J. Appl. Math. & Informatics Vol. 26(2008), No. 3 - 4, pp. 445 - 456
Website: http://www.kcam.biz

CHARACTERIZATION AND FIRST-ORDER HERMITE INTERPOLATION FOR HELICAL POLYNOMIAL SPACE CURVES

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ABSTRACT. We characterize the helical polynomial space curves and solve the first-order Hermite interpolation problem for helical polynomial space quintics.

AMS Mathematics Subject Classification : 65D17, 68U05
Key words and phrases : Pythagorean-hodograph curves, helical polynomial space curves, Hermite interpolation

1. Introduction

In this paper, we characterize the helical polynomial space curves and solve the first-order Hermite interpolation in the space for them.

In their seminal paper [9], Farouki and Sakkalis introduced the Pythagorean-hodograph (PH) curves in the plane $\mathbb{R}^2$, as curves whose component functions are polynomials and having a polynomial as their speed. The arc length and the unit normal vector field of a PH curve are expressed by a polynomial and a curve whose component functions are rational, respectively. These properties give a rational representation of the offset curves to a PH curve. Thus PH curves are very useful in computer-aided design and manufacturing applications ([1], [3], [4], [5], [8], and [10]).

Farouki, al-Kandari and Sakkalis [6] studied the first-order Hermite interpolation problem for the spatial PH quintic using the quaternion model, which was proposed by Choi et al. [2]. For given initial and final points $\mathbf{p}_i$, $\mathbf{p}_f$, and their hodographs $\mathbf{d}_i$, $\mathbf{d}_f$, the Hermite interpolants comprise, in general, a two-parameter family. To fix these free parameters and to have a desired curve to the Hermite inperpolations, Farouki et al. [7] used the helical polynomial space curves.

Received September 6, 2007. *Corresponding author.

In this paper, we characterize and solve the first order Hermite interpolation problem for helical polynomial space curves. Farouki et al. [7] solved those interoplation problem for almost all Hermite data. Here we show that for every Hermite data, there exist some Hermite interplants by helical polynomial space curves. To solve the first-order Hermite interpolation problem, we use the fact that the projection of a helical PH curve onto the plane, which is perpendicular to the direction vector of the helix, is also a PH curve. For given first-order Hermite interpolation data, we first find out the direction of helix and obtain Hermite interpolation data on the above plane. Now we solve the interpolation on the plane, for example from [8], and obtain the Hermite interpolants to the original data.

The organization of this paper is as follows. In section 2 we characterize the helical polynomial space curves and study some basic properties of the helical polynomial space curve. In section 3 we solve the first-order Hermite interpolation in the space for helical polynomial space curves.

2. Helical polynomial space curves

Let $\mathbb{R}^n$ be the Euclidean space of dimension n with $n \geq 2$. Let $\mathbb{R}[t]$ be the set of polynomials with real coefficients. By a polynomial curve in $\mathbb{R}^n$, we mean a curve $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^n$ from the space of real numbers $\mathbb{R}$ to the Euclidean space $\mathbb{R}^n$, whose component functions $x_1(t), x_2(t), \ldots, x_n(t)$ are members of $\mathbb{R}[t]$.

A polynomial curve $\mathbf{r}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ in $\mathbb{R}^n$ is said to be *Pythagorean* if there is a polynomial $\sigma(t)$ in $\mathbb{R}[t]$ such that

$$x_1(t)^2 + x_2(t)^2 + \cdots + x_n(t)^2 = \sigma(t)^2.$$

A polynomial curve $\mathbf{r(t)} = (x_1(t), x_2(t), \ldots, x_n(t))$ is called a *Pythagorean-hodograph (PH)* curve if its velocity vector or *hodograph* $\mathbf{r}'(t) = (x_1'(t), \ldots x_n'(t))$ is Pythagorean.

We present the characterization of the PH curves in $\mathbb{R}^{n+1}$:

Theorem 1 ([11]). *The polynomial curve* $\mathbf{r}(t) = (x_0(t), x_1(t), \ldots, x_n(t))$ *is a PH curve in the Euclidean space* $\mathbb{R}^{n+1}$ *with*

$$x_0'(t)^2 + x_1'(t)^2 + \cdots + x_n'(t)^2 = \sigma(t)^2$$

for some polynomial $\sigma(t)$ *if and only if there exist polynomial functions* $h(t)$, $u(t)$, $v(t)$, $a(t)$, $b(t)$ *and* $\alpha_k(t)$ *for* $1 \leq k \leq n$ *with*

$$\gcd(u(t), v(t)b(t)) = 1 = \gcd(v(t), u(t)a(t)) = \gcd(\alpha_1(t), \ldots, \alpha_n(t)),$$

and $a(t)b(t) = \alpha_1(t)^2 + \cdots + \alpha_n(t)^2$ *so that*

$$x_0'(t) = h(t)\Big[u(t)^2a(t) - v(t)^2b(t)\Big],$$

$$x_k'(t) = h(t)\Big[2u(t)\alpha_k(t)v(t)\Big], \quad 1 \leq k \leq n,$$

$$\sigma(t) = \pm h(t)\Big[u(t)^2a(t) + v(t)^2b(t)\Big].$$

We now characterize the helical polynomial space curves. At first we give the definition and basic properties of helices:

A *helix* is a regular space curve $\mathbf{r}(t)$ such that for some fixed unit vector $\mathbf{u}$, $\left\langle \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \mathbf{u} \right\rangle$ is constant. $\mathbf{u}$ is called the *direction vector of helix.* For more information, see [12].

Lemma 1. *Let* $\mathbf{r}(t) = (x(t), y(t), z(t))$ *be a polynomial space curve with* $\mathbf{r}'(t) \neq 0$ *for all* $t \in \mathbb{R}$. *Let* $h(t) = \gcd\left\{x'(t), y'(t), z'(t)\right\}$ *so that* $\mathbf{r}'(t) = h(t)\mathbf{s}'(t)$ *with some primitive space curve* $\mathbf{s}'(t)$. *Then* $\mathbf{r}(t)$ *is a helix if and only if* $\mathbf{s}(t) = \int \mathbf{s}'(t)\,dt$ *is a helix.*

Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a helical polynomial space curve with its direction vector $(1, 0, 0)$ of helix. In [7], Farouki *et al.* proved a helical polynomial space curve is PH. So we can write

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma(t)^2$$

for some positive-valued polynomial $\sigma(t)$. Since the direction vector of the helix is $(1, 0, 0)$, we have $x'(t) = c\sigma(t)$ for some constant c. In particular, we have $\deg(\mathbf{r}(t)) = \deg(x(t))$.

Lemma 2. *Let* $\mathbf{r}(t) = (x(t), y(t), z(t))$ *be a helical polynomial space curve with the direction vector* $(1, 0, 0)$ *of helix. Then the plane curve* $\mathbf{r}_P(t) = (y(t), z(t))$ *is PH.*

Proof. Let $\sigma(t)$ be a positive-valued polynomial such that

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma(t)^2.$$

Since $\mathbf{r}(t)$ is a helical polynomial space curve with its direction vector $(1, 0, 0)$, we have $x'(t) = c\sigma(t)$ for some constant c with $-1 \leq c \leq 1$. It implies that

$$\left(1 - c^2\right)x'(t)^2 = c^2\left(y'(t)^2 + z'(t)^2\right).$$

If $c = 1$ or $c = -1$, then $y'(t) \equiv 0$, $z'(t) \equiv 0$, and $x'(t)^2 \equiv 1$. In this case, $\mathbf{r}(t) = (t + a, b, d)$ or $\mathbf{r}(t) = (-t + a, b, d)$ for some real numbers a, b, and d. Therefore, the plane curve $\mathbf{r}_P(t) = (b, d)$ is a PH plane curve as a constant curve.

Suppose that $c \neq 1$ and $c \neq -1$. we have

$$y'(t)^2 + z'(t)^2 = \frac{1 - c^2}{c^2}x'(t)^2 = \left(\sqrt{1 - c^2}\sigma(t)\right)^2.$$

Therefore, the plane curve $\mathbf{r}_P(t) = (y(t), z(t))$ is PH. □

We have a constant c such that $x'(t) = c\sigma(t)$. If $c = 1$ or $c = -1$, then we have $\mathbf{r}(t) = (x(t), a, b)$, which is a straight line. So we assume that $c \neq 1$

and $c \neq -1$. Since $y'(t)^2 + z'(t)^2 = \left(\sqrt{1-c^2}\sigma(t)\right)^2$, we may assume that $\deg(\mathbf{r}(t)) = \deg(y(t))$.

By Theorem 1, we have

$$\begin{aligned} x'(t) &= h(t)\Big\{u(t)^2 a(t) - v(t)^2 b(t)\Big\} \\ y'(t) &= h(t) 2u(t)\alpha_1(t) v(t), \\ z'(t) &= h(t) 2u(t)\alpha_2(t) v(t), \\ \sigma(t) &= h(t)\Big\{u(t)^2 a(t) + v(t)^2 b(t)\Big\}, \end{aligned}$$

where $\gcd\Big(u(t), v(t)b(t)\Big) = 1 = \gcd\Big(u(t)a(t), v(t)\Big) = \gcd(\alpha_1(t), \alpha_2(t))$ and $\alpha_1(t)^2 + \alpha_2(t)^2 = a(t)b(t)$. Moreover, by Lemma 2, we have

$$\begin{aligned} \alpha_1(t) &= \tilde{u}(t)^2 - \tilde{v}(t)^2, \\ \alpha_2(t) &= 2\tilde{u}(t)\tilde{v}(t), \\ \alpha_1(t)^2 + \alpha_2(t)^2 &= \Big(\tilde{u}(t)^2 + \tilde{v}(t)^2\Big)^2, \end{aligned}$$

where $\gcd(\tilde{u}(t), \tilde{v}(t)) = 1$.

Since $x'(t) = c \cdot \sigma(t)$ for some constant c, we have

$$(1-c)u(t)^2 a(t) = (1+c)v(t)^2 b(t).$$

Since $\gcd(u(t), v(t)b(t)) = 1$, we have $u(t) \equiv u$ for some nonzero constant u. Moreover, since $\gcd\Big(a(t), v(t)\Big) \equiv 1$, $v(t) \equiv v$ is a nonzero constant.

We summarize the above result:

Theorem 2. *Let* $\mathbf{r}(t) = \Big(x(t), y(t), z(t)\Big)$ *be a helical polynomial space curve with the direction vector* $(1,0,0)$ *of helix and*

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma(t)^2, \quad \textit{and} \quad x'(t) = c \cdot \sigma(t),$$

where c *is a constant with* $-1 < c < 1$ *and* $\sigma(t)$ *is a polynomial. Then we have polynomials* $h(t)$, $\tilde{u}(t)$, $\tilde{v}(t)$ *with* $\gcd\Big(\tilde{u}(t), \tilde{v}(t)\Big) = 1$ *such that*

$$\begin{aligned} x'(t) &= h(t)\left(\frac{\pm c(\tilde{u}(t)^2 + \tilde{v}(t)^2)}{\sqrt{1-c^2}}\right), \\ y'(t) &= h(t)(\tilde{u}(t)^2 - \tilde{v}(t)^2), \\ z'(t) &= h(t)\Big(2\tilde{u}(t)\tilde{v}(t)\Big), \\ \sigma(t) &= h(t)\left(\frac{\pm(\tilde{u}(t)^2 + \tilde{v}(t)^2)}{\sqrt{1-c^2}}\right). \end{aligned}$$

3. First-order Hermite interpolation

In this section we solve the first-order Hermite interpolation problem for helical polynomial quintics.

Let $\mathbf{p}_i = (0,0,0)$, $\mathbf{p}_f = (p_{fx}, p_{fy}, p_{fz})$, $\mathbf{d}_i = (d_{ix}, d_{iy}, d_{iz}) \neq (0,0,0)$, and $\mathbf{d}_f = (d_{fx}, d_{fy}, d_{fz}) \neq (0,0,0)$ be data for the first-order Hermite interpolation. If $\mathbf{p}_i$, $\mathbf{p}_f$, $\mathbf{d}_i$, and $\mathbf{d}_f$ are in the same plane, then we can find the first-order Hermite interpolation in the plane [8]. Therefore we assume that $\mathbf{p}_i$, $\mathbf{p}_f$, $\mathbf{d}_i$, and $\mathbf{d}_f$ are not in a plane.

At first, we will change the coordinates with an orthonormal transformation $O \in \mathrm{SO}(n)$ so that

$$O(\mathbf{p}_i) = \mathbf{p}_i, \quad O(\mathbf{p}_f) = (\tilde{p}_{fx}, \tilde{p}_{fy}, \tilde{p}_{fz}),$$
$$O(\mathbf{d}_i) = (\tilde{d}_{ix}, 0, \tilde{d}_{iz}), \quad O(\mathbf{d}_f) = (\tilde{d}_{fx}, 0, \tilde{d}_{fz}),$$

and

$$\frac{O(\mathbf{d}_i)}{\|O(\mathbf{d}_i)\|} \cdot (1,0,0) = \frac{O(\mathbf{d}_f)}{\|O(\mathbf{d}_f)\|} \cdot (1,0,0) > 0.$$

We can find

$$O = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$

as follows: Since $\mathbf{p}_i$, $\mathbf{p}_f$, $\mathbf{d}_i$, and $\mathbf{d}_f$ are not in a plane, we have

$$\frac{\mathbf{d}_i}{\|\mathbf{d}_i\|} + \frac{\mathbf{d}_f}{\|\mathbf{d}_f\|} \neq (0,0,0), \quad \frac{\mathbf{d}_i}{\|\mathbf{d}_i\|} \times \frac{\mathbf{d}_f}{\|\mathbf{d}_f\|} \neq \mathbf{0}.$$

We set

$$(e_{11}, e_{12}, e_{13}) = \frac{\dfrac{\mathbf{d}_i}{\|\mathbf{d}_i\|} + \dfrac{\mathbf{d}_f}{\|\mathbf{d}_f\|}}{\left\| \dfrac{\mathbf{d}_i}{\|\mathbf{d}_i\|} + \dfrac{\mathbf{d}_f}{\|\mathbf{d}_f\|} \right\|}, \quad (e_{21}, e_{22}, e_{23}) = \frac{\dfrac{\mathbf{d}_i}{\|\mathbf{d}_i\|} \times \dfrac{\mathbf{d}_f}{\|\mathbf{d}_f\|}}{\left\| \dfrac{\mathbf{d}_i}{\|\mathbf{d}_i\|} \times \dfrac{\mathbf{d}_f}{\|\mathbf{d}_f\|} \right\|},$$

and $(e_{31}, e_{32}, e_{33}) = (e_{11}, e_{12}, e_{13}) \times (e_{21}, e_{22}, e_{23})$.

Here, we may assume that $\tilde{p}_{fx} \geq 0$ without loss of generality, because in this case of $\tilde{p}_{fx} < 0$, we can interchange the initial point $\mathbf{p}_i$ and the terminal point $\mathbf{p}_f$. Let α be an angle such that $\cos\alpha = \dfrac{\tilde{d}_{ix}}{\|O(\mathbf{d}_i)\|}$ and $0 \leq \alpha \leq \dfrac{\pi}{2}$.

Then we have

$$\frac{\tilde{d}_{ix}}{\|O(\mathbf{d}_i)\|} = \frac{\tilde{d}_{fx}}{\|O(\mathbf{d}_f)\|} > 0, \quad \frac{\tilde{d}_{iz}}{\|O(\mathbf{d}_i)\|} = -\frac{\tilde{d}_{fz}}{\|O(\mathbf{d}_f)\|} \neq 0.$$

We want to obtain a helical polynomial quintic $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$ with $\mathbf{r}'(t)^2 = \sigma(t)^2$ for some polynomial $\sigma(t)$ such that

$$\mathbf{r}(0) = \mathbf{p}_i, \quad \mathbf{r}(1) = \mathbf{p}_f, \quad \mathbf{r}'(0) = \mathbf{d}_i, \quad \mathbf{r}'(1) = \mathbf{d}_f. \tag{1}$$

Suppose that $\mathbf{r}(t)$ be a helical polynomial quintic satisfying (1). Let $\mathbf{u} = (u_x, u_y, u_z)$ be the direction vector of $O(\mathbf{r}(t))$ so that the constant

$c = \left\langle \mathbf{u}, \dfrac{O(\mathbf{r}'(t))}{\|O(\mathbf{r}'(t))\|} \right\rangle$ satisfies $0 < c < 1$. Then since

$$\left\langle \mathbf{u}, \frac{O(\mathbf{d}_i)}{\|O(\mathbf{d}_i)\|} \right\rangle = \left\langle \mathbf{u}, \frac{O(\mathbf{d}_f)}{\|O(\mathbf{d}_f)\|} \right\rangle,$$

and

$$\frac{\tilde{d}_{ix}}{\|O(\mathbf{d}_i)\|} = \frac{\tilde{d}_{fx}}{\|O(\mathbf{d}_f)\|} > 0, \quad \frac{\tilde{d}_{iz}}{\|O(\mathbf{d}_i)\|} = -\frac{\tilde{d}_{fz}}{\|O(\mathbf{d}_f)\|} \neq 0,$$

we have $u_z = 0$. Therefore we can write $\mathbf{u} = (\cos(-\theta), \sin(-\theta), 0)$ for some constant θ with $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

We change the coordinates with the rotation

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and we set

$$R(O(\mathbf{p}_i)) = \mathbf{p}_i, \quad R(O(\mathbf{p}_f)) = (\hat{p}_{fx}, \hat{p}_{fy}, \hat{p}_{fz}),$$

and

$$R(O(\mathbf{d}_i)) = (\hat{d}_{ix}, \hat{d}_{iy}, \hat{d}_{iz}), \quad R(O(\mathbf{d}_f)) = (\hat{d}_{fx}, \hat{d}_{fy}, \hat{d}_{fz}).$$

Therefore $\hat{\mathbf{r}}(t) = R(O(\mathbf{r}(t))) = (\hat{x}(t), \hat{y}(t), \hat{z}(t))$ is a helical polynomial quintic with the direction $(1,0,0)$ of helix. Since c is positive, we have

$$\hat{p}_{fx} > 0 \quad \text{and} \quad \hat{d}_{ix} > 0.$$

Thus we have two subcases:

Subcase 1.: If $\tilde{p}_{fy} < 0$, then $\beta_1 < \theta < \beta_2$,

where $\beta_1 = -\arccos\left(\dfrac{-\tilde{p}_{fy}}{\sqrt{\tilde{p}_{fx}^2 + \tilde{p}_{fy}^2}}\right)$, $(-\frac{\pi}{2} < \beta_1 < 0)$ and $\beta_2 = \frac{\pi}{2}$;

Subcase 2.: If $\tilde{p}_{fy} > 0$, then $\beta_1 < \theta < \beta_2$, where $\beta_2 = \arccos\left(\dfrac{\tilde{p}_{fy}}{\sqrt{\tilde{p}_{fx}^2 + \tilde{p}_{fy}^2}}\right)$,

$(0 < \beta_2 < \frac{\pi}{2})$ and $\beta_1 = -\frac{\pi}{2}$.

Also we have the constant

$$c = \cos\theta \cdot \cos\alpha = \frac{\hat{d}_{ix}}{\|R(O(\mathbf{d}_i))\|},$$

which satisfies $\hat{x}'(t) = c\sigma(t)$ and $0 < c < 1$.

We consider the projection $\hat{\mathbf{r}}_P(t)$ of $\hat{\mathbf{r}}(t)$ onto the y-z plane. By Lemma 2, $\hat{\mathbf{r}}_P(t)$ is a PH curve in the plane, which satisfies

$$\hat{\mathbf{r}}_P(0) = (\hat{p}_{iy}, \hat{p}_{iz}),\ \hat{\mathbf{r}}_P(1) = (\hat{p}_{fy}, \hat{p}_{fz}),\ \hat{\mathbf{r}}'_P(0) = (\hat{d}_{iy}, \hat{d}_{iz}),\ \hat{\mathbf{r}}'_P(1) = (\hat{d}_{fy}, \hat{d}_{iz}).$$

Until now we have assumed that $\hat{\mathbf{r}}(t)$ is a helical polynomial curve and have chosen θ by the direction vector of $\tilde{\mathbf{r}}(t)$. From now on we do not take this assumption and will obtain helical polynomial curves $\mathbf{r}(t)$, which satisfies (1). We take θ as a variable with $\beta_1 \leq \theta \leq \beta_2$, and from the transformation O and R, we have the first-order Hermite data

$$(\hat{p}_{iy}(\theta), \hat{p}_{iz}(\theta)), \quad (\hat{p}_{fy}(\theta), \hat{p}_{fz}(\theta)), \quad (\hat{d}_{iy}(\theta), \hat{d}_{iz}(\theta)), \quad (\hat{d}_{fy}(\theta), \hat{d}_{iz}(\theta)),$$

and $c(\theta) = \cos\theta \cdot \cos\alpha$. Then from [8] we have PH curves $\mathbf{s}_\theta(t)$ satisfying

$$\mathbf{s}_\theta(0) = (\hat{p}_{iy}(\theta), \hat{p}_{iz}(\theta)), \quad \mathbf{s}_\theta(1) = (\hat{p}_{fy}(\theta), \hat{p}_{fz}(\theta)),$$

and

$$\mathbf{s}'_\theta(0) = (\hat{d}_{iy}(\theta), \hat{d}_{iz}(\theta)), \quad \mathbf{s}'_\theta(1) = (\hat{d}_{fy}(\theta), \hat{d}_{iz}(\theta)).$$

We also know the length $S(\theta)$ of the curve $\mathbf{s}_\theta(t)$. For the first-order Hermite interpolation in the plane, see Appendix.

Here we note that the curve $\mathbf{s}_\theta(t)$ is the projection onto the y-z plane of a helical polynomial curve $\hat{\mathbf{r}}(t)$, whose direction vector is $(1,0,0)$, if and only if we have

$$\hat{p}_{fx}(\theta) = \frac{c(\theta)}{\sqrt{1-c(\theta)^2}} \cdot S(\theta). \tag{2}$$

For the curve $\hat{\mathbf{r}}$ is a helix with the direction $(1,0,0)$ of helix, which satisfies

$$\hat{x}(t)^2 + \hat{y}(t)^2 + \hat{z}(t)^2 = \sigma(t)^2$$

for some positive polynomial $\sigma(t)$ if and only if

$$\hat{p}_{fx}(\theta) = c(\theta) \int_0^1 \sigma(t)\, dt.$$

Here we know that

$$\int_0^1 \sigma(t)\, dt = \frac{1}{\sqrt{1-c(\theta)^2}} S(\theta).$$

From now on our main goal is to find θ for which Equation (2) satisfies, so that we obtain the helical polynomial quintic $\hat{\mathbf{r}}(t)$ with the direction vector $(1,0,0)$ of helix. We note that such θ always exists: Consider *Subcase 1.* If $\theta = -\beta_1$, then $\hat{p}_{fx}(\theta) = 0$ and $c(\theta) \int_0^1 \sigma(t)\, dt > 0$, and if $\theta = \frac{\pi}{2}$, then $\hat{p}_{fx}(\theta) > 0$ and $c(\theta) \int_0^1 \sigma(t)\, dt = 0$. And as θ changes from $-\beta_1$ to $\frac{\pi}{2}$, $\hat{p}_{fx}(\theta)$ and $c(\theta) \int_0^1 \sigma(t)\, dt > 0$ change continuously. Therefore for some θ, we have $\hat{p}_{fx}(\theta) = c(\theta) \int_0^1 \sigma(t)\, dt > 0$. By the same method, we can have the result for *Subcase 2.*

For each solution $\mathbf{s}_\theta(t)$, we can find θ for which Equation (2) satisfies by the Newton method: let

$$f(\theta) = \frac{c(\theta)}{\sqrt{1-c(\theta)^2}} \cdot S(\theta) - \hat{p}_{fx}(\theta).$$

From the graph of $f(\theta)$, we pick up a point θ_1 near a zero of $f(\theta)$. For a natural number m, we set

$$\theta_{m+1} = \theta_m - \frac{f(\theta_m)}{f'(\theta_m)}.$$

With this iterative procedure, θ_m conveges a zero of $f(\theta)$ very quickly. We then take for example $\theta = \theta_{10}$.

From the θ above, we obtain the solution $\mathbf{s}_\theta(t) = (\hat{y}(t), \hat{z}(t))$ and the solution $\hat{\mathbf{r}}(t) = (\hat{x}(t), \hat{y}(t), \hat{z}(t))$ for the space with

$$\hat{x}(t) = \frac{c}{\sqrt{1-c^2}} \int_0^t \sqrt{\hat{y}'(s)^2 + \hat{z}'(s)^2}\, ds = c \int_0^t \sigma(s)\, ds.$$

Then we have

$$\hat{x}(0) = 0 = \hat{p}_{ix}, \qquad \hat{x}(1) = c \int_0^1 \sigma(s)\, ds = \hat{p}_{fx},$$

and from $\dfrac{\hat{d}_{ix}}{\sqrt{\hat{d}_{ix}^2 + \hat{d}_{iy}^2 + \hat{d}_{iz}^2}} = c = \dfrac{\hat{d}_{fx}}{\sqrt{\hat{d}_{fx}^2 + \hat{d}_{fy}^2 + \hat{d}_{fz}^2}}$, we see that

$$\hat{x}'(0) = \frac{c}{\sqrt{1-c^2}} \sqrt{\hat{y}'(0)^2 + \hat{z}'(0)^2} = \frac{c}{\sqrt{1-c^2}} \sqrt{\hat{d}_{iy}^2 + \hat{d}_{iz}^2} = \hat{d}_{ix},$$

and

$$\hat{x}'(0) = \frac{c}{\sqrt{1-c^2}} \sqrt{\hat{y}'(0)^2 + \hat{z}'(t)^2} = \frac{c}{\sqrt{1-c^2}} \sqrt{\hat{d}_{iy}^2 + \hat{d}_{iz}^2} = \hat{d}_{ix}.$$

Finally we obtain the helical polynomial quintic $\mathbf{r}(t) = O^{-1}(R^{-1}(\hat{\mathbf{r}}(t)))$.

Example 1. Let $\mathbf{p}_i = (0,0,0)$, $\mathbf{d}_i = (10,0,10)$ and $\mathbf{p}_f = (10,10,10)$, $\mathbf{d}_f = (0,10,10)$ be given Hermite data. We can find the orthogonal matrices

$$O = \begin{pmatrix} 0.4082482905 & 0.4082482905 & 0.8164965810 \\ -0.5773502693 & -0.5773502693 & 0.5773502693 \\ 0.7071067814 & -0.7071067814 & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $(\hat{p}_{ix}, \hat{p}_{iy}, \hat{p}_{iz}) = R(O(\mathbf{p}_i))$, $\left(\hat{d}_{ix}, \hat{d}_{iy}, \hat{d}_{iz}\right) = R(O(\mathbf{d}_i))$, $\left(\hat{p}_{fx}, \hat{p}_{fy}, \hat{p}_{fz}\right) = R(O(\mathbf{p}_f))$, and $\left(\hat{d}_{fx}, \hat{d}_{fy}, \hat{d}_{fz}\right) = R(O(\mathbf{d}_f))$. For Hermite data $\hat{\mathbf{P}}_i = (\hat{p}_{iy}, \hat{p}_{iz})$, $\hat{\mathbf{D}}_i = (\hat{d}_{iy}, \hat{d}_{iz})$, and $\hat{\mathbf{P}}_f = (\hat{p}_{fy}, \hat{p}_{fz})$, $\hat{\mathbf{D}}_f = (\hat{d}_{fy}, \hat{d}_{iz})$, we have four solutions. Since $\tilde{p}_{fy} < 0$, we obtain

$$\beta_1 = -1.230959417, \qquad \beta_2 = \frac{\pi}{2}.$$

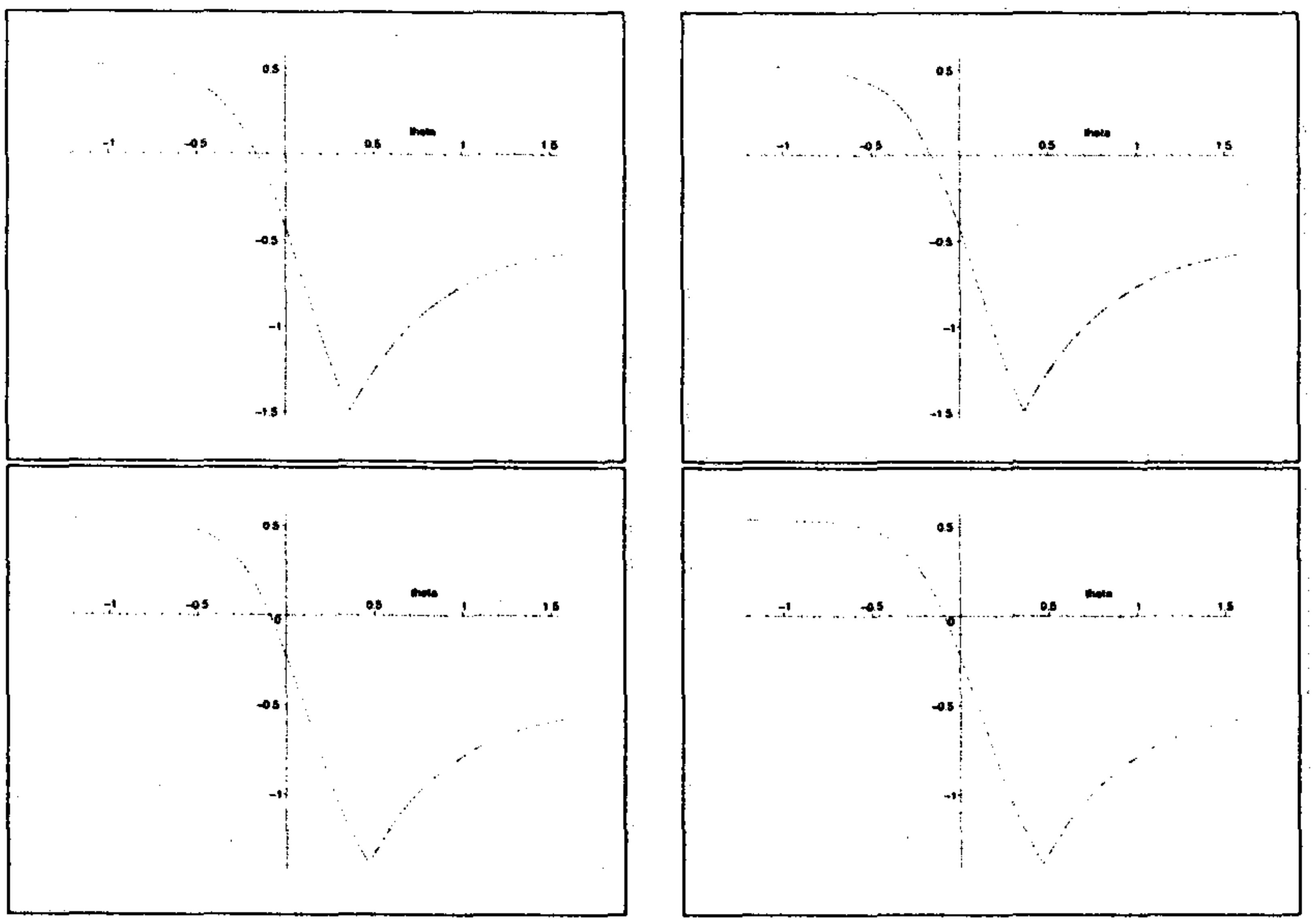

FIGURE 1. The graphs of $f(\theta)$

For each solutions to Hermite interpolation in the y-z plane, we can find θ satisfying (2) by the Newton method: −0.16179, −0.16179, −0.09007, −0.09007. See Fig. 1 and the following table:

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
1	0	-0.14051	-0.16116	-0.16179	-0.16179	-0.16179
2	0	-0.14051	-0.16116	-0.16179	-0.16179	-0.16179
3	0	-0.08328	-0.09001	-0.09007	-0.09007	-0.09007
4	0	-0.08328	-0.09001	-0.09007	-0.09007	-0.09007

Also, the energies associated with the rotation-minimizing frame for each solution

$$E = \int_0^1 \kappa(\mathbf{r}(\xi))^2 |\mathbf{r}'(\xi)| \, d\xi$$

are 32.230, 32.230, 0.127, and 8.958, where κ is the curvature of the curve. Therefore we choose the third one for the solution.

4. Concluding remark

We characterize the PH curves in the general Euclidean spaces. Upon this characterization, we have solved the first-order Hermite interpolation problem in the space for the helical polynomial space curves. The key idea comes from the fact that the projection of a helical polynomial space curve onto the plane, which is perpendicular to the direction vector of the helix, is a PH curve. Then we

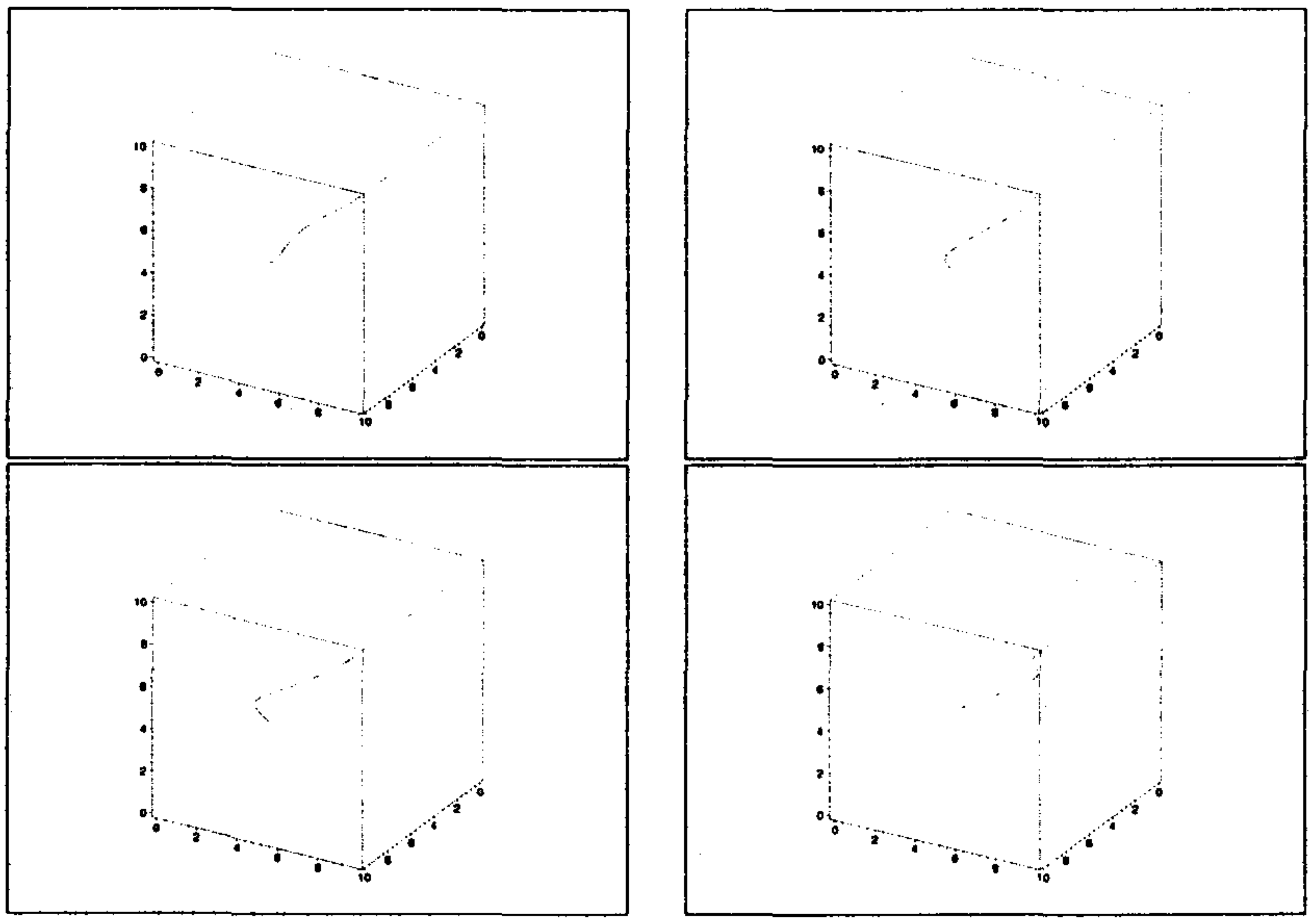

FIGURE 2. The PH quintic Hermite interpolations

make the Hermite interpolation problem in the space to that in the plane. Since the solution of the Hermite interpolation problem in the plane is well-known, we could obtain the solution in the space. The characterization of the PH curves would produce further investigations on the PH curve theory.

Appendix

Let $\hat{\mathbf{P}}_i = (\hat{p}_{iy}, \hat{p}_{iz})$, $\hat{\mathbf{D}}_i = (\hat{d}_{iy}, \hat{d}_{iz})$, and $\hat{\mathbf{P}}_f = (\hat{p}_{fy}, \hat{p}_{fz})$, $\hat{\mathbf{D}}_f = (\hat{d}_{fy}, \hat{d}_{iz})$ be data for the Hermite interpolation in the plane. From [8] we always find four solutions. The solutions are

$$\mathbf{s}(t) = \sum_{k=0}^{5} \mathbf{p}_k \binom{5}{k} (1-t)^{5-k} t^k,$$

where

$$\mathbf{p}_0 = \hat{\mathbf{P}}_i, \quad \mathbf{p}_1 = \hat{\mathbf{P}}_i + \frac{1}{5}\hat{\mathbf{D}}_i, \quad \mathbf{p}_4 = \hat{\mathbf{P}}_f - \frac{1}{5}\hat{\mathbf{D}}_f, \quad \mathbf{p}_5 = \hat{\mathbf{P}}_f,$$

and

$$\mathbf{p}_2 = \mathbf{p}_1 + \frac{1}{5}(u_0u_1 - v_0v_1, u_0v_1 + u_1v_0),$$
$$\mathbf{p}_3 = \mathbf{p}_2 + \frac{2}{15}(u_1^2 - v_1^2, 2u_1v_1) + \frac{1}{15}(u_0u_2 - v_0v_2, u_0v_2 + u_2v_0).$$

Here, u_0, u_1, u_2, v_0, v_1 and v_2 are given by

$$(u_0, v_0) = \sqrt{\frac{5}{2}}\left(\sqrt{|\Delta \mathbf{p}_0| + \Delta x_0},\ \mathrm{sign}(\Delta y_0)\sqrt{|\Delta \mathbf{p}_0| - \Delta x_0}\right),$$

$$(u_2, v_2) = \pm\sqrt{\frac{5}{2}}\left(\sqrt{|\Delta \mathbf{p}_4| + \Delta x_4},\ \mathrm{sign}(\Delta y_4)\sqrt{|\Delta \mathbf{p}_4| - \Delta x_4}\right),$$

$$(u_1, v_1) = -\frac{3}{4}(u_0 + u_2,\ v_0 + v_0) \pm \sqrt{\frac{1}{2}}(\sqrt{d+a},\ \mathrm{sign}(b)\sqrt{d-a}),$$

where $\Delta \mathbf{p}_0 = (\Delta x_0, \Delta y_0), \Delta \mathbf{p}_4 = (\Delta x_4, \Delta y_4), a, b$ and d are defined by

$$\Delta \mathbf{p}_0 = (\Delta x_0, \Delta y_0) = \mathbf{p}_1 - \mathbf{p}_0,$$
$$\Delta \mathbf{p}_4 = (\Delta x_4, \Delta y_4) = \mathbf{p}_5 - \mathbf{p}_4,$$
$$a = \frac{9}{16}(u_0^2 - v_0^2 + u_2^2 - v_2^2) + \frac{5}{8}(u_0u_2 - v_0v_2) + \frac{15}{2}(x_4 - x_1),$$
$$b = \frac{9}{8}(u_0v_0 + u_2v_2) + \frac{5}{8}(u_0v_2 + u_2v_0) + \frac{15}{2}(y_4 - y_1),$$
$$d = \sqrt{a^2 + b^2}.$$

Moreover the length S of each solution have a closed form:

$$S = \frac{5}{8}(|\Delta \mathbf{p}_0| + |\Delta \mathbf{p}_4|) - \frac{1}{12}(u_0u_2 + v_0v_2) + \frac{2}{15}d.$$

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