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LEAST SQUARES SOLUTIONS OF THE MATRIX EQUATION $AXB = D$ OVER GENERALIZED REFLEXIVE X

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ABSTRACT. Let $R \in \mathbf{C}^{m\times m}$ and $S \in \mathbf{C}^{n\times n}$ be nontrivial unitary involutions, i.e., $R^* = R = R^{-1} \neq I_m$ and $S^* = S = S^{-1} \neq I_n$. We say that $G \in \mathbf{C}^{m\times n}$ is a generalized reflexive matrix if $RGS = G$. The set of all $m \times n$ generalized reflexive matrices is denoted by $\mathbf{GRC}^{m\times n}$. In this paper, an efficient method for the least squares solution $X \in \mathbf{GRC}^{m\times n}$ of the matrix equation $AXB = D$ with arbitrary coefficient matrices $A \in \mathbf{C}^{p\times m}, B \in \mathbf{C}^{n\times q}$ and the right-hand side $D \in \mathbf{C}^{p\times q}$ is developed based on the canonical correlation decomposition(CCD) and, an explicit formula for the general solution is presented.

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1. Introduction

In this paper we shall adopt the following notation. $\mathbf{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices. $\mathbf{UC}^{n\times n}$ denotes the set of all unitary matrices in $\mathbf{C}^{n\times n}$. A^* and $\|A\|$ stand for the conjugate transpose and the Frobenius norm of a complex matrix A, respectively. For $A, B \in \mathbf{C}^{m\times n}$, we define an inner product in $\mathbf{C}^{m\times n}$: $\langle A, B\rangle = \text{trace}(B^*A)$, then $\mathbf{C}^{m\times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. I_n represents the identity matrix of size n. For $A = (a_{ij}), B = (b_{ij}) \in \mathbf{C}^{m\times n}$, $A * B$ represents the Hadamard product of the matrices A and B, i.e., $A * B = (a_{ij}b_{ij}) \in \mathbf{C}^{m\times n}$.

Throughout this paper $R \in \mathbf{C}^{m\times m}$ and $S \in \mathbf{C}^{n\times n}$ are nontrivial unitary involutions, i.e., $R^* = R = R^{-1} \neq I_m$ and $S^* = S = S^{-1} \neq I_n$. We say that $G \in \mathbf{C}^{m\times n}$ is a generalized reflexive matrix [5] if $RGS = G$. Let $J_n = (j_{i,k})$ represent the exchange matrix of order n defined by $j_{i,k} = \delta_{i,n-k+1}$ for $1 \leq i, k \leq n$, where $\delta_{i,k}$ is the Kronecker delta, i.e., J_n is a matrix with ones on the cross-diagonal and zeros elsewhere. By taking $m = n, R = S = J_n$, then the generalized reflexive matrices reduce to the centrosymmetric matrices [18] which

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play an important role in many areas [6, 8, 9, 16]. Therefore, centrosymmetric matrices, whose special properties have been under extensive study [1, 2, 3, 4, 11, 15, 18, 19], are a special case of generalized reflexive matrices. Chen [5] discussed applications that give rise to these matrices and considered least squares problems involving them. In the following we denote the set of all $m \times n$ generalized reflexive matrices by $\mathbf{GRC}^{m\times n}$. The linear matrix equation

$$AXB = D \tag{1}$$

has been considered by many authors. In [14] Penrose provided a sufficient and necessary condition for consistency of this equation and, for the consistent case, gave a representation of its general solution. Yuan [20, 21], Khatri and Mitra [12] got necessary and sufficient conditions for the existence of symmetric solutions and symmetric positive semidefinite solutions as well as explicit formulae using generalized inverses. Wang and Chang [17] studied least squares symmetric solutions to the equation using the generalized singular value decomposition(GSVD), and a sufficient and necessary condition for its solvability and a representation of its general solution were also established therein. Recently, Cvetkovic [7] discussed the reflexive solutions of (1). Unfortunately, this paper didn't provide the explicit solution formula for the general case.

In the present paper, we will consider least squares solutions of the matrix equation (1) over generalized reflexive X, where $A \in \mathbf{C}^{p\times m}, B \in \mathbf{C}^{n\times q}$ and $D \in \mathbf{C}^{p\times q}$. Using the canonical correlation decomposition(CCD), we present an explicit formula for the general solution. As a by-product of our results, we obtain a necessary and sufficient condition on A, B, D for existence of $X \in \mathbf{GRC}^{m\times n}$ such that the equation of (1) holds, and a general form for all such X. Clearly, the results obtained are shown to include those given in [3, 7, 13] as particular cases.

2. The least squares solutions of the matrix equation (1)

If λ is an eigenvalue of $K \in \mathbf{C}^{m\times m}$, let $V_K(\lambda)$ denote the eigenspace of K corresponding to the eigenvalue λ. We will say that a vector $z \in \mathbf{C}^m$ is R-symmetric (R-skew symmetric) if $Rz = z$ ($Rz = -z$); thus, $V_R(1)$ and $V_R(-1)$ are the subspaces of $\mathbf{C}^{m\times m}$ consisting respectively of R-symmetric and R-skew symmetric vectors. Let $r = \dim[V_R(1)], s = \dim[V_R(-1)]$. Since a unitary involution is diagonalizable and $R \neq \pm I_m$, then $r, s \geq 1$, and $r + s = m$. Let $\{p_1, \cdots, p_r\}$ and $\{q_1, \cdots, q_s\}$ be the orthonormal bases for $V_R(1)$ and $V_R(-1)$ respectively, and define

$$P = [p_1, \cdots, p_r] \in \mathbf{C}^{m\times r}, Q = [q_1, \cdots, q_s] \in \mathbf{C}^{m\times s},$$

then $[P, Q]$ is a unitary matrix and R has the following spectral decomposition:

$$R = [P, Q] \begin{bmatrix} I_r & 0 \\ 0 & -I_s \end{bmatrix} \begin{bmatrix} P^* \\ Q^* \end{bmatrix}. \tag{2}$$

Similarly, there are positive integers k and l such that $k + l = n$ and the matrices $U \in \mathbf{C}^{n\times k}$ and $V \in \mathbf{C}^{n\times l}$ whose column vectors form the orthonormal bases for the eigenspaces $V_S(1)$ and $V_S(-1)$, respectively. Thus, $[U, V]$ is a unitary matrix and S has the spectral decomposition:

$$S = [U,V]\begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix}\begin{bmatrix} U^* \\ V^* \end{bmatrix}. \tag{3}$$

In the following P, Q, U, V are always defined by (2) and (3).

(2) and (3) yield the following characterization of $m \times n$ generalized reflexive matrices.

Lemma 1. *G is a generalized reflexive matrix if and only if*

$$G = [P,Q]\begin{bmatrix} G_{PU} & 0 \\ 0 & G_{QV} \end{bmatrix}\begin{bmatrix} U^* \\ V^* \end{bmatrix}, \tag{4}$$

where $G_{PU} = P^*GU, G_{QV} = Q^*GV$.

Proof. It follows from (2) and (3) that

$$RGS = G \Longleftrightarrow \begin{bmatrix} I_r & 0 \\ 0 & -I_s \end{bmatrix}\begin{bmatrix} P^* \\ Q^* \end{bmatrix}G[U,V]\begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix} = \begin{bmatrix} P^* \\ Q^* \end{bmatrix}G[U,V]. \tag{5}$$

Let

$$\begin{bmatrix} P^* \\ Q^* \end{bmatrix}G[U,V] = \begin{bmatrix} G_{PU} & G_{PV} \\ G_{QU} & G_{QV} \end{bmatrix}.$$

Then the relation of (5) holds if and only if

$$G_{PV} = 0, \quad G_{QU} = 0,$$

which implies the conclusion. □

For given matrices $A_1 \in \mathbf{C}^{p\times r}$ and $A_2 \in \mathbf{C}^{p\times s}$, without loss of generality, we assume that $\operatorname{rank}(A_1) \geq \operatorname{rank}(A_2)$, then the canonical correlation decomposition(CCD) ([10, Theorem 2.1]) of the matrix pair $[A_1, A_2]$ is

$$A_1 = W[\Sigma_1, 0]M, \; A_2 = W[\Omega_1, 0]N, \tag{6}$$

where $M \in \mathbf{C}^{r\times r}, N \in \mathbf{C}^{s\times s}$ are nonsingular matrices and

$$\Sigma_1 = \begin{array}{c} \left[\begin{array}{ccc} I_{r_1} & 0 & 0 \\ 0 & \Lambda_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Theta_1 & 0 \\ 0 & 0 & I_{t_1} \end{array}\right] \\ \begin{array}{ccc} r_1 & s_1 & t_1 \end{array} \end{array} \begin{array}{c} r_1 \\ s_1 \\ l_1 \\ p - h_1 - s_1 - t_1 \\ s_1 \\ t_1 \\ \end{array},$$

$$\Omega_1 = \begin{array}{c} \left[\begin{array}{ccc} I_{r_1} & 0 & 0 \\ 0 & I_{s_1} & 0 \\ 0 & 0 & I_{l_1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} r_1 \\ s_1 \\ l_1 \\ p-h_1-s_1-t_1 \\ s_1 \\ t_1 \end{array} \\ \begin{array}{ccc} r_1 & s_1 & l_1 \end{array} \end{array},$$

$$\begin{aligned} &l_1 = h_1 - r_1 - s_1,\ \text{rank}(A_1) = r_1 + s_1 + t_1,\ h_1 = \text{rank}(A_2), \\ &r_1 = \text{rank}(A_1) + \text{rank}(A_2) - \text{rank}([A_1, A_2]), \\ &\Lambda_1 = \text{diag}\{\lambda_1^{(1)}, \cdots, \lambda_{s_1}^{(1)}\},\ \Theta_1 = \text{diag}\{\theta_1^{(1)}, \cdots, \theta_{s_1}^{(1)}\} \end{aligned}$$

with

$$\begin{aligned} &1 > \lambda_1^{(1)} \geq \lambda_2^{(1)} \geq \cdots \geq \lambda_{s_1}^{(1)} > 0,\ \ 0 < \theta_1^{(1)} \leq \theta_2^{(1)} \leq \cdots \leq \theta_{s_1}^{(1)} < 1, \\ &(\lambda_i^{(1)})^2 + (\theta_i^{(1)})^2 = 1\ (i = 1, \cdots, s_1). \end{aligned}$$

$W = [W_1,\ W_2,\ W_3,\ W_4,\ W_5,\ W_6] \in \mathbf{UC}^{p\times p}$ is unitary with its columns partitioned with that of the row partitions of Σ_1, i.e., $W_1 \in \mathbf{C}^{p\times r_1}, W_2 \in \mathbf{C}^{p\times s_1}$, and so on.

Likewise, for given matrices $B_1 \in \mathbf{C}^{k\times q}$ and $B_2 \in \mathbf{C}^{l\times q}$, we assume that $\text{rank}(B_1) \geq \text{rank}(B_2)$, then the CCD of the matrix pair $[B_1^*, B_2^*]$ is

$$B_1^* = H[\Sigma_2, 0]E,\ B_2^* = H[\Omega_2, 0]F, \tag{7}$$

where $E \in \mathbf{C}^{k\times k}$, $F \in \mathbf{C}^{l\times l}$ are nonsingular matrices, and

$$\Sigma_2 = \begin{array}{c} \left[\begin{array}{ccc} I_{r_2} & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & I_{t_2} \end{array}\right] \begin{array}{c} r_2 \\ s_2 \\ l_2 \\ q-h_2-s_2-t_2 \\ s_2 \\ t_2 \end{array} \\ \begin{array}{ccc} r_2 & s_2 & t_2 \end{array} \end{array},$$

$$\Omega_2 = \begin{array}{c} \left[\begin{array}{ccc} I_{r_2} & 0 & 0 \\ 0 & I_{s_2} & 0 \\ 0 & 0 & I_{l_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} r_2 \\ s_2 \\ l_2 \\ q-h_2-s_2-t_2 \\ s_2 \\ t_2 \end{array} \\ \begin{array}{ccc} r_2 & s_2 & l_2 \end{array} \end{array},$$

$$\begin{aligned} &l_2 = h_2 - r_2 - s_2,\ \text{rank}(B_1) = r_2 + s_2 + t_2,\ h_2 = \text{rank}(B_2), \\ &r_2 = \text{rank}(B_1) + \text{rank}(B_2) - \text{rank}([B_1^*, B_2^*]), \\ &\Lambda_2 = \text{diag}\{\lambda_1^{(2)}, \cdots, \lambda_{s_2}^{(2)}\},\ \Theta_2 = \text{diag}\{\theta_1^{(2)}, \cdots, \theta_{s_2}^{(2)}\} \end{aligned}$$

with

$$\begin{aligned} &1 > \lambda_1^{(2)} \geq \lambda_2^{(2)} \geq \cdots \geq \lambda_{s_2}^{(2)} > 0,\ \ 0 < \theta_1^{(2)} \leq \theta_2^{(2)} \leq \cdots \leq \theta_{s_2}^{(2)} < 1, \\ &(\lambda_i^{(2)})^2 + (\theta_i^{(2)})^2 = 1\ (i = 1, \cdots, s_2). \end{aligned}$$

$H = [H_1,\ H_2,\ H_3,\ H_4,\ H_5,\ H_6] \in \mathbf{UC}^{q\times q}$ is unitary with its columns partitioned in conformal sizes.

Theorem 1. *For given matrices $A \in \mathbf{C}^{p\times m}, B \in \mathbf{C}^{n\times q}$ and $D \in \mathbf{C}^{p\times q}$, denote AP, AQ, U^*B and V^*B by A_1, A_2, B_1 and B_2, respectively. Suppose that the CCDs of the matrix pairs $[A_1, A_2]$ and $[B_1^*, B_2^*]$ are given by (6) and (7), respectively. Partition W^*DH into the following form:*

$$W^*DH = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix}, \tag{8}$$

where $D_{ij} = W_i^*DH_j$, $i,\ j = 1,2,3,4,5,6$. Then the least squares solution $X \in \mathbf{GRC}^{m\times n}$ of the matrix equation (1) can be expressed as

$$X = [P, Q] \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} U^* \\ V^* \end{bmatrix}, \tag{9}$$

where

$$Y = M^{-1} \begin{bmatrix} D_{11} - Z_{11} & D_{15}\Theta_2^{-1} & D_{16} & Y_{14} \\ \Theta_1^{-1}D_{51} & Y_{22} & \Lambda_1 D_{26} + \Theta_1 D_{56} & Y_{24} \\ D_{61} & D_{62}\Lambda_2 + D_{65}\Theta_2 & D_{66} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} (E^*)^{-1}, \tag{10}$$

$$Z = N^{-1} \begin{bmatrix} Z_{11} & D_{12} - D_{15}\Theta_2^{-1}\Lambda_2 & D_{13} & Z_{14} \\ D_{21} - \Lambda_1\Theta_1^{-1}D_{51} & D_{22} - \Lambda_1 Y_{22}\Lambda_2 & D_{23} & Z_{24} \\ D_{31} & D_{32} & D_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{bmatrix} (F^*)^{-1}, \tag{11}$$

$$Y_{22} = \Phi * (\Lambda_1 D_{25}\Theta_2 + \Theta_1 D_{52}\Lambda_2 + \Theta_1 D_{55}\Theta_2) \tag{12}$$

with

$$\Phi = [\phi_{ij}],\ \phi_{ij} = \frac{1}{(\lambda_i^{(1)})^2(\theta_j^{(2)})^2 + (\theta_i^{(1)})^2}, i = 1, \cdots, s_1;\ j = 1, \cdots, s_2,$$

and $Z_{11}, Y_{i4}, Z_{i4}, Y_{4j}, Z_{4j}$ $(i = 1,2,3,4;\ j = 1,2,3)$ are arbitrary matrices.

Proof. If $X \in \mathbf{GRC}^{m\times n}$, it follows from Lemma 1 that there exist $Y \in \mathbf{C}^{r\times k}$ and $Z \in \mathbf{C}^{s\times l}$ satisfying

$$X = [P, Q] \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} U^* \\ V^* \end{bmatrix}. \tag{13}$$

Therefore,

$$\|AXB - D\| = \|A_1 Y B_1 + A_2 Z B_2 - D\|. \tag{14}$$

It follows from (6), (7) and the unitary invariance of Frobenius norm that the relation of (14) is equivalent to

$$\|AXB - D\| = \|[\Sigma_1, 0]MYE^*[\Sigma_2, 0]^* + [\Omega_1, 0]NZF^*[\Omega_2, 0]^* - W^*DH\|. \tag{15}$$

Write

$$MYE^* = \begin{array}{c} \left[\begin{matrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{matrix}\right] \\ \begin{matrix} r_2 & \quad s_2 & \quad t_2 & \quad b_2 \end{matrix} \end{array} \begin{array}{c} r_1 \\ s_1 \\ t_1 \\ a_1 \\ \end{array}, \tag{16}$$

$$NZF^* = \begin{array}{c} \left[\begin{matrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{matrix}\right] \\ \begin{matrix} r_2 & \quad s_2 & \quad l_2 & \quad a_2 \end{matrix} \end{array} \begin{array}{c} r_1 \\ s_1 \\ l_1 \\ b_1 \\ \end{array}, \tag{17}$$

where $a_1 = r - r_1 - s_1 - t_1$, $b_2 = k - r_2 - s_2 - t_2$, $b_1 = s - r_1 - s_1 - l_1$, $a_2 = l - r_2 - s_2 - l_2$. Inserting (8), (16) and (17) into (15), we get

$$\|AXB - D\| = \left\| \begin{matrix} Y_{11} + Z_{11} - D_{11} & Y_{12}\Lambda_2 + Z_{12} - D_{12} \\ \Lambda_1 Y_{21} + Z_{21} - D_{21} & \Lambda_1 Y_{22}\Lambda_2 + Z_{22} - D_{22} \\ Z_{31} - D_{31} & Z_{32} - D_{32} \\ -D_{41} & -D_{42} \\ \Theta_1 Y_{21} - D_{51} & \Theta_1 Y_{22}\Lambda_2 - D_{52} \\ Y_{31} - D_{61} & Y_{32}\Lambda_2 - D_{62} \end{matrix} \right.$$

$$\left. \begin{matrix} Z_{13} - D_{13} & -D_{14} & Y_{12}\Theta_2 - D_{15} & Y_{13} - D_{16} \\ Z_{23} - D_{23} & -D_{24} & \Lambda_1 Y_{22}\Theta_2 - D_{25} & \Lambda_1 Y_{23} - D_{26} \\ Z_{33} - D_{33} & -D_{34} & -D_{35} & -D_{36} \\ -D_{43} & -D_{44} & -D_{45} & -D_{46} \\ -D_{53} & -D_{54} & \Theta_1 Y_{22}\Theta_2 - D_{55} & \Theta_1 Y_{23} - D_{56} \\ -D_{63} & -D_{64} & Y_{32}\Theta_2 - D_{65} & Y_{33} - D_{66} \end{matrix} \right\| \tag{18}$$

Therefore, $\|AXB - D\| = \min$ if and only if

$$Y_{13} = D_{16},\ Y_{31} = D_{61},\ Y_{33} = D_{66}, \tag{19}$$

$$Z_{13} = D_{13},\ Z_{23} = D_{23},\ Z_{31} = D_{31},\ Z_{32} = D_{32},\ Z_{33} = D_{33}, \tag{20}$$

$$\|Y_{11} + Z_{11} - D_{11}\| = \min, \tag{21}$$

$$\|Y_{12}\Lambda_2 + Z_{12} - D_{12}\|^2 + \|Y_{12}\Theta_2 - D_{15}\|^2 = \min, \tag{22}$$

$$\|\Lambda_1 Y_{21} + Z_{21} - D_{21}\|^2 + \|\Theta_1 Y_{21} - D_{51}\|^2 = \min, \tag{23}$$

$$\begin{aligned}&\|\Lambda_1 Y_{22}\Lambda_2 + Z_{22} - D_{22}\|^2 + \|\Lambda_1 Y_{22}\Theta_2 - D_{25}\|^2 \\ &\qquad + \|\Theta_1 Y_{22}\Lambda_2 - D_{52}\|^2 + \|\Theta_1 Y_{22}\Theta_2 - D_{55}\|^2 = \min.\end{aligned} \tag{24}$$

$$\|\Lambda_1 Y_{23} - D_{26}\|^2 + \|\Theta_1 Y_{23} - D_{56}\|^2 = \min, \tag{25}$$

$$\|Y_{32}\Lambda_2 - D_{62}\|^2 + \|Y_{32}\Theta_2 - D_{65}\|^2 = \min, \tag{26}$$

From (21), (22) and (23), we have

$$Y_{11} = D_{11} - Z_{11}, \tag{27}$$

$$Y_{12} = D_{15}\Theta_2^{-1}, \ Z_{12} = D_{12} - D_{15}\Theta_2^{-1}\Lambda_2, \tag{28}$$

$$Y_{21} = \Theta_1^{-1}D_{51}, \ Z_{21} = D_{21} - \Lambda_1\Theta_1^{-1}D_{51}. \tag{29}$$

Clearly, the minimization problem (24) is equivalent to

$$Z_{22} = D_{22} - \Lambda_1 Y_{22}\Lambda_2 \tag{30}$$

and

$$f(Y_{22}) = \|\Lambda_1 Y_{22}\Theta_2 - D_{25}\|^2 + \|\Theta_1 Y_{22}\Lambda_2 - D_{52}\|^2 + \|\Theta_1 Y_{22}\Theta_2 - D_{55}\|^2 = \min.$$

Let $D_{25} = [d_{ij}^{(25)}]$, $D_{52} = [d_{ij}^{(52)}]$, $D_{55} = [d_{ij}^{(55)}] \in \mathbf{C}^{s_1 \times s_2}$ and $Y_{22} = [y_{ij}] \in \mathbf{C}^{s_1 \times s_2}$, then

$$\begin{aligned} f(Y_{22}) &= \sum_{i=1}^{s_1}\sum_{j=1}^{s_2}\left(\left|\lambda_i^{(1)}y_{ij}\theta_j^{(2)} - d_{ij}^{(25)}\right|^2\right. \\ &\quad \left. + \left|\theta_i^{(1)}y_{ij}\lambda_j^{(2)} - d_{ij}^{(52)}\right|^2 + \left|\theta_i^{(1)}y_{ij}\theta_j^{(2)} - d_{ij}^{(55)}\right|^2\right).\end{aligned}$$

Now we minimize the quantities

$$\begin{aligned} q_{ij} &= \left|\lambda_i^{(1)}y_{ij}\theta_j^{(2)} - d_{ij}^{(25)}\right|^2 + \left|\theta_i^{(1)}y_{ij}\lambda_j^{(2)} - d_{ij}^{(52)}\right|^2 + \left|\theta_i^{(1)}y_{ij}\theta_j^{(2)} - d_{ij}^{(55)}\right|^2, \\ &\quad 1 \le i \le s_1; \ 1 \le j \le s_2.\end{aligned}$$

It is easy to obtain the minimizers

$$y_{ij} = \frac{\lambda_i^{(1)}d_{ij}^{(25)}\theta_j^{(2)} + \theta_i^{(1)}d_{ij}^{(52)}\lambda_j^{(2)} + \theta_i^{(1)}d_{ij}^{(55)}\theta_j^{(2)}}{(\lambda_i^{(1)})^2(\theta_j^{(2)})^2 + (\theta_i^{(1)})^2}, \ 1 \le i \le s_1; \ 1 \le j \le s_2. \tag{31}$$

By rewriting (31) in matrix form, we immediately obtain (12).
In a similar way, from (25) and (26), we get

$$Y_{23} = \Lambda_1 D_{26} + \Theta_1 D_{56}, \tag{32}$$

$$Y_{32} = D_{62}\Lambda_2 + D_{65}\Theta_2. \tag{33}$$

Substituting (19), (20), (27), (28), (29), (30), (32) and (33) into (16) and (17) yields (9). □

From (18), we can easily obtain the following corollary.

Corollary 1. *Under the same assumptions as in Theorem 1, then matrix equation (1) has a solution* $X \in \mathbf{GRC}^{m \times n}$ *if and only if*

$$\begin{gathered} D_{53} = 0,\ D_{63} = 0,\ D_{35} = 0,\ D_{36} = 0, \\ D_{i4} = 0,\ D_{4j} = 0,\ i, j = 1, 2, 3, 4, 5, 6; \\ D_{62}\Lambda_2^{-1} = D_{65}\Theta_2^{-1},\ \Lambda_1^{-1}D_{26} = \Theta_1^{-1}D_{56}, \\ \Lambda_1^{-1}D_{25}\Theta_2^{-1} = \Theta_1^{-1}D_{55}\Theta_2^{-1} = \Theta_1^{-1}D_{52}\Lambda_2^{-1}. \end{gathered}$$

In this case, the general solution can expressed as

$$X = [P, Q] \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} U^* \\ V^* \end{bmatrix},$$

where

$$Y = M^{-1} \begin{bmatrix} D_{11} - Z_{11} & D_{15}\Theta_2^{-1} & D_{16} & Y_{14} \\ \Theta_1^{-1}D_{51} & \Lambda_1^{-1}D_{25}\Theta_2^{-1} & \Lambda_1^{-1}D_{26} & Y_{24} \\ D_{61} & D_{62}\Lambda_2^{-1} & D_{66} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} (E^*)^{-1},$$

$$Z = N^{-1} \begin{bmatrix} Z_{11} & D_{12} - D_{15}\Theta_2^{-1}\Lambda_2 & D_{13} & Z_{14} \\ D_{21} - \Lambda_1\Theta_1^{-1}D_{51} & D_{22} - D_{25}\Theta_2^{-1}\Lambda_2 & D_{23} & Z_{24} \\ D_{31} & D_{32} & D_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{bmatrix} (F^*)^{-1},$$

and $Z_{11}, Y_{i4}, Z_{i4}, Y_{4j}, Z_{4j}$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3)$ *are arbitrary matrices.*

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