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SOME ITERATIVE ALGORITHMS FOR THE GENERALIZED MIXED EQUILIBRIUM-LIKE PROBLEMS

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ABSTRACT. In this paper, we introduce and analyze a new class of generalized mixed equilibrium-like problems. By using the auxiliary principle technique, we suggest three iterative algorithms for the generalized mixed equilibrium-like problem. Under certain conditions, we establish the convergence of the iterative algorithms. Our results extend, improve and unify several known results in this field.

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1. Introduction

Equilibrium problems theory provide us a natural, novel and unified framework to study a wide class of problems arising in economic, finance, transportation, network and structural analysis, elastic and optimization. It is well known that equilibrium problems include variational inequalities and related optimization problem as special cases [1-14].

In 2002, Moudafi [12] studied the sensitivity and algorithm for a class of mixed equilibrium problems. In 2004 and 2005, Ding [5, 6] used the auxiliary principle technique to suggest several predictor-corrector iterative algorithms for a few classes of generalized and general mixed variational inequality problems and generalized mixed implicit equilibrium-like problems.

Inspired and motivated by the recent results in [3, 5, 8, 10, 14], in this paper, we introduce a new class of generalized mixed implicit equilibrium-like problems, which include the generalized mixed equilibrium-like problem, the generalized mixed variational-like inequality problem, the generalized mixed equilibrium problem and the mixed variational inequality problem in [3, 5, 8, 10, 14]

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as special cases. By applying the auxiliary principle technique, three iterative algorithms for solving the generalized mixed equilibrium-like problems are suggested and analyzed. The convergence of the iterative sequences generated by the algorithms are also investigated. Our results improve and generalize many known results in [3-14].

2. Preliminaries

Let H be a real Hilbert space endowed with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$,respectively. Let K be a nonempty convex subset of H. Let $T, A, B : H \to H$ and $\eta : H \times H \to H$ be nonlinear continuous mappings. Let $F : H \times H \times H \to (-\infty, +\infty]$, φ and $a : H \times H \to (-\infty, +\infty]$ be functionals. Now we consider the following generalized mixed equilibrium-like problem (in short, GMELP):

Find $u \in H$ such that

$$F(Tu, Au, v) + \langle Bu, \eta(v,u)\rangle + \varphi(v,u) - \varphi(u,u) + a(u, v-u) \geq 0, \quad \forall v \in K. \quad (2.1)$$

Special cases.

(A) If $\varphi(v,u) = f(v)$, $a(u,v) = 0$ for all $u, v \in H$, then the GMELP(2.1) reduces to the following mixed equilibrium-like problem: find $u \in H$ such that

$$F(Tu, Au, v) + \langle Bu, \eta(v,u)\rangle \geq 0, \quad \forall v \in K,$$

which is new and includes previously known *equilibrium problems* as special cases.

(B) If $Bu = a(u,v) = 0$ for all $u, v \in H$, then the GMELP(2.1) collapses to the following problem: find $u \in H$ such that

$$F(Tu, Au, v) + \varphi(v,u) - \varphi(u,u) \geq 0, \quad \forall v \in K,$$

which is called a *mixed equilibrium-like problem* and appears to be a new one.

(C) If $A = I$, where I is an identity mapping, and $F(Tu, u, v) = \langle Tu, gv - gu\rangle$, $\eta(u,v) = gu - gv$, $a(u,v) = 0$, $\varphi(v,u) = f(v)$ for all $u, v \in H$, then the GMELP(2.1) reduces to the following mixed variational-like inequality problem: find $u \in H$ such that

$$\langle Tu - Bu, gv - gu\rangle + f(v) - f(v) \geq 0, \quad \forall v \in K,$$

which is known as the *generalized variational inequality* and studied by Yao [14].

(D) If $F(Tu, Au, v) = 0$, $\eta(u,v) = v - u$, $\varphi(v,u) = f(v)$ and $a(u,v) = 0$ for all $u, v \in H$, then the GMELP(2.1) collapses to the following problem: find $u \in H$ such that

$$\langle Bu, v - u\rangle + f(v) - f(u) \geq 0, \quad \forall v \in K,$$

which was introduced and studied by Cohen [3].

For a suitable and appropriate choice of $F, \eta, T, A, B, \varphi$ and a, one can obtain various classes of equilibrium and variational inequality problems as special cases of the GMELP(2.1).

We need the following concepts.

Definition 2.1. Let $F : H \times H \times H \to (-\infty, +\infty]$, φ and $a : H \times H \to (-\infty, +\infty]$ be three functionals. Let $T, A, B : H \to H$ and $\eta : H \times H \to H$ be nonlinear continuous mappings.

(1) F is said to be *partially relaxed strongly monotone* with respect to T and A if there exists a constant $r > 0$ such that

$$F(Tu, Au, v) + F(Tv, Av, z) \leq r\|z - u\|, \quad \forall u, v, z \in H;$$

(2) F is said to be *mixed pseudmonotone* with respect to T, A and B if

$$\begin{aligned} &F(Tu, Au, v) + \langle Bu, \eta(v, u)\rangle + \varphi(v, u) - \varphi(u, u) + a(u, v - u) \geq 0 \\ &\Rightarrow -F(Tv, Av, u) - \langle Bv, \eta(u, v)\rangle + \varphi(v, u) \\ &\quad - \varphi(u, u) + a(u, v - u) \geq 0, \quad \forall u, v \in H; \end{aligned}$$

(3) B is said to be *partially relaxed η-strongly monotone* if there exists a constant $s > 0$ such that

$$\langle Bu - Bv, \eta(w, v)\rangle \geq -s\|w - u\|^2, \quad u, v, w \in H;$$

(4) φ is said to be *skew-symmetric* if

$$\varphi(u, u) + \varphi(v, v) - \varphi(u, v) - \varphi(v, u) \geq 0, \quad \forall u, v \in H;$$

(5) a is said to be a *coercive continuous bilinear form* if there exist $c > 0$ and $d > 0$ such that

$$a(v, v) \geq c\|v\|^2 \quad \text{and} \quad a(u, v) \leq d\|u\|\|v\|, \quad \forall u, v \in H.$$

Definition 2.2. ([1, 8]) Let K be a nonempty convex subset of a Hilbert H and $E : K \to \mathbb{R}$ be a Fréchet differentiable. E is said to be

(1) *η-convex* if

$$E(v) - E(u) \geq \langle E'(u), \eta(v, u)\rangle, \forall u, v \in K;$$

(2) *η-strongly convex* if there exists a constant $\alpha > 0$ such that

$$E(v) - E(u) - \langle E'(u), \eta(v, u)\rangle \geq \alpha\|u - v\|^2, \quad \forall u, v \in K.$$

3. Iterative algorithms and convergence theorems

In this section, we introduce and analyze some new iterative algorithms for the GMELP(2.1) by using the auxiliary principe technique. For a given $u \in$

H, consider the problem of finding a solution $w \in H$ satisfying the auxiliary problem:

$$\begin{aligned}&\langle w-u, v-w\rangle + \rho F(Tu, Au, v) + \rho\langle Bu, \eta(v,u)\rangle \\ &\quad + \rho\varphi(v,u) - \rho\varphi(u,u) + \rho a(u, v-u) \geq 0, \quad \forall v \in K,\end{aligned} \tag{3.1}$$

where $\rho > 0$ is a constant. We notice that if $w = u$, then clearly, w is a solution of the GMELP(2.1).

Now we suggest the following new predictor-corrector iterative algorithm for the GMELP(2.1).

Algorithm 3.1. *Let $F : H \times H \times H \to (-\infty, +\infty]$, φ and $a : H \times H \to (-\infty, +\infty]$ be three functionals. Let $T, A, B : H \to H$ and $\eta : H \times H \to H$ be nonlinear continuous mappings. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the following iterative schemes:*

$$\begin{aligned}&\langle u_{n+1} - w_n, v - u_{n+1}\rangle + \rho F(Tw_n, Aw_n, v) \\ &\quad + \rho\langle Bw_n, \eta(v, u_{n+1})\rangle + \rho\varphi(v, u_{n+1}) - \rho\varphi(u_{n+1}, u_{n+1}) \\ &\quad + \rho a(u_{n+1}, v - u_{n+1}) \geq 0, \quad \forall v \in K,\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\langle w_n - y_n, v - w_n\rangle + \beta F(Ty_n, Ay_n, v) \\ &\quad + \beta\langle By_n, \eta(v, w_n)\rangle + \beta\varphi(v, w_n) - \beta\varphi(w_n, w_n) \\ &\quad + \beta a(w_n, v - w_n) \geq 0, \quad \forall v \in K,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\langle y_n - u_n, v - y_n\rangle + \mu F(Tu_n, Au_n, v) \\ &\quad + \mu\langle Bu_n, \eta(v, y_n)\rangle + \mu\varphi(v, y_n) - \mu\varphi(y_n, y_n) \\ &\quad + \mu a(y_n, v - y_n) \geq 0, \quad \forall v \in K,\end{aligned} \tag{3.4}$$

For the convergence of Algorithm 3.1, we have the following result.

Theorem 3.1. *Let H be a real Hilbert space and K be a nonempty convex subset of H. Let $T, A, B : H \to H$ and $\eta : H \times H \to H$ be nonlinear continuous mappings, where $\eta(x,y) = -\eta(y,x)$, $\forall x, y \in H$. Assume that $F : H \times H \times H \to (-\infty, +\infty]$, $\varphi : H \times H \to (-\infty, +\infty]$ and $a : H \times H \to (-\infty, +\infty]$ are continuous functionals. Assume that F is partially relaxed strongly monotone with respect to T and A with constant $r > 0$, B is partially relaxed η-strongly monotone with constant $s > 0$, φ is skew-symmetric and a is a coercive continuous bilinear form. If $u \in H$ is an exact solution of the GMELP(2.1), then for any $u_0 \in K$,* *the iterative sequence* $\{u_n\}_{n\geq 0}$ *generated by Algorithm 3.1 converges strongly to* u.

Proof. Let $u \in H$ be an exact solution of the GMELP(2.1). It follows that

$$\begin{aligned}&\rho F(Tu, Au, v) + \rho\langle Bu, \eta(v, u)\rangle + \rho\varphi(v, u)\\&\quad - \rho\varphi(u, u) + \rho a(u, v - u) \geq 0, \quad \forall v \in K,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\beta F(Tu, Au, v) + \beta\langle Bu, \eta(v, u)\rangle + \beta\varphi(v, u)\\&\quad - \beta\varphi(u, u) + \beta a(u, v - u) \geq 0, \quad \forall v \in K,\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\mu F(Tu, Au, v) + \mu\langle Bu, \eta(v, u)\rangle + \mu\varphi(v, u)\\&\quad - \mu\varphi(u, u) + \mu a(u, v - u) \geq 0, \quad \forall v \in K.\end{aligned} \tag{3.7}$$

Let n be a nonnegative integer. Taking $v = u_{n+1}$ in (3.5) and $v = u$ in (3.2), we infer that

$$\begin{aligned}&\rho F(Tu, Au, u_{n+1}) + \rho\langle Bu, \eta(u_{n+1}, u)\rangle + \rho\varphi(u_{n+1}, u)\\&\quad - \rho\varphi(u, u) + \rho a(u, u_{n+1} - u) \geq 0\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\langle u_{n+1} - w_n, u - u_{n+1}\rangle + \rho F(Tw_n, Aw_n, u) + \rho\langle Bw_n, \eta(u, u_{n+1})\rangle\\&\quad + \rho\varphi(u, u_{n+1}) - \rho\varphi(u_{n+1}, u_{n+1}) + \rho a(u_{n+1}, u - u_{n+1}) \geq 0.\end{aligned} \tag{3.9}$$

Adding (3.8) and (3.9), we obtain that

$$\begin{aligned}&\langle u_{n+1} - w_n, u - u_{n+1}\rangle\\&\geq -\rho[F(Tw_n, Aw_n, u) + F(Tu, Au, u_{n+1})]\\&\quad - \rho[\langle Bw_n, \eta(u, u_{n+1}) + \langle Bu, \eta(u_{n+1}, u)\rangle] + \rho a(u - u_{n+1}, u - u_{n+1})\\&\geq -\rho r\|u_{n+1} - w_n\|^2 - \rho s\|u_{n+1} - w_n\|^2 + \rho c\|u - u_{n+1}\|^2,\end{aligned} \tag{3.10}$$

where we have used the fact that F is partially relaxed strongly monotone mappings with constant $r > 0$ and and B is partially relaxed η-monotone mappings with constant $s > 0$, respectively, φ is skew-symmetric and a is a coercive continuous bilinear form. Note that

$$\|u - w_n\|^2 = \|u - u_{n+1}\|^2 + \|u_{n+1} - w_n\|^2 + 2\langle u_{n+1} - w_n, u - u_{n+1}\rangle. \tag{3.11}$$

In light of (3.10) and (3.11), we have

$$\begin{aligned}\|u_{n+1} - u\|^2 &\leq \frac{1}{1 + 2\rho c}\|w_n - u\|^2 - \frac{1 - 2\rho(s + r)}{1 + 2\rho c}\|u_{n+1} - w_n\|^2\\&\leq \frac{1}{1 + 2\rho c}\|w_n - u\|^2.\end{aligned} \tag{3.12}$$

Taking $v = w_n$ in (3.6) and $v = u$ in (3.3), we deduce that

$$\begin{aligned}&\beta F(Tu, Au, w_n) + \beta\langle Bu, \eta(w_n, u)\rangle + \beta\varphi(w_n, u)\\&\quad - \beta\varphi(u, u) + \beta a(u, w_n - u) \geq 0\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}&\langle w_n - y_n, u - w_n\rangle + \beta F(Ty_n, Ay_n, u) + \beta\langle By_n, \eta(u, w_n)\rangle\\&\quad + \beta\varphi(u_{n+1}, w_n) - \beta\varphi(w_n, w_n) + \beta a(w_n, u - w_n) \geq 0.\end{aligned} \tag{3.14}$$

Adding (3.13) and (3.14), we infer that

$$\begin{aligned}&\langle w_n - u_n, u - w_n\rangle\\&\geq -\beta[F(Ty_n, Ay_n, u) + F(Tu, Au, w_n)]\\&\quad - \beta[\langle By_n, \eta(u, w_n)\rangle + \langle Bu, \eta(w_n, u)\rangle] + \beta a(u - w_n, u - w_n)\\&\geq -\beta(s + r)\|u_{n+1} - w_n\|^2 + \beta c\|u - u_{n+1}\|^2\end{aligned} \tag{3.15}$$

because F is partially relaxed strongly monotone mappings with constant $r > 0$ and and B is partially relaxed η-monotone mappings with constant $s > 0$, respectively, φ is skew-symmetric and a is a coercive continuous bilinear form. In terms of (3.15) and

$$\|u - y_n\|^2 = \|w_n - u\|^2 + \|w_n - y_n\|^2 + 2\langle w_n - y_n, u - w_n\rangle, \tag{3.16}$$

we obtain that

$$\begin{aligned}\|w_n - u\|^2 &\leq \frac{1}{1 + 2\beta c}\|y_n - u\|^2 - \frac{1 - 2\beta(s + r)}{1 + 2\beta c}\|y_n - w_n\|^2\\&\leq \frac{1}{1 + 2\beta c}\|y_n - u\|^2.\end{aligned} \tag{3.17}$$

Similarly, taking $v = y_n$ in (3.7) and $v = u$ in (3.4), and using the partially relaxed strong monotonicity of F, and partially relaxed η-monotonicity of B, we conclude that

$$\langle y_n - u_n, u - y_n\rangle \geq -\mu(r + s)\|y_n - u_n\|^2 + \mu c\|u - y_n\|^2.$$

As in the proof of (3.12) and (3.17), we have

$$\begin{aligned}\|y_n - u\|^2 &\leq \frac{1}{1 + 2\mu c}\|u_n - u\|^2 - \frac{1 - 2\mu(s + r)}{1 + 2\mu c}\|u_n - y_n\|^2\\&\leq \frac{1}{1 + 2\mu c}\|u_n - u\|^2.\end{aligned} \tag{3.18}$$

It follows from (3.12) (3.17) and (3.18) that

$$\|u_{n+1} - u\| \leq t\|u_n - u\|, \tag{3.19}$$

where $t = \dfrac{1}{\sqrt{(1+2\rho c)(1+2\beta c)(1+2\mu c)}} \in (0,1)$. It follows from (3.19) that

$$\|u_{n+1} - u\| \leq t^{n+1}\|u_0 - u\|, \quad \forall n \geq 0,$$

that is, $\{u_n\}_{n\geq 0}$ converges strongly to the solution u of the GMELP(2.1). This completes the proof. □

Now, we suggest and analyze a new iterative method for the GMELP(2.1) by using the auxiliary principle technique. For a given $u \in K$, consider the problem of finding $w \in K$ satisfying the auxiliary generalized mixed equilibrium-like problem

$$\begin{aligned}&\langle E'(w) - E'(u) + \lambda Bu, \eta(v,w)\rangle + \lambda F(Tu, Au, v)\\ &\quad \geq \lambda[\varphi(w,w) - \varphi(v,w)] - \lambda a(w, v-w), \quad \forall v \in K,\end{aligned}$$

where $\lambda > 0$ is a constant and E' is a differential of a strongly convex functional E. We note that if $w = u$, then w is a solution of the GMELP(2.1). This observation enables us to suggest the following iterative method for solving the GMELP(2.1).

Algorithm 3.2. *Let $F : H \times H \times H \to (-\infty, +\infty]$, φ and $a : H \times H \to (-\infty, +\infty]$ be three functionals. Let $T, A, B : H \to H$ and $\eta : H \times H \to H$ be nonlinear continuous mappings. For a given $u_0 \in K$, compute the approximate solution $u_{n+1} \in K$ by the following iterative scheme:*

$$\begin{aligned}&\langle E'(u_{n+1}) - E'(u_n) + \lambda Bu_n, \eta(v, u_{n+1})\rangle + \lambda F(Tu_n, Au_n, v)\\ &\quad \geq \lambda[\varphi(u_{n+1}, u_{n+1}) - \varphi(v, u_{n+1})]\\ &\quad - \lambda a(u_{n+1}, v - u_{n+1}), \quad \forall v \in K,\ n \geq 0,\end{aligned} \tag{3.20}$$

where $\lambda > 0$ is a constant.

Theorem 3.2. *Let $H, K, T, A, B, F, \varphi$ and a be as in Theorem 3.1. Let $\eta : H \times H \to H$ satisfy that $\eta(x,y) = \eta(x,z) + \eta(z,y), \forall x, y, z \in H$. Assume that $E : K \to \mathbb{R}$ is a Fréchet differentiable and η-strongly convex with constant $l > 0$. If $\lambda \in \left(0, \dfrac{l}{s+r}\right)$ and $u \in K$ is an exact solution of the GMELP(2.1), then the sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.2 converges strongly to u.*

Proof. Let $u \in K$ be a solution of the GMELP(2.1). Then

$$\begin{aligned}&\lambda F(Tu, Au, v) + \lambda\langle Bu, \eta(v,u)\rangle\\ &\quad \geq \lambda\varphi(u,u) - \lambda\varphi(v,u) - \lambda a(u, v-u), \quad \forall v \in K,\end{aligned} \tag{3.21}$$

where $\lambda > 0$ is a constant. Now taking $v = u_{n+1}$ in (3.21) and $v = u$ in (3.20), we have

$$\begin{aligned} &\lambda F(Tu, Au, u_{n+1}) + \lambda\langle Bu, \eta(u_{n+1}, u)\rangle \\ &\quad \geq \lambda\varphi(u, u) - \lambda\varphi(u_{n+1}, u) - \lambda a(u, u_{n+1} - u), \quad \forall n \geq 0 \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} &\Big\langle E'(u_{n+1}) - E'(u_n) + \lambda Bu_n, \eta(u, u_{n+1})\Big\rangle + \lambda F(Tu_n, Au_n, u) \\ &\quad \geq \lambda\Big[\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\Big] - \lambda a(u_{n+1}, u - u_{n+1}), \quad \forall n \geq 0. \end{aligned} \tag{3.23}$$

Put

$$G(u, w) = E(u) - E(w) - \langle E'(w), \eta(u, w)\rangle, \quad \forall w \in K.$$

Using η-strong convexity of E, by (3.22) and (3.23) we deduce

$$\begin{aligned} &G(u, u_n) - G(u, u_{n+1}) \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), \eta(u_{n+1}, u_n)\rangle \\ &\quad + \Big\langle E'(u_{n+1}) - E'(u_n), \eta(u, u_{n+1})\Big\rangle \\ &\geq l\|u_{n+1} - u_n\|^2 - \lambda\Big[F(Tu_n, Au_n, u) + F(Tu, Au, u_{n+1})\Big] \\ &\quad - \lambda\Big\langle Bu - Bu_n, \eta(u_{n+1}, u)\Big\rangle + \lambda c\|u - u_{n+1}\|^2 \\ &\geq [l - \lambda(s + r)]\|u_{n+1} - u_n\|^2 + \lambda c\|u - u_{n+1}\|^2 \\ &\geq \lambda c\|u - u_{n+1}\|^2. \end{aligned} \tag{3.24}$$

Suppose that $u_{n_0+1} = u_{n_0}$ for some $n_0 \geq 0$. It follows from (3.24) that $u_n = u$ for all $n \geq n_0$. Suppose that $u_{n+1} \neq u_n$ for any $n \geq 0$. Thus (3.24) ensures that the sequence $\{G(u, u_n)\}_{n\geq 0}$ is strictly decreasing. It follows from η-strong convexity of E that $G(u, \cdot)$ is nonnegative in K. Consequently, the sequence $\{G(u, u_n)\}_{n\geq 0}$ converges to some number. It follows from (3.24) that the sequence $\{u_n\}_{n\geq 0}$ converges to the solution u of the GMELP(2.1). This completes the proof. □

We now use the auxiliary principle technique to suggest a proximal method for the GMELP(2.1), and prove that the convergence of the proximal method requires only pseudomonotonicity, which is a weaker condition than monotonicity. For a given $u \in K$, consider the auxiliary problem of finding $w \in K$ such that

$$\begin{aligned} &\Big\langle E'(w) - E'(u) + \gamma Bw, \eta(v, w)\Big\rangle + \gamma F(Tw, Aw, v) \\ &\quad \geq \gamma[\varphi(w, w) - \varphi(v, w)] - \gamma a(w, v - w), \quad \forall v \in K, \end{aligned}$$

where $\gamma > 0$ is a constant and E' is the differential of a strongly convex functional E.

Algorithm 3.3. *Let $F : H \times H \times H \to (-\infty, +\infty]$, φ and $a : H \times H \to (-\infty, +\infty]$ be three functionals. Let $T, A, B : H \to H$ and $\eta : H \times H \to H$ be nonlinear continuous mappings. For a given $u_0 \in K$, compute the approximate solution $u_{n+1} \in K$ by the following iterative scheme:*

$$\begin{aligned}&\Big\langle E'(u_{n+1}) - E'(u_n) + \gamma B u_{n+1}, \eta(v, u_{n+1})\Big\rangle + \gamma F(Tu_{n+1}, Au_{n+1}, v)\\ &\quad \geq \gamma\Big[\varphi(u_{n+1}, u_{n+1}) - \varphi(v, u_{n+1})\Big]\\ &\quad - \gamma a(u_{n+1}, v - u_{n+1}), \quad \forall v \in K, n \geq 0,\end{aligned} \tag{3.25}$$

where $\gamma > 0$ is a constant.

Theorem 3.3. *Let $F : H \times H \times H \to (-\infty, +\infty]$, φ and $a : H \times H \to (-\infty, +\infty]$ be three functionals, where φ is skew-symmetric. Let $T, A, B : H \to H$ and $\eta : H \times H \to H$ be nonlinear continuous mappings. Assume that F is mixed pseudomonotone with respect to T, A and B, and $E : K \to \mathbb{R}$ is a Fréchet differentiable and η-strongly convex with constant $l > 0$ and $\eta(x, y) = \eta(x, z) + \eta(z, y), \forall x, y, z \in H$. If $u \in K$ is an exact solution of the GMELP(2.1), then the sequence $\{u_n\}_{n \geq 0}$ generalized by Algorithm 3.3 converges strongly to u.*

Proof. Let $u \in K$ be a solution of the GMELP(2.1). Then

$$F(Tu, Au, v) + \langle Bu, \eta(v, u)\rangle \geq [\varphi(u, u) - \varphi(v, u)] - a(u, v - u), \quad \forall v \in K, \tag{3.26}$$

which implies that

$$-F(Tv, Av, u) - \langle Bv, \eta(u, v)\rangle \geq [\varphi(u, u) - \varphi(v, u)] - a(u, v - u), \quad \forall v \in K \tag{3.27}$$

since F is mixed pseudomonotone with respect to T, A and B.

Taking $v = u_{n+1}$ in (3.27), we have

$$\begin{aligned}&- F(Tu_{n+1}, Au_{n+1}, u) - \langle Bu_{n+1}, \eta(u, u_{n+1})\rangle\\ &\quad \geq [\varphi(u, u) - \varphi(u_{n+1}, u)] - a(u, u_{n+1} - u).\end{aligned} \tag{3.28}$$

Now as in Theorem 3.2, by (3.24), (3.25) with $v = u$ and (3.28), we have

$$\begin{aligned}&G(u, u_n) - G(u, u_{n+1})\\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), \eta(u_{n+1}, u_n)\rangle\\ &\quad + \Big\langle E'(u_{n+1}) - E'(u_n), \eta(u, u_{n+1})\Big\rangle\\ &\geq l\|u_{n+1} - u_n\|^2 - \gamma F(Tu_{n+1}, Au_{n+1}, u) - \gamma\langle Bu_{n+1}, \eta(u, u_{n+1})\rangle\\ &\quad + \gamma\Big[\varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\Big] - \gamma a(u_{n+1}, u - u_{n+1})\\ &\geq l\|u_{n+1} - u_n\|^2 + \gamma\Big[\varphi(u, u) - \varphi(u_{n+1}, u) + \varphi(u_{n+1}, u_{n+1}) - \varphi(u, u_{n+1})\Big]\\ &\quad + \gamma a(u - u_{n+1}, u - u_{n+1})\\ &\geq l\|u_{n+1} - u_n\|^2 + \gamma c\|u - u_{n+1}\|^2\\ &\geq \gamma c\|u - u_{n+1}\|^2.\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.2, and is omitted. This completes the proof. □

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