.1 implies that the coefficient matrix A is monotone. The following theorem presents a convergence result for the generalized two-stage splitting method when A is an H-matrix which need not be monotone.

Theorem 3.2. *Let $A = M - N_1 - N_2$ be a composite splitting of A. Suppose that $A = M_1 - N_1$ is an H-splitting of A and $M_1 = M - N_2$ is an H-compatible splitting of M_1. If $M = F - G$ is an H-compatible splitting of M, then the generalized two-stage splitting method converges for any initial vector x_0.*

Proof. We only need to show that $\rho(T) < 1$, where $T = (B - N_2)^{-1}(C + N_1)$, and B and C are defined as in (2). Since $A = M_1 - N_1$ is an H-splitting, A and M_1 are H-matrices, and $\rho(M_1^{-1}N_1) \le \rho(\langle M_1\rangle^{-1}|N_1|) < 1$. Let $\tilde{A} = \langle M\rangle - |N_1| - |N_2|$. Since $M_1 = M - N_2$ is an H-compatible splitting of an H-matrix M_1, $\tilde{A} = \langle M_1\rangle - |N_1|$, M is an H-matrix, and $\rho(\langle M\rangle^{-1}|N_2|) < 1$. It follows that both

$$\tilde{A} = \langle M_1\rangle - |N_1| \quad \text{and} \quad \langle M_1\rangle = \langle M\rangle - |N_2|$$

are convergent regular splittings. Hence, $\tilde{A} = \langle M\rangle - |N_1| - |N_2|$ is a convergent regular composite splitting of $\tilde{A}$. Since $M = F - G$ is an H-compatible splitting of M, $\langle M\rangle = \langle F\rangle - |G|$ is a regular splitting of $\langle M\rangle$. Let $\tilde{T}$ denote an iteration matrix of the generalized two-stage splitting method with p inner iterations corresponding to splittings $\tilde{A} = \langle M\rangle - |N_1| - |N_2|$ and $\langle M\rangle = \langle F\rangle - |G|$. Then, it is easy to show that $\tilde{T} = (\tilde{B} - |N_2|)^{-1}(\tilde{C} + |N_1|)$, where $\tilde{B} = \langle M\rangle(I - \tilde{H}^p)^{-1}$, $\tilde{C} = \langle M\rangle(I - \tilde{H}^p)^{-1}\tilde{H}^p$, and $\tilde{H} = \langle F\rangle^{-1}|G|$. From Theorem 3.1,

$$\rho(\tilde{T}) < 1. \tag{3}$$

From Lemma 2.1, it can be easily seen that $\tilde{H}^p$ induces the unique splitting $\langle M\rangle = \tilde{B} - \tilde{C}$. Let $H = F^{-1}G$. Since $|H| \le \tilde{H}$ and $\rho(H) \le \rho(\tilde{H}) < 1$, by Theorem 2.2

$$\begin{aligned} |B^{-1}| &= |(I - H^p)(I - H)^{-1}F^{-1}| \\ &= \left|\sum_{s=0}^{p-1} H^s F^{-1}\right| \\ &\le \sum_{s=0}^{p-1} \tilde{H}^s \langle F\rangle^{-1} = \tilde{B}^{-1}. \end{aligned} \tag{4}$$

Since $\tilde{H}^p \ge 0$ and $\rho(\tilde{H}^p) < 1$,

$$\tilde{B}^{-1} \le (I - \tilde{H}^p)^{-1}\tilde{B}^{-1} = \langle M\rangle^{-1}. \tag{5}$$

From (4) and (5), one obtains

$$\rho\Big(B^{-1}N_2\Big) \le \rho\Big(\tilde{B}^{-1}|N_2|\Big) \le \rho\Big(\langle M\rangle^{-1}|N_2|\Big) < 1. \tag{6}$$

From equation (6) and Theorem 2.2,

$$\begin{aligned} T &= (B - N_2)^{-1}(C + N_1) \\ &= (I - B^{-1}N_2)^{-1}(B^{-1}C + B^{-1}N_1) \\ &= \sum_{j=0}^{\infty}(B^{-1}N_2)^j(H^p + B^{-1}N_1). \end{aligned} \tag{7}$$

From (4), (6) and (7),

$$\begin{aligned} |T| &= \left|\sum_{j=0}^{\infty}(B^{-1}N_2)^j(H^p + B^{-1}N_1)\right| \\ &\le \sum_{j=0}^{\infty}(\tilde{B}^{-1}|N_2|)^j(\tilde{H}^p + \tilde{B}^{-1}|N_1)| \\ &= \sum_{j=0}^{\infty}(\tilde{B}^{-1}|N_2|)^j\tilde{B}^{-1}(\tilde{C} + |N_1|) \\ &= (I - \tilde{B}^{-1}|N_2|)^{-1}\tilde{B}^{-1}(\tilde{C} + |N_1|) \\ &= (\tilde{B} - |N_2|)^{-1}(\tilde{C} + |N_1|) = \tilde{T}. \end{aligned} \tag{8}$$

From (3) and (8), $\rho(T) < 1$ which completes the proof. □

Corollary 3.3. *Let $A = M - N_1 - N_2$ be a composite splitting of an H-matrix A. Suppose that $A = M_1 - N_1$ is an H-compatible splitting of A and $M_1 = M - N_2$ is an H-compatible splitting of M_1. If $M = F - G$ is an H-compatible splitting of M, then the generalized two-stage splitting method converges for any initial vector x_0.*

Proof. Since A is an H-matrix and $A = M_1 - N_1$ is an H-compatible splitting of A, $A = M_1 - N_1$ is an H-splitting of A. Hence, the proof is complete from Theorem 3.2. □

Now we provide a typical example of the generalized two-stage splitting method for solving the linear system (1).

Example 3.4. Suppose that a matrix A is partitioned into $q \times q$ blocks, i.e.,

$$A = \begin{pmatrix} D_1 & -U_{12} & \cdots & -U_{1q} \\ -L_{21} & D_2 & \cdots & -U_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ -L_{q1} & -L_{q2} & \cdots & D_q \end{pmatrix}.$$

Let $A = D - U - L$ be a composite splitting of A, where D is a block diagonal matrix, U is a strictly block upper triangular matrix and L is a strictly block lower triangular matrix, i.e.,

$$D = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_q \end{pmatrix}, L = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ L_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{q1} & L_{q2} & \cdots & 0 \end{pmatrix}, U = \begin{pmatrix} 0 & U_{12} & \cdots & U_{1q} \\ 0 & 0 & \cdots & U_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $D = F - G$ be a splitting of D such that each diagonal block D_i has a splitting $D_i = F_i - G_i$, i.e.,

$$F = \begin{pmatrix} F_1 & 0 & \cdots & 0 \\ 0 & F_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_q \end{pmatrix}, \quad G = \begin{pmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_q \end{pmatrix}.$$

Then, the generalized two-stage splitting method corresponding to this splitting for solving $Ax = b$ is as follows.

Given an initial vector x_0
For $i = 1, 2, \ldots,$ until convergence
  $y_0 = x_{i-1}$
  For $k = 1$ to q
    For $j = 1$ to p
      $F_k y_j^{(k)} = G_k y_{j-1}^{(k)} + (Ux_{i-1} + Lx_i + b)^{(k)}$
  $x_i^{(k)} = y_p^{(k)}$

In this algorithm, the superscript (k) of a vector stands for the kth block of the vector. Notice that x_i at each iteration i is computed one block at a time. Since L is a strictly block lower triangular matrix, $(Lx_i)^{(k)}$ can be computed explicitly.

4. Convergence of the generalized multisplitting and two-stage multisplitting methods

In this section, we consider convergence of both the generalized multisplitting method (Algorithm 2) and the generalized two-stage multisplitting method (Algorithm 3) for solving the linear system (1). Algorithm 3 using outer splittings $A = M_k - N_1^k - N_2^k$ and inner splittings $M_k = F_k - G_k$ can be written as

$$x_i = \sum_{k=1}^{\ell} E_k T_k \, x_{i-1} + \sum_{k=1}^{\ell} (B_k - N_2^k)^{-1} \, b, \; i = 1, 2, \ldots, \tag{9}$$

where $T_k = (B_k - N_2^k)^{-1}(C_k + N_1^k)$, $B_k = M_k(I - (F_k^{-1}G_k)^p)^{-1}$ and $C_k = M_k(I - (F_k^{-1}G_k)^p)^{-1}(F_k^{-1}G_k)^p$. Here $S = \sum_{k=1}^{\ell} E_k T_k$ is called an iteration matrix for Algorithm 3 with p inner iterations. In equation (9), it is assumed that $I - (F_k^{-1}G_k)^p$ and $B_k - N_2^k$ are nonsingular for $k = 1, 2, \ldots, \ell$.

Theorem 4.1 ([5]). *Let A be a monotone matrix and (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, be a multisplitting of A. If $A = M_k - N_k$ is a weak regular splitting of A for $k = 1, 2, \ldots, \ell$, then the multisplitting method converges for any initial vector x_0.*

Theorem 4.2. *Let $\left(M_k, N_1^k, N_2^k, E_k\right)$, $k = 1, 2, \ldots, \ell$, be a generalized multisplitting of A. If $A = M_k - N_1^k - N_2^k$ is a convergent regular composite splitting of A and $M_k = F_k - G_k$ is a weak regular splitting of M_k for $k = 1, 2, \ldots, \ell$, then the generalized two-stage multisplitting method converges for any initial vector x_0.*

Proof. It is easy to see that A is monotone. Let $S = \sum_{k=1}^{\ell} E_k T_k$ be an iteration matrix of the generalized two-stage multisplitting method, where each T_k is defined as in (9). Notice that each T_k is an iteration matrix of the generalized two-stage splitting method corresponding to an outer splitting $A = M_k - N_1^k - N_2^k$ and an inner splitting $M_k = F_k - G_k$. From Theorem 3.1, $A = \left(B_k - N_2^k) - (C_k + N_1^k\right)$ is a weak regular splitting of A for each $1 \le k \le \ell$. Since S can be an iteration matrix of the multisplitting method using splittings $A = (B_k - N_2^k) - (C_k + N_1^k)$, from Theorem 4.1 $\rho(S) < 1$ which completes the proof. □

Theorem 4.3. *Let $\left(M_k, N_1^k, N_2^k, E_k\right)$, $k = 1, 2, \ldots, \ell$, be a generalized multisplitting of an H-matrix A. Suppose that $A = M_1^k - N_1^k$ is an H-compatible splitting of A and $M_1^k = M_k - N_2^k$ is an H-compatible splitting of M_1^k for every $1 \le k \le \ell$. If $M_k = F_k - G_k$ is an H-compatible splitting of M_k for $k = 1, 2, \ldots, \ell$, then the generalized two-stage multisplitting method converges for any initial vector x_0.*

Proof. Let $S = \displaystyle\sum_{k=1}^{\ell} E_k T_k$ be an iteration matrix of the generalized two-stage multisplitting method, where each T_k is defined as in (9). Since A is an H-matrix, the H-compatible splittings $A = M_1^k - N_1^k$ and $M_1^k = M_k - N_2^k$ are also H-splittings. Hence from the process of the proof of Theorem 3.2, $\langle A \rangle = \langle M_k \rangle - |N_1^k| - |N_2^k|$ is a convergent regular composite splitting of $\langle A \rangle$ and $\langle M_k \rangle = \langle F_k \rangle - |G_k|$ is a regular splitting of $\langle M_k \rangle$. For each $1 \le k \le \ell$, let $\tilde{T}_k$ be an iteration matrix of the generalized two-stage splitting method using an outer splitting $\langle A \rangle = \langle M_k \rangle - |N_1^k| - |N_2^k|$ and an inner splitting $\langle M_k \rangle = \langle F_k \rangle - |G_k|$. Then, $\tilde{S} = \displaystyle\sum_{k=1}^{\ell} E_k \tilde{T}_k$ is an iteration matrix of the generalized two-stage multisplitting method using outer splittings $\langle A \rangle = \langle M_k \rangle - |N_1^k| - |N_2^k|$ and inner splittings $\langle M_k \rangle = \langle F_k \rangle - |G_k|$. From Theorem 4.2 and the process of the proof of Theorem 3.2,

$$\rho(\tilde{S}) < 1 \text{ and } |T_k| \le \tilde{T}_k. \tag{10}$$

From (10), $|S| \le \tilde{S}$ and hence $\rho(S) < 1$ which completes the proof. □

Since the generalized multisplitting method can be viewed as the generalized two-stage multisplitting method with $p = 1$ and the inner splittings $M_k = M_k - 0$, Corollaries 4.4 and 4.5 are obtained from Theorems 4.2 and 4.3, respectively.

Corollary 4.4. *Let $\left(M_k, N_1^k, N_2^k, E_k\right)$, $k = 1, 2, \ldots, \ell$, be a generalized multisplitting of A. If $A = M_k - N_1^k - N_2^k$ is a convergent regular composite splitting of A, then the generalized multisplitting method converges for any initial vector x_0.*

Corollary 4.5. *Let $\left(M_k, N_1^k, N_2^k, E_k\right)$, $k = 1, 2, \ldots, \ell$, be a generalized multisplitting of an H-matrix A. If $A = M_1^k - N_1^k$ is an H-compatible splitting of A and $M_1^k = M_k - N_2^k$ is an H-compatible splitting of M_1^k for every $1 \leq k \leq \ell$, then the generalized multisplitting method converges for any initial vector x_0.*

Now we provide an example of the generalized multisplitting method for solving the linear system (1).

Example 4.6. For simplicity, suppose that $\ell = 2$, i.e., the number of processors to be used is 2. Then A is partitioned into $2\ell \times 2\ell$ blocks, i.e.,

$$A = \begin{pmatrix} A_{11} & -A_{12} & -A_{13} & -A_{14} \\ -A_{21} & A_{22} & -A_{23} & -A_{24} \\ -A_{31} & -A_{32} & A_{33} & -A_{34} \\ -A_{41} & -A_{42} & -A_{43} & A_{44} \end{pmatrix},$$

where A_{ii}'s are assumed to be nonsingular matrices. Let

$$M = \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & 0 & A_{44} \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} \\ 0 & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & 0 & 0 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{43} & 0 \end{pmatrix}.$$

Then $A = M - N_1 - N_2$ is clearly a composite splitting of A. Let $M_k = M$, $N_1^k = N_1$, and $N_2^k = N_2$ for $k = 1, 2$. Let

$$E_1 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Then (M_k, N_1^k, N_2^k, E_k), $k = 1, 2$, is a generalized multisplitting of A. The generalized multisplitting method associated with this generalized multisplitting for solving $Ax = b$ can be written as

Given an initial vector x_0
For $i = 1, 2, \ldots,$ until convergence
For $k = 1$ to $\ell = 2$
$My_k = N_1 x_{i-1} + N_2 x_i + b$
$x_i = \sum_{k=1}^{\ell} E_k y_k$

In this algorithm, notice that x_i is required in the computation of the right-hand side of equation. Nevertheless, since M is a block diagonal matrix and N_2 is a special strictly block lower triangular matrix, $x_i = \sum_{k=1}^{\ell} E_k y_k$ can be computed completely in parallel among the ℓ processors as follows:

$$\begin{pmatrix} x_i^{(1)} \\ x_i^{(2)} \end{pmatrix} = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix} \left(\begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} \\ 0 & 0 & A_{23} & A_{24} \end{pmatrix} x_{i-1} + \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} \begin{pmatrix} x_i^{(1)} \\ x_i^{(2)} \end{pmatrix} + \begin{pmatrix} b^{(1)} \\ b^{(2)} \end{pmatrix} \right),$$

$$\begin{pmatrix} x_i^{(3)} \\ x_i^{(4)} \end{pmatrix} = \begin{pmatrix} A_{33}^{-1} & 0 \\ 0 & A_{44}^{-1} \end{pmatrix} \left(\begin{pmatrix} A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & 0 & 0 \end{pmatrix} x_{i-1} + \begin{pmatrix} 0 & 0 \\ A_{43} & 0 \end{pmatrix} \begin{pmatrix} x_i^{(3)} \\ x_i^{(4)} \end{pmatrix} + \begin{pmatrix} b^{(3)} \\ b^{(4)} \end{pmatrix} \right).$$

In this equation, the superscript (k) of a vector stands for the kth block of the vector.

REFERENCES

1. A. Frommer and G. Mayer, Convergence of relaxed parallel multisplitting methods. Linear Algebra Appl. 119 (1989), 141–152.
2. A. Frommer and D.B. Szyld, H-splittings and two-stage iterative methods. Numer. Math. 63 (1992), 345–356.
3. P.J. Lanzkron, R.J. Rose and D.B. Szyld, Convergence of nested classical iterative methods for linear systems. Numer. Math. 58 (1991), 685–702.
4. M. Neumann and R.J. Plemmons, Convergence of parallel multisplitting methods for M-matrices. Linear Algebra Appl. 88-89 (1987), 559–574.
5. D.P. O'Leary and R.E. White, Multisplittings of matrices and parallel solution of linear systems. SIAM J. Algebraic Discrete Meth. 6 (1985), 630–640.
6. R.S. Varga, Matrix iterative analysis, Prentice-Hall, Englewood Cliffs, 1962.
7. R.E. White, Multisplitting of a symmetric positive definite matrix. SIAM J. Matrix Anal. Appl. 11 (1990), 69–82.

Jae Heon Yun received M.Sc. from Kyungpook National University, and Ph.D. from Iowa State University. He is currently a professor at Chungbuk National University since 1991. His research interests are computational mathematics, iterative method and parallel computation.

Department of Mathematics, Institute for Basic Sciences & College of Natural Sciences, Chungbuk National University, Cheongju 361-763, Korea
e-mail: gmjae@chungbuk.ac.kr

SeYoung Oh received M.Sc. from Seoul National University and Ph.D at University of Minnesota. Since 1992 he has been at Chungnam National University. His research interests include numerical optimization and biological computation.

Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea
e-mail: soh@cnu.ac.kr

Yu Du Han received M.Sc. in Applied Mathematics from Chungbuk National University. She is currently a temporary instructor at Chungbuk National University.

Department of Mathematics, College of Natural Sciences, Chungbuk National University, Cheongju 361-763, Korea