

# Locally Optimal and Robust Backstepping Design for Systems in Strict Feedback Form with $C^1$ Vector Fields

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**Abstract:** Due to the difficulty in solving the Hamilton-Jacobi-Isaacs equation, the nonlinear optimal control approach is not very practical in general. To overcome this problem, Ezal et al. (2000) first solved a linear optimal control problem for the linearized model of a nonlinear system given in the strict-feedback form. Then, using the backstepping procedure, a nonlinear feedback controller was designed where the linear part is same as the linear feedback obtained from the linear optimal control design. However, their construction is based on the cancellation of the high order nonlinearity, which limits the application to the smooth ( $C^\infty$ ) vector fields. In this paper, we develop an alternative method for backstepping procedure, so that the vector field can be just  $C^1$ , which allows this approach to be applicable to much larger class of nonlinear systems.

**Keywords:** Disturbance attenuation, inverse optimality, optimal control, robust control.

## 1. INTRODUCTION

In many cases including the optimal disturbance attenuation problem, it is not easy to design the optimal controller for general nonlinear systems because one has to solve the so-called Hamilton-Jacobi-Isaacs (HJI) equation or its generalized version [5,8,11]. One solution to this problem is the linearization approach; that is, to obtain the local optimal controller for the linearized system (near equilibrium) which is well-known to be solvable, provided that some assumptions, such as controllability, are satisfied [11], and to apply it to the original system. One main drawback of this approach is that the region where the optimal controller is valid can be excessively small, and in general one does not know how large (or small) it is. Thus, it might happen that the closed loop system is unstable outside the local region.

Recently, for strict-feedback systems, a new solution is provided in [3] where the authors constructed a local optimal controller and developed a

robust backstepping [4,6] guaranteeing that the closed loop system is locally optimal and globally stable. In particular, the global stability is assured by achieving the inverse optimality which is known to have desirable stability margins [7]. This solution is also extended to the output-feedback control problem that achieves local near-optimality and semiglobal inverse optimality [2].

The main objective of this paper is to alleviate the somewhat stringent assumption made in [3], that is, the *smoothness* property of vector fields. By modifying the backstepping tool developed in [3], we allow the vector fields to be  $C^1$ . The problem is meaningful since many systems have vector fields which are not necessarily smooth in the domain of interest. Note that the system should have  $C^1$  vector field, since the local optimal control for linearized system is used. Thus, our approach is not applicable to systems with  $C^0$  or discontinuous vector field, in general. The decisive factor which makes it different from [3] is that the virtual controls are selected at each step so that the nonlinearities of the system are cancelled approximately rather than exactly.

This paper is organized as follows. In Section 2, the problem is formulated, and several important facts are presented. Section 3.1 describes the nonlinear backstepping approach to design the optimal controller, and the main result follows in Section 3.2. Illustrative examples are presented in Section 4, and some concluding remarks are given in Section 5.

**Notation:** Given a matrix  $A \in R^{n \times n}$ ,  $A_{[i]}$  is a matrix consisting of the first  $i$  columns and the first  $i$  rows of  $A$ . Similarly, for a given vector  $x =$

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$[x_1 \cdots x_n]^T \in R^n$ , we define  $x_{[i]} := [x_1 \cdots x_i]^T$ . The  $n \times 1$  zero vector is denoted by  $0_n$ . For a positive definite matrix  $P = P^T > 0$ ,  $\lambda_{\min}(P)$  means the smallest eigenvalue of  $P$ .

## 2. PROBLEM STATEMENT

Consider a nonlinear system

$$\dot{x} = f(x) + G_1(x)w + B_2u$$

in strict-feedback form:

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(x_1) + g_1(x_1)w, \\ \dot{x}_2 &= x_3 + f_2(x_1, x_2) + g_2(x_1, x_2)w, \\ &\vdots \\ \dot{x}_n &= f_n(x) + g_n(x)w + u, \end{aligned} \quad (1)$$

where  $x = [x_1 \cdots x_n]^T \in R^n$  is the state,  $u \in R$  is the control input,  $w(\cdot): [0, \infty) \mapsto R^q$  is an unknown disturbance of either  $L_2$  or  $L_\infty$  and  $B_2 = [0 \cdots 0 \ 1]^T$ . The vector fields  $f_i$  and  $g_i$  are assumed to be  $C^1$  with  $f_i(0) = 0$ , and  $b_i := g_i(0) \in R^{1 \times q}$ .

The aim of this paper is to develop a recursive design procedure for a nonlinear system of the form (1) with  $C^1$  vector fields, so that a globally-defined state-feedback controller  $u = \mu(x)$  is constructed which guarantees Local Optimality and Global Inverse Optimality.

Before describing these objectives in detail, it is noted that the linear part can be extracted from (1) as

$$\dot{x} = Ax + B_1w + B_2u + f^H(x) + G_1^H(x)w, \quad (2)$$

where  $B_1 = G_1(0)$ ,  $f^H(x) = f(x) - Ax$ ,  $G_1^H(x) = G_1(x) - B_1$ , and

$$A = \frac{\partial f}{\partial x}(0) = \begin{bmatrix} a_{11} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & 1 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

The linearized dynamics of (2) is

$$\dot{x} = Ax + B_1w_l + B_2u_l, \quad (3)$$

where the subscript  $l$  identifies the local property. Note that  $(A, B_2)$  is controllable.

In addition, suppose a locally-defined cost functional is given by

$$J_l(u_l, w_l) = \int_0^\infty [x^T Qx + Ru_l^2 - \gamma^2 w_l^T w_l] dt, \quad (4)$$

where  $Q = Q^T > 0$  and  $R > 0$ . Note that  $(A, Q)$  is observable.

The dual properties that the controller  $u$  should satisfy are described as follows.

**Local Optimality:** Let  $u_l$  be the  $H_\infty$ -optimal controller in the region where the linear dynamics dominates around the origin. Equivalently,  $u_l$  is the solution of the *dynamic game*  $\min_{u_l} \max_{w_l} J_l(u_l, w_l)$  of the system (3) for the cost functional (4), that is, it minimizes the cost for the worst case disturbance  $w_l$ . Because  $(A, B_2)$  is controllable and  $(A, Q)$  is observable, it is well known that there exist the optimal disturbance attenuation level  $\gamma^* > 0$  and the unique solution  $P = P^T > 0$  to the *generalized algebraic Riccati equation* (GARE)

$$PA + A^T P + P \left( \frac{1}{\gamma^2} B_1 B_1^T - B_2 R^{-1} B_2^T \right) P + Q = 0 \quad (5)$$

for  $\gamma > \gamma^* > 0$ , which implies that the optimal controller  $u_l$  can be found with respect to the cost functional (4) for a disturbance attenuation level  $\gamma > \gamma^* > 0$ , and the value function [7,8] of the game  $\min_{u_l} \max_{w_l} J_l(u_l, w_l)$  is  $V(x) = x^T P x$ . In this case, the controller  $u_l$  is called *suboptimal  $H_\infty$  controller* [11] of the form

$$u_l = \mu_l(x) = -R^{-1} B_2^T P x$$

and the corresponding worst case disturbance is

$$w_l = v_l(x) = \frac{1}{\gamma^2} B_1^T P x.$$

Therefore, for local optimality, the controller  $u = \mu(x)$  should satisfy

$$\frac{\partial \mu}{\partial x}(0)x = -R^{-1} B_2^T P x. \quad (6)$$

**Global Inverse Optimality:** We design the controller  $u$  so that it achieves the global optimality for the original nonlinear system (1) with respect to a globally-defined cost functional

$$J(u, w) = \int_0^\infty [q(x) + r(x)u^2 - \gamma^2 w^T w] dt \quad (7)$$

for some positive definite function  $q(x)$  and strictly positive function  $r(x)$ . This is equivalent to

satisfying

$$\min_u \max_w \left[ q(x) + r(x)u^2 - \gamma^2 w^T w + \dot{V}(x) \right] = 0 \quad (8)$$

for a value function  $V(\cdot): R^n \mapsto R$ . Furthermore, for the local optimality to be meaningful, they should satisfy

$$\frac{1}{2} \frac{\partial^2 q}{\partial x^2}(0) = Q, \quad r(0) = R. \quad (9)$$

Our design relies on the robust backstepping developed in [3] where it is essential to factorize  $P$  into the form  $P = L^T \Delta L$  where

$$L := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\alpha_{11} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\alpha_{n-1,1} & \cdots & -\alpha_{n-1,n-1} & 1 \end{bmatrix},$$

$$\Delta := \text{diag}(\delta_1, \dots, \delta_n).$$

Clearly,  $L_{[k]}$  is invertible for  $1 \leq k \leq n$ . Associated with this factorization, we define the following.

$$\begin{aligned} \alpha_{[i]} &:= [\alpha_{i1} \ \cdots \ \alpha_{ii}], \quad 1 \leq i \leq n \\ \alpha_{[i]} &:= [\alpha_{i1} \ \cdots \ \alpha_{ii}], \quad 1 \leq i \leq n-1 \\ \alpha_{[0]} &:= 0 \quad \alpha_{[n]} := 0_n^T \\ \bar{\alpha}_{[i]} &:= \alpha_{[i]} L_{[i]}^{-1}, \quad L_{[i]}^{-1} := (L^{-1})_{[i]} = (L_{[i]})^{-1}, \quad 1 \leq i \leq n. \end{aligned}$$

Some important properties related to this factorization are recalled below. See [3] for details.

**Lemma 1:** Under the linear transformation  $z = Lx$ ,

1. For  $1 \leq k \leq n$ ,  $z_{[k]} = L_{[k]} x_{[k]}$ .
2. Let  $\bar{A} = LAL^{-1}$  and  $\bar{B}_1 = LB_1$ . Then,  $\bar{A}$  has the same structure as  $A$ , that is

$$\bar{A} = \begin{bmatrix} \bar{a}_{11} & 1 & 0 & \cdots & 0 \\ \bar{a}_{21} & \bar{a}_{22} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n-1,1} & \bar{a}_{n-1,2} & \bar{a}_{n-1,3} & \cdots & 1 \\ \bar{a}_{n1} & \bar{a}_{n2} & \bar{a}_{n3} & \cdots & \bar{a}_{nn} \end{bmatrix}$$

and the linearized dynamics of  $z_{[k]}$  becomes

$$\dot{z}_{[k]} = \bar{A}_{[k]} z_{[k]} + \begin{bmatrix} 0_{k-1} \\ z_{k+1} \end{bmatrix} + \bar{B}_{1[k]} w_l, \quad 1 \leq k < n$$

where  $\bar{B}_1 = [\bar{b}_1^T \ \cdots \ \bar{b}_n^T]^T$  and  $\bar{B}_{1[k]}$  is defined by

$$\bar{B}_{1[k]} = (\bar{B}_1)_{[k]} := [\bar{b}_1^T \ \cdots \ \bar{b}_k^T]^T.$$

3. Let  $\bar{a}_{[i]} := [\bar{a}_{i1} \ \cdots \ \bar{a}_{ii}]$  for  $1 \leq i \leq n$ , then

$$\begin{aligned} \bar{a}_{[i]} &= \alpha_{[i]} L_{[i]}^{-1} + \bar{\alpha}_{[i]} - [\bar{\alpha}_{[i-1]} \bar{A}_{[i-1]} \quad \bar{\alpha}_{i-1,i-1}] \\ \bar{b}_i &= b_i - \bar{\alpha}_{[i-1]} \bar{B}_{1[i-1]}. \end{aligned} \quad (10)$$

4. Let  $\bar{Q} = (L^T)^{-1} Q L^{-1}$ , then the GARE (5) is modified in  $z$ -coordinates as, for  $1 \leq k < n$

$$\begin{aligned} 2z_{[k]}^T \Delta_{[k]} \bar{A}_{[k]} z_{[k]} &= -z_{[k]}^T \bar{Q}_{[k]} z_{[k]} - \gamma^2 \bar{v}_{lk}^T \bar{v}_{lk} \\ 2z^T \Delta \bar{A} z &= -z^T \bar{Q} z - \gamma^2 \bar{v}_l^T \bar{v}_l + R^{-1} \delta_n^2 z_n^2, \end{aligned} \quad (11)$$

where  $\bar{v}_{lk}(z_{[k]}) := \frac{1}{\gamma^2} \bar{B}_{1[k]}^T \Delta_{[k]} z_{[k]}$  and  $\bar{v}_l(z) := \frac{1}{\gamma^2} \bar{B}_1^T \Delta z$ . This property implies that  $\bar{A}_{[k]}$  is Hurwitz for  $1 \leq k < n$ , and the value function associated with the dynamic game  $\min_{u_l} \max_{w_l} J_l(u_l, w_l)$  is  $\bar{V}(z) = z^T \Delta z$  which is the same form as the one in  $x$ -coordinates.

5. For  $1 < k \leq n$ ,  $\bar{v}_{lk}(z_{[k]}) = \bar{v}_{l,k-1}(z_{[k-1]}) + \frac{1}{\gamma^2} \bar{b}_k^T \delta_k z_k$ , and  $\bar{v}_{l1}(z_1) = \frac{1}{\gamma^2} \bar{b}_1^T \delta_1 z_1$ .

During the derivation, we need second-order derivatives of a  $C^1$  function multiplied by another  $C^1$  function. The following result is a tool regarding this.

**Lemma 2:** Let  $F(\cdot), G(\cdot): R^n \rightarrow R$  be  $C^1$  functions with  $F(0) = G(0) = 0$  and  $\frac{\partial F}{\partial z_i}(0) = 0$ ,  $1 \leq i \leq n$ . Then,

$$\left. \frac{\partial^2}{\partial z_j \partial z_i} [F(z)G(z)] \right|_{z=0} = 0, \quad 1 \leq i, j \leq n.$$

**Proof:** Since  $F, G \in C^1$ , one has

$$\frac{\partial}{\partial z_i} [F(z)G(z)] = \frac{\partial F}{\partial z_i}(z)G(z) + F(z) \frac{\partial G}{\partial z_i}(z).$$

Define  $F_G^i(z) := \frac{\partial F}{\partial z_i}(z)G(z)$  and  $G_F^i(z) := F(z) \frac{\partial G}{\partial z_i}(z)$ .

Note that the property  $F(0) = G(0) = 0$  guarantees the existence of the continuous functions  $F_k(\cdot), G_k(\cdot): R^n \mapsto R$ ,  $1 \leq k \leq n$ , such that

$$\begin{aligned} F(z) &= F_1(z)z_1 + \cdots + F_n(z)z_n, \\ G(z) &= G_1(z)z_1 + \cdots + G_n(z)z_n, \end{aligned}$$

where  $F_k(z) = \int_0^1 \frac{\partial}{\partial z_k} F(\tau z) d\tau$ ,  $G_k(z) = \int_0^1 \frac{\partial}{\partial z_k} G(\tau z) d\tau$ . Note that  $F_1(0) = \dots = F_n(0) = 0$ , since  $\frac{\partial F}{\partial z_k}(0) = 0$ . Let  $e_j$  be the  $j$ th elementary basis in  $R^n$ , then

$$\frac{\partial F_G^i}{\partial z_j}(0) = \lim_{\varepsilon \rightarrow 0} \frac{F_G^i(\varepsilon e_j) - F_G^i(0)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{F_G^i(\varepsilon e_j)}{\varepsilon}.$$

Since  $F_G^i(\varepsilon e_j) = \frac{\partial F}{\partial z_i}(\varepsilon e_j)G(\varepsilon e_j)$ , it follows that

$$F_G^i(\varepsilon e_j) = \left[ \frac{\partial F_j}{\partial z_i}(\varepsilon e_j)\varepsilon + F_i(\varepsilon e_j) \right] G_j(\varepsilon e_j)\varepsilon, \text{ and that } \frac{\partial F_G^i}{\partial z_j}(0) = 0. \text{ Similarly, one has } \frac{\partial G_F^i}{\partial z_j}(0) = 0. \text{ Thus,}$$

the assertion follows.  $\square$

### 3. LOCALLY OPTIMAL AND GLOBALLY INVERSE OPTIMAL CONTROLLER

One solution to achieve two goals, local optimality (6) and global inverse optimality (8) and (9), is the robust nonlinear backstepping procedure which provides a flexible design framework via the appropriate choices of the virtual controls and the control Lyapunov function (*clf*, [1]). As mentioned above, the overall flow to solve the problem is similar to that of [3]. That is to say, the virtual controls are selected properly at each step such that the backstepping design results in a nonlinear system whose linearized dynamics near the origin become

$$\dot{z} = \bar{A}z + \bar{B}_1 w_1 + B_2 u_1 \quad (12)$$

in  $z$ -coordinates where  $z = Lx$ . On the other hand, the *clf* is chosen as  $\bar{V}(z) = z^T \Delta z$  which is the value function of the dynamic game for the linearized system (12). Using the virtual controls, we construct a global diffeomorphism  $z = \Phi(x)$ . With  $\Phi$  and the *clf*  $\bar{V}(x)$ , the optimal controller is designed and furthermore, it will be shown that the *clf* happens to be the value function of a game for the original nonlinear system. Section 3.1 focuses on the nonlinear backstepping which successively constructs a global diffeomorphism  $z = \Phi(x)$ , and the design of optimal controller is described in Section 3.2.

#### 3.1. Nonlinear backstepping

Conventional backstepping relies on cancelling the nonlinearities, which requires the smoothness of vector fields. However, note that the backstepping approach provides lots of flexibility during the design step. In this subsection, keeping this advantage in mind, we relax the smoothness assumption by

choosing new virtual controls and a *clf*. The main idea is to cancel the nonlinearities approximately rather than exactly.

Define

$$\delta := \max_{1 \leq i \leq n} \{\delta_i\}, \quad \varepsilon^* := \frac{\lambda_{\min}(\bar{Q})}{(n+1)\delta} \quad (13)$$

and choose  $\varepsilon > 0$  such that  $\varepsilon < \varepsilon^*$ .

**Step 1:** Let  $z_1 = \varphi_1(x_1) := x_1$ . Then, dynamics of the subsystem  $z_{[1]}$  is obtained as

$$\begin{aligned} \dot{z}_1 &= a_{11}x_1 + x_2 + f_1^H(x_1) + g_1(x_1)w \\ &= a_{[1]}z_{[1]} + x_2 + f_1^H(z_{[1]}) + \bar{g}_1(z_{[1]})w \end{aligned}$$

in which  $\bar{g}_1 := g_1$ . The virtual control  $\bar{\alpha}_1$  for  $x_2$  is chosen as  $\bar{\alpha}_1(z_{[1]}) = \bar{\alpha}_{[1]}z_{[1]} + \bar{\alpha}_1^H(z_{[1]})$  where  $\bar{\alpha}_1^H(\cdot)$  is to be defined later. Using (10), the  $z_1$ -dynamics is derived as

$$\begin{aligned} \dot{z}_1 &= a_{[1]}z_{[1]} + \bar{\alpha}_1(z_{[1]}) + f_1^H(z_{[1]}) \\ &\quad + (x_2 - \bar{\alpha}_1(z_{[1]})) + \bar{g}_1(z_{[1]})w \\ &= \bar{a}_{[1]}z_{[1]} + (x_2 - \bar{\alpha}_1(z_{[1]})) + \bar{f}_1^H(z_{[1]}) + \bar{g}_1(z_{[1]})w, \end{aligned}$$

where

$$\begin{aligned} \bar{f}_1^H(z_{[1]}) &:= \bar{\alpha}_1^H(z_{[1]}) + \rho_1^H(z_{[1]}), \\ \bar{\rho}_1^H(z_{[1]}) &:= f_1^H(z_{[1]}). \end{aligned}$$

By (11) and by choosing the value function as  $\bar{V}_1(z_{[1]}) = z_{[1]}^T \Delta_{[1]} z_{[1]}$ , the time derivative of  $\bar{V}_1$  along the trajectory of (1) becomes

$$\begin{aligned} \dot{\bar{V}}_1 &= 2z_{[1]}^T \Delta_{[1]} [\bar{a}_{[1]}z_{[1]} + \bar{f}_1^H(z_{[1]}) \\ &\quad + (x_2 - \bar{\alpha}_1(z_{[1]})) + \bar{g}_1(z_{[1]})w] \\ &= -z_{[1]}^T \bar{Q}_{[1]} z_{[1]} + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_1|^2 \\ &\quad + 2z_1 \delta_1 [x_2 - \bar{\alpha}_1 + \bar{\alpha}_1^H(z_{[1]}) + \rho_1^H(z_{[1]}) + \bar{h}_1^H(z_{[1]})] \end{aligned}$$

where  $\bar{v}_1(z_{[1]}) := \frac{1}{\gamma^2} \bar{g}_1^T(z_{[1]}) \delta_1 z_1 = \frac{1}{\gamma^2} \bar{G}_{1[1]}^T(z_{[1]}) \Delta_{[1]} z_{[1]}$

and  $\bar{h}_1^H(z_{[1]}) := \frac{1}{2\gamma^2} (\bar{g}_1 \bar{g}_1^T - \bar{b}_1 \bar{b}_1^T) \Delta_{[1]} z_{[1]}$ . Since  $\bar{\rho}_1^H$

and  $\bar{h}_1^H$  are  $C^1$  functions vanishing at the origin, a continuous function  $\bar{\kappa}_{11}$  given by

$$\bar{\kappa}_{11}(z_{[1]}) = \int_0^1 \frac{\partial}{\partial z_1} [\bar{\rho}_1^H(\tau z_{[1]}) + \bar{h}_1^H(\tau z_{[1]})] d\tau$$

satisfies  $\bar{\rho}_1^{-H}(z_{[1]}) + \bar{h}_1^{-H}(z_{[1]}) = \bar{\kappa}_{11}(z_{[1]})z_1$ . Moreover, it holds that  $\frac{\partial}{\partial z_{[1]}}(\bar{\rho}_1^{-H} + \bar{h}_1^{-H})(0) = 0$ , since

$$\begin{aligned}\frac{\partial \bar{\rho}_1^{-H}}{\partial z_{[1]}}(0) &= \frac{\partial f_1^H}{\partial z_{[1]}}(0) = 0 \\ \frac{\partial \bar{h}_1^{-H}}{\partial z_{[1]}}(0) &= \frac{1}{2\gamma^2} \left( \bar{g}_1(0)\bar{g}_1^{-T}(0) - \bar{b}_1\bar{b}_1^T \right) \Delta_{[1]} = 0\end{aligned}$$

where  $\bar{g}_1(0) = g_1(0) = \bar{b}_1$  is used. Hence,  $\bar{\kappa}_{11}(0) = 0$ .

Now, choose a smooth function  $\bar{\sigma}_{11}(\cdot)$  such that

$$|\bar{\sigma}_{11}(z_{[1]}) - \bar{\kappa}_{11}(z_{[1]})| \leq \varepsilon, \quad \bar{\sigma}_{11}(0) = 0,$$

and select  $\bar{\alpha}_1^{-H}(z_{[1]})$  as  $\bar{\alpha}_1^{-H}(z_{[1]}) := -\bar{\sigma}_{11}(z_{[1]})z_1$ .

Then, it follows that  $\bar{f}_1^{-H}(0) = 0$ ,  $\frac{\partial \bar{f}_1^{-H}}{\partial z_{[1]}}(0) = 0$  since

$$\bar{\alpha}_1^{-H}(0) = 0 \quad \text{and} \quad \frac{\partial \bar{\alpha}_1^{-H}}{\partial z_{[1]}}(0) = 0.$$

Thus, we have

$$\begin{aligned}\dot{\bar{V}}_1 &= -\bar{q}_1(z_{[1]}) + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_1|^2 \\ &\quad + 2z_1 \delta_1 (x_2 - \bar{\alpha}_1(z_{[1]})),\end{aligned}$$

where the function  $\bar{q}_1(z_{[1]})$  is defined by

$$\begin{aligned}\bar{q}_1(z_{[1]}) &:= z_{[1]}^T \bar{Q}_{[1]} z_{[1]} + \bar{q}_1^{-H}(z_{[1]}) \\ \bar{q}_1^{-H}(z_{[1]}) &:= -2z_1 \delta_1 \left[ \bar{\alpha}_1^{-H}(z_{[1]}) + \bar{\rho}_1^{-H}(z_{[1]}) + \bar{h}_1^{-H}(z_{[1]}) \right].\end{aligned}$$

It is easy to check that  $\frac{\partial^2 \bar{q}_1^{-H}}{\partial (z_{[1]})^2}(0) = 0$  which is guaranteed by Lemma 2, and it entails  $\frac{1}{2} \frac{\partial^2 \bar{q}_1}{\partial (z_{[1]})^2}(0) = \bar{Q}_{[1]}$

Note that  $\bar{q}_1^{-H}$  can be written as

$$\bar{q}_1^{-H}(z_{[1]}) = z_{[1]}^T \left[ \Delta_{[1]} \bar{\Pi}_{[1]}(z_{[1]}) + \bar{\Pi}_{[1]}^T(z_{[1]}) \Delta_{[1]} \right] z_{[1]}$$

where  $\bar{\Pi}_{[1]}(z_{[1]}) := \bar{\pi}_{11}(z_{[1]}) = \bar{\sigma}_{11}(z_{[1]}) - \bar{\kappa}_{11}(z_{[1]})$  and that  $|\bar{\pi}_{11}(z_{[1]})| \leq \varepsilon$ . It can be shown that the function

$$\bar{q}_1(z_{[1]}) = z_{[1]}^T \left[ \bar{Q}_{[1]} + \Delta_{[1]} \bar{\Pi}_{[1]} + \bar{\Pi}_{[1]}^T \Delta_{[1]} \right] z_{[1]}$$

is positive definite and radially unbounded with respect to  $z_{[1]}$ . Indeed, from the fact that  $\bar{Q}_{[1]} \geq \lambda_{\min}(\bar{Q})$ , the claim is proved since for

$z_{[1]} \neq 0$ ,

$$\begin{aligned}\bar{q}_1(z_{[1]}) &\geq (\bar{Q}_{[1]} - 2\delta_1 \varepsilon) \|z_{[1]}\|^2 \\ &> (\lambda_{\min}(\bar{Q}) - \delta \varepsilon^*(n+1)) \|z_{[1]}\|^2 = 0.\end{aligned}$$

Hence,  $\dot{\bar{V}}_1$  satisfies the desired dissipation inequality

$$\begin{aligned}\dot{\bar{V}}_1 &= -\bar{q}_1(z_{[1]}) + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_1|^2 \\ &\leq -\bar{q}_1(z_{[1]}) + \gamma^2 w^T w\end{aligned}\tag{14}$$

when  $x_2 \equiv \bar{\alpha}_1(z_{[1]})$ . Therefore, the subsystem  $z_{[1]}$  is stabilized. Indeed,  $z_{[1]}(t) \rightarrow 0$  as  $t \rightarrow \infty$  for  $w(t) \in L_2$  [10] and  $z_{[1]}(t) \in L_\infty$  for  $w(t) \in L_\infty$ . Moreover, the subsystem has the globally exponentially stable equilibrium  $z_{[1]} = 0$  in the case of  $w(t) \equiv 0$ .

**Inductive Assumption:** Suppose there exists a coordinate transformation denoted by

$$z_{[i-1]} = \Phi_{[i-1]}(x_{[i-1]}) := \left[ \varphi_1(x_{[1]}) \quad \cdots \quad \varphi_{i-1}(x_{[i-1]}) \right]^T$$

such that the  $x_{[i-1]}$ -dynamics in new coordinates becomes

$$\begin{aligned}\dot{z}_{[i-1]} &= \bar{A}_{[i-1]} z_{[i-1]} + \begin{bmatrix} 0_{i-2} \\ z_i \end{bmatrix} \\ &\quad + \bar{f}_{[i-1]}^H(z_{[i-1]}) + \bar{G}_{[i-1]}(z_{[i-1]}) w\end{aligned}$$

where

$$\begin{aligned}\bar{G}_{[i-1]}(z_{[i-1]}) &= (\bar{G}_1)_{[i-1]}(z_{[i-1]}) \\ &:= \left[ \bar{g}_1(z_{[1]}) \quad \cdots \quad \bar{g}_{i-1}(z_{[i-1]}) \right]^T\end{aligned}\tag{15}$$

with  $\bar{G}_{[i-1]}(0) = \bar{B}_{[i-1]}$ , and  $\bar{f}_{[i-1]}^H = \left[ \bar{f}_1^H \quad \cdots \quad \bar{f}_{i-1}^H \right]^T$

with  $\bar{f}_{[i-1]}^H(0) = 0$ ,  $\frac{\partial \bar{f}_{[i-1]}^H}{\partial z_{[i-1]}}(0) = 0$ . It is obvious that

$z_{[i-1]} = \Phi_{[i-1]}(x_{[i-1]})$  is a global diffeomorphism.

In addition, suppose that

$$\begin{aligned}\dot{\bar{V}}_{i-1} &= -\bar{q}_{i-1}(z_{[i-1]}) + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_{i-1}|^2 \\ &\quad + 2z_{i-1} \delta_{i-1} z_i\end{aligned}$$

where  $\bar{q}_{i-1}$  and  $\bar{v}_{i-1}$  are given by

$$\begin{aligned}\bar{q}_{i-1}(z_{[i-1]}) &:= z_{[i-1]}^T \bar{Q}_{[i-1]} z_{[i-1]} + \bar{q}_{i-1}^{-H}(z_{[i-1]}) \\ \bar{v}_{i-1}(z_{[i-1]}) &:= \frac{1}{\gamma^2} \bar{G}_{[i-1]}^T(z_{[i-1]}) \Delta_{[i-1]} z_{[i-1]},\end{aligned}$$

and that  $\bar{q}_{i-1}^{-H}$  satisfies  $\frac{\partial^2 \bar{q}_{i-1}^{-H}}{\partial (z_{[i-1]})^2}(0) = 0$  and has the following structure

$$\begin{aligned} & \bar{q}_{i-1}^{-H}(z_{[i-1]}) \\ & := z_{[i-1]}^T \left[ \Delta_{[i-1]} \bar{\Pi}_{[i-1]}(z_{[i-1]}) + \bar{\Pi}_{[i-1]}^T(z_{[i-1]}) \Delta_{[i-1]} \right] z_{[i-1]} \\ & \bar{\Pi}_{[i-1]}(z_{[i-1]}) \\ & := \begin{bmatrix} \bar{\pi}_{11}(z_{[1]}) & 0 & \cdots & 0 \\ \bar{\pi}_{21}(z_{[2]}) & \bar{\pi}_{22}(z_{[2]}) & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\pi}_{i-1,1}(z_{[i-1]}) & \bar{\pi}_{i-1,2}(z_{[i-1]}) & \cdots & \bar{\pi}_{i-1,i-1}(z_{[i-1]}) \end{bmatrix} \end{aligned} \quad (16)$$

with  $|\bar{\pi}_{jk}(z_{[j]})| \leq \varepsilon < \varepsilon^*$ ,  $\forall z_{[j]} \in R^j$ ,  $1 \leq k \leq j \leq i-1$ .

Finally, suppose that the  $(i-1)$ th virtual control can be constructed of the form

$$\bar{\alpha}_{i-1}(z_{[i-1]}) = \bar{\alpha}_{[i-1]} z_{[i-1]} + \bar{\alpha}_{i-1}^{-H}(z_{[i-1]})$$

with  $\bar{\alpha}_{i-1}^{-H}(0) = 0$  and  $\frac{\partial \bar{\alpha}_{i-1}^{-H}}{\partial z_{[i-1]}}(0) = 0$ .

**Step  $i$  ( $i < n$ ):** Let  $z_i = \varphi_i(x_{[i]}) := x_i - \bar{\alpha}_{i-1}(z_{[i-1]})$ ,

and define  $z_{[i]} = \Phi_{[i]}(x_{[i]}) := \begin{bmatrix} \Phi_{[i-1]}(x_{[i-1]}) \\ \varphi_i(x_{[i]}) \end{bmatrix}$ .

By the inductive assumption, we have

$$\begin{aligned} \dot{z}_i &= x_{i+1} + f_i(x_{[i]}) + g_i(x_{[i]})w - \frac{\partial \bar{\alpha}_{i-1}}{\partial z_{[i-1]}} \dot{z}_{[i-1]} \\ &= x_{i+1} + a_{[i]} \Phi_{[i]}^{-1}(z_{[i]}) + f_i^H(\Phi_{[i]}^{-1}(z_{[i]})) \\ & \quad + g_i(\Phi_{[i]}^{-1}(z_{[i]}))w - \frac{\partial \bar{\alpha}_{i-1}}{\partial z_{[i-1]}} \dot{z}_{[i-1]}. \end{aligned}$$

Thus,

$$\begin{aligned} \dot{z}_i &= (x_{i+1} - \bar{\alpha}_i(z_{[i]})) + \bar{\alpha}_i(z_{[i]}) + a_{[i]} L_{[i]}^{-1} z_{[i]} \\ & \quad + a_{[i]} \left( \Phi_{[i]}^{-1}(z_{[i]}) - L_{[i]}^{-1} z_{[i]} \right) \\ & \quad + f_i^H(\Phi_{[i]}^{-1}(z_{[i]})) + g_i(\Phi_{[i]}^{-1}(z_{[i]}))w \\ & \quad - \frac{\partial \bar{\alpha}_{i-1}}{\partial z_{[i-1]}} \left( \bar{A}_{[i-1]} z_{[i-1]} + \begin{bmatrix} 0_{i-2} \\ z_i \end{bmatrix} \right) \\ & \quad + \bar{f}_{[i-1]}^H(z_{[i-1]}) + \bar{G}_{[i-1]}(z_{[i-1]})w. \end{aligned}$$

If the virtual control  $\bar{\alpha}_i(z_{[i]})$  for  $x_{i+1}$  is chosen as

$$\bar{\alpha}_i(z_{[i]}) = \bar{\alpha}_{[i]} z_{[i]} + \bar{\alpha}_i^{-H}(z_{[i]}),$$

where  $\bar{\alpha}_i^{-H}$  will be chosen later,  $\dot{z}_i$  is reduced to (by (10))

$$\dot{z}_i = \bar{a}_{[i]} z_{[i]} + (x_{i+1} - \bar{\alpha}_i(z_{[i]})) + \bar{f}_i^H(z_{[i]}) + \bar{g}_i(z_{[i]})w,$$

where

$$\bar{f}_i^H(z_{[i]}) := \bar{\alpha}_i^{-H}(z_{[i]}) + \bar{\rho}_i^H(z_{[i]})$$

$$\begin{aligned} \bar{\rho}_i^H(z_{[i]}) &:= a_{[i]} \left( \Phi_{[i]}^{-1}(z_{[i]}) - L_{[i]}^{-1} z_{[i]} \right) + f_i^H(\Phi_{[i]}^{-1}(z_{[i]})) \\ & \quad - \frac{\partial \bar{\alpha}_{i-1}^{-H}}{\partial z_{[i-1]}} \left( \bar{A}_{[i-1]} z_{[i-1]} + \begin{bmatrix} 0_{i-2} \\ z_i \end{bmatrix} \right) \\ & \quad - \frac{\partial \bar{\alpha}_{i-1}^{-H}}{\partial z_{[i-1]}} \bar{f}_{[i-1]}^H(z_{[i-1]}) \end{aligned}$$

$$\bar{g}_i(z_{[i]}) := g_i(\Phi_{[i]}^{-1}(z_{[i]})) - \frac{\partial \bar{\alpha}_{i-1}^{-H}}{\partial z_{[i-1]}} \bar{G}_{[i-1]}(z_{[i-1]}).$$

Note that  $\bar{\rho}_i^H(0) = 0$  and  $\frac{\partial \bar{\rho}_i^H}{\partial z_j}(0) = 0$ ,  $1 \leq j \leq i$ .

Indeed, consider the identity:

$$\begin{aligned} \frac{\partial \bar{\rho}_i^H}{\partial z_{[i]}}(z_{[i]}) &= a_{[i]} \left( \frac{\partial \Phi_{[i]}^{-1}}{\partial z_{[i]}}(z_{[i]}) - L_{[i]}^{-1} \right) + \frac{\partial f_i^H}{\partial z_{[i]}}(\Phi_{[i]}^{-1}(z_{[i]})) \\ & \quad - \left( \bar{A}_{[i-1]} z_{[i-1]} + \begin{bmatrix} 0_{i-2} \\ z_i \end{bmatrix} \right)^T \left[ \frac{\partial}{\partial z_{[i]}} \left( \frac{\partial \bar{\alpha}_{i-1}^{-H}}{\partial z_{[i-1]}} \right)^T \right] \\ & \quad - \frac{\partial \bar{\alpha}_{i-1}^{-H}}{\partial z_{[i-1]}} \left\{ \frac{\partial}{\partial z_{[i]}} \left( \bar{A}_{[i-1]} z_{[i-1]} + \begin{bmatrix} 0_{i-2} \\ z_i \end{bmatrix} \right) \right\} \\ & \quad - \left[ \bar{f}_{[i-1]}^H(z_{[i-1]}) \right]^T \left[ \frac{\partial}{\partial z_{[i]}} \left( \frac{\partial \bar{\alpha}_{i-1}^{-H}}{\partial z_{[i-1]}} \right)^T \right] \\ & \quad - \frac{\partial \bar{\alpha}_{i-1}^{-H}}{\partial z_{[i-1]}} \left( \frac{\partial \bar{f}_{[i-1]}^H}{\partial z_{[i]}}(z_{[i-1]}) \right). \end{aligned}$$

By the inductive assumption, the assertion follows since

$$\begin{aligned} \frac{\partial \bar{\rho}_i^H}{\partial z_{[i]}}(0) &= a_{[i]} \left( \left[ \frac{\partial \Phi_{[i]}^{-1}}{\partial x_{[i]}}(x_{[i]}) \right]^{-1} \Big|_{x_{[i]}=0} - L_{[i]}^{-1} \right) \\ & \quad + \frac{\partial f_i^H}{\partial z_{[i]}}(\Phi_{[i]}^{-1}(z_{[i]})) \Big|_{z_{[i]}=0} \quad (17) \\ &= \frac{\partial f_i^H}{\partial x_{[i]}}(x_{[i]}) \Big|_{x_{[i]}=0} \left[ \frac{\partial \Phi_{[i]}^{-1}}{\partial z_{[i]}}(z_{[i]}) \right] \Big|_{z_{[i]}=0} \\ &= 0_i^T. \end{aligned}$$

As a result, the dynamics of  $z_{[i]}$  is deduced as

$$\dot{z}_{[i]} = \bar{A}_{[i]}z_{[i]} + \bar{f}_{[i]}^H(z_{[i]}) + \bar{G}_{1[i]}(z_{[i]})w + \begin{bmatrix} 0_{i-1} \\ x_{i+1} - \bar{\alpha}_i(z_{[i]}) \end{bmatrix},$$

where  $\bar{G}_{1[i]}(z_{[i]}) := \begin{bmatrix} \bar{G}_{1[i-1]}(z_{[i-1]}) \\ \bar{g}_i(z_{[i]}) \end{bmatrix}$ , which preserves the structure of (15) and satisfies  $\bar{G}_{1[i]}(0) = \bar{B}_{1[i]}$  since, by (10), it holds that

$$\begin{aligned} \bar{g}_i(0) &= g_i(0) - \frac{\partial \alpha_{i-1}(0)}{\partial z_{[i-1]}} \bar{G}_{1[i-1]}(0) \\ &= b_i - \bar{\alpha}_{[i-1]} \bar{B}_{1[i-1]} = \bar{b}_i. \end{aligned}$$

Define  $\bar{V}_i(z_{[i]}) = \bar{V}_{i-1}(z_{[i-1]}) + \delta_i z_i^2 = z_{[i]}^T \Delta_{[i]} z_{[i]}$ . Then, the time derivative of  $\bar{V}_i$  along the trajectory of (1) is

$$\begin{aligned} \dot{\bar{V}}_i &= \dot{\bar{V}}_{i-1} + 2\delta_i z_i \dot{z}_i \\ &= -\bar{q}_{i-1}(z_{[i-1]}) \\ &\quad + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_{i-1}|^2 + 2z_{i-1} \delta_{i-1} z_i \\ &\quad + 2\delta_i z_i \left[ \bar{a}_{[i]} z_{[i]} + (x_{i+1} - \bar{\alpha}_i(z_{[i]})) \right. \\ &\quad \left. + \bar{f}_i^H(z_{[i]}) + \bar{g}_i(z_{[i]})w \right] \\ &= -z_{[i-1]}^T \bar{Q}_{[i-1]} z_{[i-1]} + 2z_{i-1} \delta_{i-1} z_i + 2\delta_i z_i \bar{a}_{[i]} z_{[i]} \\ &\quad + 2\gamma^2 \bar{v}_i^T w - \gamma^2 \bar{v}_{i-1}^T \bar{v}_{i-1} - \bar{q}_{i-1}^H(z_{[i-1]}) \\ &\quad + 2\delta_i z_i \left[ x_{i+1} - \bar{\alpha}_i(z_{[i]}) + \bar{f}_i^H(z_{[i]}) \right] \end{aligned} \quad (18)$$

where

$$\begin{aligned} \bar{v}_i(z_{[i]}) &= \bar{v}_{i-1}(z_{[i-1]}) + \frac{1}{\gamma^2} \bar{g}_i^T(z_{[i]}) \delta_i z_i \\ &= \frac{1}{\gamma^2} \bar{G}_{1[i]}^T(z_{[i]}) \Delta_{[i]} z_{[i]}. \end{aligned}$$

By applying (11) with some matrix algebra, one has

$$\begin{aligned} &-z_{[i-1]}^T \bar{Q}_{[i-1]} z_{[i-1]} + 2z_{i-1} \delta_{i-1} z_i + 2\delta_i z_i \bar{a}_{[i]} z_{[i]} \\ &= 2z_{[i-1]}^T \Delta_{[i-1]} \bar{A}_{[i-1]} z_{[i-1]} + 2z_{i-1} \delta_{i-1} z_i \\ &\quad + 2\delta_i z_i \bar{a}_{[i]} z_{[i]} + \gamma^2 \bar{v}_{l,i-1}^T \bar{v}_{l,i-1} \\ &= 2z_{[i]}^T \Delta_{[i]} \bar{A}_{[i]} z_{[i]} + \gamma^2 \bar{v}_{l,i-1}^T \bar{v}_{l,i-1} \\ &= -z_{[i]}^T \bar{Q}_{[i]} z_{[i]} - \gamma^2 (\bar{v}_{li}^T \bar{v}_{li} - \bar{v}_{l,i-1}^T \bar{v}_{l,i-1}). \end{aligned}$$

Furthermore, substituting this result into (18) and completing the squares with respect to  $w$  yield

$$\begin{aligned} \dot{\bar{V}}_i &= -z_{[i]}^T \bar{Q}_{[i]} z_{[i]} + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_i|^2 \\ &\quad + \gamma^2 (\bar{v}_i^T \bar{v}_i - \bar{v}_{i-1}^T \bar{v}_{i-1}) - \gamma^2 (\bar{v}_{li}^T \bar{v}_{li} - \bar{v}_{l,i-1}^T \bar{v}_{l,i-1}) \\ &\quad - \bar{q}_{[i-1]}^H(z_{[i-1]}) + 2z_i \delta_i \left[ x_{i+1} - \bar{\alpha}_i(z_{[i]}) + \bar{f}_i^H(z_{[i]}) \right]. \end{aligned}$$

If one recalls the definitions of  $\bar{v}_i$ ,  $\bar{v}_{i-1}$ ,  $\bar{v}_{l,i}$  and  $\bar{v}_{l,i-1}$ , then it follows that

$$\begin{aligned} &\gamma^2 (\bar{v}_i^T \bar{v}_i - \bar{v}_{i-1}^T \bar{v}_{i-1}) - \gamma^2 (\bar{v}_{li}^T \bar{v}_{li} - \bar{v}_{l,i-1}^T \bar{v}_{l,i-1}) \\ &= 2z_i \delta_i \bar{v}_{i-1}^T \bar{g}_i + \frac{1}{\gamma^2} z_i \delta_i \bar{g}_i^T \bar{g}_i \delta_i z_i \\ &\quad - (2z_i \delta_i \bar{v}_{l,i-1}^T \bar{b}_i + \frac{1}{\gamma^2} z_i \delta_i \bar{b}_i^T \bar{b}_i \delta_i z_i) \\ &= \frac{1}{\gamma^2} z_i \delta_i (\bar{g}_i^T \bar{g}_i - \bar{b}_i^T \bar{b}_i) \delta_i z_i + 2z_i \delta_i (\bar{g}_i^T \bar{v}_{i-1} - \bar{b}_i^T \bar{v}_{l,i-1}) \\ &= 2z_i \delta_i \bar{h}_i^H(z_{[i]}), \end{aligned}$$

where

$$\begin{aligned} \bar{h}_i^H(z_{[i]}) &:= \frac{1}{\gamma^2} \left[ \bar{g}_i^T \bar{G}_{1[i-1]} - \bar{b}_i^T \bar{B}_{1[i-1]} \quad \frac{1}{2} (\bar{g}_i^T \bar{g}_i - \bar{b}_i^T \bar{b}_i) \right] \\ &\quad \times \Delta_{[i]} z_{[i]}. \end{aligned}$$

It can be easily shown that  $\bar{h}_i^H(0) = 0$  and  $\frac{\partial \bar{h}_i^H}{\partial z_j}(0) = 0$ ,  $1 \leq j \leq i$ . Thus,  $\dot{\bar{V}}_i$  becomes

$$\begin{aligned} \dot{\bar{V}}_i &= -z_{[i]}^T \bar{Q}_{[i]} z_{[i]} + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_i|^2 - \bar{q}_{i-1}^H(z_{[i-1]}) \\ &\quad + 2\delta_i z_i \left[ x_{i+1} - \bar{\alpha}_i(z_{[i]}) + \bar{f}_i^H(z_{[i]}) + \bar{h}_i^H(z_{[i]}) \right]. \end{aligned}$$

We define

$$\begin{aligned} \bar{q}_i(z_{[i]}) &:= z_{[i]}^T \bar{Q}_{[i]} z_{[i]} + \bar{q}_i^H(z_{[i]}) \\ \bar{q}_i^H(z_{[i]}) &:= \bar{q}_{i-1}^H(z_{[i-1]}) \\ &\quad - 2\delta_i z_i \left[ \bar{\alpha}_i^H(z_{[i]}) + \bar{\rho}_i^H(z_{[i]}) + \bar{h}_i^H(z_{[i]}) \right]. \end{aligned} \quad (19)$$

Then,  $\dot{\bar{V}}_i$  becomes

$$\begin{aligned} \dot{\bar{V}}_i &= -\bar{q}_i(z_{[i]}) + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_i|^2 \\ &\quad + 2\delta_i z_i \left[ x_{i+1} - \bar{\alpha}_i(z_{[i]}) \right]. \end{aligned}$$

Now, we choose  $\bar{\alpha}_i^H$  so that a dissipation inequality such as (14) is satisfied and the virtual control  $\bar{\alpha}_i$  is smooth. The design of  $\bar{\alpha}_i^H$  begins with factoring out the function  $\bar{\kappa}_{ij}$  from the high order terms of

$\bar{\rho}_i^H + \bar{h}_i^H$ , namely

$$\bar{\rho}_i^H(z_{[i]}) + \bar{h}_i^H(z_{[i]}) = \bar{\kappa}_{i1}(z_{[i]})z_1 + \cdots + \bar{\kappa}_{ii}(z_{[i]})z_i \quad (20)$$

with  $\bar{\kappa}_{ij}(z_{[i]})$ ,  $1 \leq j \leq i$ , being continuous functions.

Again, this is always possible, because  $\bar{\rho}_i^H(z_{[i]}) + \bar{h}_i^H(z_{[i]}) \in C^1$  with  $\bar{\rho}_i^H(0) + \bar{h}_i^H(0) = 0$ , and  $\bar{\kappa}_{ij}$  is given by

$$\bar{\kappa}_{ij}(z_{[i]}) = \int_0^1 \frac{\partial}{\partial z_j} [\bar{\rho}_i^H(\tau z_{[i]}) + \bar{h}_i^H(\tau z_{[i]})] d\tau.$$

Furthermore, it is guaranteed that  $\bar{\kappa}_{ij}(0) = 0$  since

$$\frac{\partial}{\partial z_j} [\bar{\rho}_i^H + \bar{h}_i^H](0) = 0.$$

Choose a smooth function  $\bar{\sigma}_{ij}(\cdot) : R^i \mapsto R$  such that

$$|\bar{\sigma}_{ij}(z_{[i]}) - \bar{\kappa}_{ij}(z_{[i]})| \leq \varepsilon, \quad \bar{\sigma}_{ij}(0) = 0. \quad (21)$$

With these functions  $\bar{\sigma}_{ij}$ ,  $\bar{\alpha}_i^H(z_{[i]})$  is designed as

$$\bar{\alpha}_i^H(z_{[i]}) := -\bar{\sigma}_{i1}(z_{[i]})z_1 - \cdots - \bar{\sigma}_{ii}(z_{[i]})z_i, \quad (22)$$

which guarantees  $\frac{\partial^2 \bar{q}_i^H}{\partial (z_{[i]})^2}(0) = 0$  and  $\frac{1}{2} \frac{\partial^2 \bar{q}_i^H}{\partial (z_{[i]})^2}(0) = \bar{Q}_{[i]}$ .

**Remark 1:** The key point to approximate the non-smooth nonlinearity  $\bar{\rho}_i^H + \bar{h}_i^H$  is on the construction of the smooth function  $\bar{\sigma}_{ij}$  satisfying the condition (21). However, this is not easy, because the constant  $\varepsilon > 0$  is very small and  $\bar{\kappa}_{ij}$  is a multi-variable function, generally speaking.

One way to this end is to make the best use of a class of smooth functions  $y(\cdot) : R \mapsto R$  of the form

$$y(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

which is sometimes called a ‘‘bump function’’. See [9] for another useful smooth functions. Note that these functions can be patched with some continuous functions to construct a smooth one when  $\bar{\kappa}_{ij}$  is the function of one or two variables. See the examples in Section 4.

To complete the inductive argument, it remains to show that  $\bar{q}_i^H$  admits a structure of (16) and stability properties as the first step is preserved. Indeed, define

$$\bar{\pi}_{ij}(z_{[i]}) := \bar{\sigma}_{ij}(z_{[i]}) - \bar{\kappa}_{ij}(z_{[i]}).$$

Then, it follows from (21) that  $|\bar{\pi}_{ij}(z_{[i]})| \leq \varepsilon$ . The first assertion regarding the structure  $\bar{q}_i^H$  can be deduced from

$$\begin{aligned} & \bar{q}_i^H(z_{[i-1]}) \\ &= \bar{q}_{i-1}^H(z_{[i-1]}) - 2\delta_i z_i \left[ \bar{\alpha}_i^H(z_{[i]}) + \bar{\rho}_i^H(z_{[i]}) + \bar{h}_i^H(z_{[i]}) \right] \\ &= z_{[i-1]}^T \left[ \Delta_{[i-1]} \bar{\Pi}_{[i-1]}(z_{[i-1]}) + \bar{\Pi}_{[i-1]}^T(z_{[i-1]}) \Delta_{[i-1]} \right] z_{[i-1]} \\ & \quad + 2\delta_i z_i \left[ \bar{\pi}_{i1}(z_{[i]})z_1 + \cdots + \bar{\pi}_{ii}(z_{[i]})z_i \right] \\ &= z_{[i]}^T \left[ \Delta_{[i]} \bar{\Pi}_{[i]}(z_{[i]}) + \bar{\Pi}_{[i]}^T(z_{[i]}) \Delta_{[i]} \right] z_{[i]} \end{aligned}$$

where we used the following identity:

$$\begin{aligned} z_{[i]}^T \Delta_{[i]} \bar{\Pi}_{[i]} z_{[i]} &= z_{[i-1]}^T \Delta_{[i-1]} \bar{\Pi}_{[i-1]} z_{[i-1]} \\ & \quad + \delta_i z_i (\bar{\pi}_{i1} z_1 + \cdots + \bar{\pi}_{ii} z_i). \end{aligned}$$

Before proceeding, we introduce the following.

**Lemma 3:** Given  $i$  with  $1 \leq i \leq n$ , let  $\varepsilon$  and  $\varepsilon^*$  are given by (13). Suppose each function  $\bar{\pi}_{jk}$ ,  $1 \leq k \leq j \leq i$ , chosen during the steps 1 to  $i-1$ , satisfies  $|\bar{\pi}_{jk}(z_{[j]})| \leq \varepsilon$ ,  $\forall z_{[j]} \in R^j$ . Then, the function  $\bar{q}_i^H(z_{[i]})$  given in (19) is positive definite and radially unbounded w.r.t.  $z_{[i]}$ .

**Proof:** Let  $z_{[i]} \neq 0$ . Since  $|\bar{\pi}_{jk}(z_{[j]})| \leq \varepsilon$ , we have

$$\begin{aligned} \bar{q}_i^H(z_{[i]}) &= z_{[i]}^T (\bar{Q}_{[i]} + \Delta_{[i]} \bar{\Pi}_{[i]} + \bar{\Pi}_{[i]}^T \Delta_{[i]}) z_{[i]} \\ &= z_{[i]}^T \bar{Q}_{[i]} z_{[i]} + \sum_{j=1}^i \sum_{k=1}^j 2\delta_j z_j z_k \bar{\pi}_{jk}(z_{[j]}) \\ &\geq \lambda_{\min}(\bar{Q}_{[i]}) \|z_{[i]}\|^2 - 2\delta\varepsilon \sum_{j=1}^i \sum_{k=1}^j |z_j z_k| \\ &\geq \lambda_{\min}(\bar{Q}_{[i]}) \|z_{[i]}\|^2 - \delta\varepsilon(i+1) \|z_{[i]}\|^2. \end{aligned}$$

Note that since  $Q > 0$ , it follows that  $\bar{Q} > 0$  and  $\bar{Q}_{[i]} > 0$ . Considering the identity, with  $v \in R^i$  being the eigenvector associated with  $\lambda_{\min}(\bar{Q}_{[i]})$ ,

$$\begin{aligned} \lambda_{\min}(\bar{Q}_{[i]}) \|v\|^2 &= v^T \bar{Q}_{[i]} v \\ &= \begin{bmatrix} v^T & 0_{n-i}^T \end{bmatrix} \bar{Q} \begin{bmatrix} v^T & 0_{n-i}^T \end{bmatrix}^T \\ &\geq \lambda_{\min}(\bar{Q}) \|v\|^2, \end{aligned}$$



one obtains  $\lambda_{\min}(\bar{Q}_{[i]}) \geq \lambda_{\min}(\bar{Q})$ .

Therefore, we have

$$\begin{aligned} \bar{q}_i(z_{[i]}) &\geq \left[ \lambda_{\min}(\bar{Q}_{[i]}) - \delta\varepsilon(n+1) \right] \|z_{[i]}\|^2 \\ &> \left[ \lambda_{\min}(\bar{Q}) - \delta\varepsilon^*(n+1) \right] \|z_{[i]}\|^2 = 0, \end{aligned}$$

which concludes the proof.  $\square$

Note that this result ensures that for each  $i$ , there exists  $k_i > 0$  such that

$$\bar{Q}_{[i]} + \Delta_{[i]}\bar{\Pi}_{[i]} + \bar{\Pi}_{[i]}^T\Delta_{[i]} \geq k_i I_i > 0, \quad (23)$$

where  $I_i \in R^{i \times i}$  is the identity matrix.

Regarding the stability, first note that  $\bar{q}_i(z_{[i]})$  is positive definite and radially unbounded by Lemma 3. Hence,

$$\begin{aligned} \bar{V}_i &= -\bar{q}_i(z_{[i]}) + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_i|^2 \\ &\leq -\bar{q}_i(z_{[i]}) + \gamma^2 w^T w, \end{aligned}$$

which implies the stability of the  $i$  th subsystem. Particularly,  $z_{[i]}(t) \rightarrow 0$  as  $t \rightarrow \infty$  for  $w(t) \in L_2$  and  $z_{[i]}(t) \in L_\infty$  for  $w(t) \in L_\infty$ . Furthermore,  $z_{[i]} = 0$  is the globally exponentially stable equilibrium point when  $w(t) \equiv 0$ . Thus, the induction holds for  $i = 1, \dots, n-1$ .

**Step  $n$ :** Let  $z_n = \varphi_n(x) := x_n - \bar{\alpha}_{n-1}(z_{[n-1]})$ . The dynamics of  $z_n$  is given by

$$\begin{aligned} \dot{z}_n &= f_n(x) + g_n(x)w - \frac{\partial \bar{\alpha}_{n-1}}{\partial z_{[n-1]}} \dot{z}_{[n-1]} + u \\ &= a_{[n]}\Phi^{-1}(z) + f_n^H(\Phi^{-1}(z)) + g_n(\Phi^{-1}(z))w \\ &\quad - \frac{\partial \bar{\alpha}_{n-1}}{\partial z_{[n-1]}} \dot{z}_{[n-1]} + u \\ &= \bar{a}_{[n]}z + \bar{f}_n^H(z) + \bar{g}_n(z)w + u, \end{aligned}$$

where

$$\begin{aligned} \bar{f}_n^H(z) &= \bar{\rho}_n^H(z) \\ &:= a_{[n]} \left[ \Phi^{-1}(z) - L^{-1}z \right] + f_n^H(\Phi^{-1}(z)) \\ &\quad - \frac{\partial \bar{\alpha}_{n-1}^H}{\partial z_{[n-1]}} \left( \bar{A}_{[n-1]}z_{[n-1]} + \begin{bmatrix} 0_{n-2} \\ z_n \end{bmatrix} \right) \\ &\quad - \frac{\partial \bar{\alpha}_{n-1}^H}{\partial z_{[n-1]}} \bar{f}_{[n-1]}^H(z_{[n-1]}) \\ \bar{g}_n(z) &:= g_n(\Phi^{-1}(z)) - \frac{\partial \bar{\alpha}_{n-1}}{\partial z_{[n-1]}} \bar{G}_{1[n-1]}(z_{[n-1]}). \end{aligned} \quad (24)$$

Now, the global diffeomorphism  $z = \Phi(x)$  is constructed and it transforms the system (1) into

$$\dot{z} = \bar{A}z + \bar{f}^H(z) + \bar{G}_1(z)w + B_2u, \quad (25)$$

whose linearized system at the origin is of the form (12).

### 3.2. Optimal controller design

Choose the *clf* as

$$\bar{V} := \bar{V}_{n-1} + \delta_n z_n^2 = z^T \Delta z \quad (26)$$

then, the time derivative of  $\bar{V}$  becomes

$$\begin{aligned} \dot{\bar{V}}(z) &= -z_{[n-1]}^T \bar{Q}_{[n-1]} z_{[n-1]} + 2z_{n-1} \delta_{n-1} z_n \\ &\quad + 2\gamma^2 \bar{v}_{n-1}^T w - \gamma^2 \bar{v}_{n-1}^T \bar{v}_{n-1} \\ &\quad - \bar{q}_{n-1}^H(z_{[n-1]}) + 2\delta_n z_n [u + \bar{a}_{[n]} z_{[n]} + \bar{\rho}_n^H(z)], \end{aligned} \quad (27)$$

where  $\bar{v}(z) := \bar{v}_{n-1} + \frac{1}{\gamma^2} \bar{g}_n^T \delta_n z_n = \frac{1}{\gamma^2} \bar{G}_1^T \Delta z$ . From the relations (10) and (11), we have

$$\begin{aligned} &-z_{[n-1]}^T \bar{Q}_{[n-1]} z_{[n-1]} + 2z_{n-1} \delta_{n-1} z_n + 2\delta_n z_n \bar{a}_{[n]} z_{[n]} \\ &= 2z_{[n-1]}^T \Delta_{[n-1]} \bar{A}_{[n-1]} z_{[n-1]} \\ &\quad + \gamma^2 \bar{v}_{l,n-1}^T \bar{v}_{l,n-1} + 2z_{n-1} \delta_{n-1} z_n + 2\delta_n z_n \bar{a}_{[n]} z_{[n]} \\ &= 2z^T \Delta \bar{A} z + \gamma^2 \bar{v}_{l,n-1}^T \bar{v}_{l,n-1} \\ &= -z^T \bar{Q} z - \gamma^2 \bar{v}_l^T \bar{v}_l + R^{-1} \delta_n^2 z_n^2 + \gamma^2 \bar{v}_{l,n-1}^T \bar{v}_{l,n-1}. \end{aligned}$$

Besides, substituting this result into (27), adding and subtracting  $\bar{r}(z)u^2 + \gamma^2 w^T w$ , and completing the squares yield

$$\begin{aligned} \dot{\bar{V}}(z) &= -z^T \bar{Q} z - (\bar{r}^{-1}(z) - R^{-1}) \delta_n^2 z_n^2 - \bar{q}_{n-1}^H(z_{[n-1]}) \\ &\quad - \bar{r}(z)u^2 + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_n|^2 \\ &\quad + \bar{r}(z)(u - \bar{\mu}(z))^2 + 2\delta_n z_n \bar{\rho}_n^H(z) \\ &\quad + \gamma^2 (\bar{v}_n^T \bar{v}_n - \bar{v}_{n-1}^T \bar{v}_{n-1}) \\ &\quad - \gamma^2 (\bar{v}_l^T \bar{v}_l - \bar{v}_{l,n-1}^T \bar{v}_{l,n-1}) \end{aligned}$$

in which  $\bar{\mu}(z) := -\bar{r}^{-1}(z)B_2^T \Delta z$ . By similar arguments used in Step  $i$  of the derivation,

$$\begin{aligned} &\gamma^2 (\bar{v}_n^T \bar{v}_n - \bar{v}_{n-1}^T \bar{v}_{n-1}) - \gamma^2 (\bar{v}_l^T \bar{v}_l - \bar{v}_{l,n-1}^T \bar{v}_{l,n-1}) \\ &= \frac{1}{\gamma^2} z_n \delta_n (\bar{g}_n \bar{g}_n^T - \bar{b}_n \bar{b}_n^T) \delta_n z_n \end{aligned}$$

$$\begin{aligned}
 &+ 2z_n \delta_n (\bar{g}_n \bar{v}_{n-1} - \bar{b}_n \bar{v}_{l,n-1}) \\
 &= 2z_n \delta_n \bar{h}_n^H(z),
 \end{aligned}$$

where

$$\begin{aligned}
 &\bar{h}_n^H(z) \\
 &:= \frac{1}{\gamma^2} \left[ \bar{g}_n \bar{G}_{1[n-1]}^T \bar{b}_n \bar{B}_{1[n-1]}^T \quad \frac{1}{2} (\bar{g}_n \bar{g}_n^T \bar{b}_n \bar{b}_n^T) \right] \Delta z. \quad (28)
 \end{aligned}$$

Thus, by defining the function  $\bar{q}(z)$  as

$$\begin{aligned}
 \bar{q}(z) &:= z^T \bar{Q} z + \bar{q}_{n-1}^H(z_{[n-1]}) + (\bar{r}^{-1}(z) - R^{-1}) \delta_n^2 z_n^2 \\
 &\quad - 2z_n \delta_n (\bar{\rho}_n^H(z) + \bar{h}_n^H(z)), \quad (29)
 \end{aligned}$$

$\dot{\bar{V}}$  is reduced to

$$\begin{aligned}
 \dot{\bar{V}}(z) &= -\bar{q}(z) - \bar{r}(z) u^2 + \gamma^2 w^T w - \gamma^2 |w - \bar{v}_n|^2 \\
 &\quad + \bar{r}(z) (u - \bar{\mu}(z))^2. \quad (30)
 \end{aligned}$$

Since  $\bar{\rho}_n^H(z) + \bar{h}_n^H(z) \in C^1$ ,  $\bar{\rho}_n^H(0) + \bar{h}_n^H(0) = 0$ , and  $\frac{\partial}{\partial z_j} [\bar{\rho}_n^H + \bar{h}_n^H](0) = 0$ ,  $1 \leq j \leq n$  which can be shown easily (such as (17)), the existence of continuous functions  $\bar{\kappa}_{nj}(\cdot): R^n \mapsto R$  with  $\bar{\kappa}_{nj}(0) = 0$  is guaranteed such that

$$\begin{aligned}
 \bar{\rho}_n^H(z) + \bar{h}_n^H(z) &= \bar{\kappa}_{n1}(z) z_1 + \dots + \bar{\kappa}_{nn}(z) z_n \\
 &= \bar{\kappa}_{[n-1]}(z) z_{[n-1]} + \bar{\kappa}_{nn}(z) z_n,
 \end{aligned}$$

where  $\bar{\kappa}_{[n-1]}(z) := [\bar{\kappa}_{n1}(z) \quad \dots \quad \bar{\kappa}_{n,n-1}(z)]$ . Define

$$\bar{\Pi}(z) := \begin{bmatrix} \bar{\Pi}_{[n-1]}(z_{[n-1]}) & 0_{n-1} \\ 0_{n-1}^T & 0 \end{bmatrix}.$$

Then, it follows that  $\bar{Q} + \Delta \bar{\Pi} + \bar{\Pi}^T \Delta \geq k_n I_n$  for some  $k_n > 0$ . Using the identity

$$z^T \Delta \bar{\Pi} z = z_{[n-1]}^T \Delta_{[n-1]} \bar{\Pi}_{[n-1]} z_{[n-1]},$$

$\bar{q}(z)$  is reduced as

$$\begin{aligned}
 \bar{q}(z) &= z^T \bar{Q} z + z_{[n-1]}^T [\Delta_{[n-1]} \bar{\Pi}_{[n-1]} + \bar{\Pi}_{[n-1]}^T \Delta_{[n-1]}] z_{[n-1]} \\
 &\quad - 2z_n \delta_n (\bar{\kappa}_{[n-1]} z_{[n-1]} + \bar{\kappa}_{nn} z_n) \\
 &\quad + (\bar{r}^{-1}(z) - R^{-1}) \delta_n^2 z_n^2 \\
 &= z^T (\bar{Q} + \Delta \bar{\Pi} + \bar{\Pi}^T \Delta) z - 2\delta_n z_{[n-1]}^T \bar{\kappa}_{[n-1]} z_n \\
 &\quad + 2\delta_n (-\bar{\kappa}_{nn}) z_n^2 + (\bar{r}^{-1}(z) - R^{-1}) \delta_n^2 z_n^2. \quad (31)
 \end{aligned}$$

We now state the main result of this paper.

**Theorem 1:** There exist a positive definite, radially unbounded function  $\bar{q}(z)$  and a strictly positive function  $\bar{r}(z)$  such that the controller

$$u = \bar{\mu}(z) := -\bar{r}^{-1}(z) B_2^T \Delta z \quad (32)$$

is the robust optimal one for the system (25) in the local and global inverse sense with respect to the cost functionals (4) and (7) in  $z$ -coordinates, respectively, for the worst case disturbance  $w = \bar{v}(z) := \frac{1}{\gamma^2} \bar{G}_1^T(z) \Delta z$ . In addition,  $z(t) \rightarrow 0$  as  $t \rightarrow \infty$  for  $w(t) \in L_2$ , and  $z(t) \in L_\infty$  for  $w(t) \in L_\infty$ . If  $w \equiv 0$ , then the origin is globally exponentially stable by this controller.

**Proof:** It is clear that  $\bar{r}(0) = R$  renders  $\frac{1}{2} \frac{\partial^2 \bar{q}}{\partial z^2}(0) = \bar{Q}$ , since  $\frac{\partial^2 \bar{q}_{n-1}^H}{\partial (z_{[n-1]})^2}(0) = 0$  by induction and Lemma 2. So, it is only needed to find a strictly positive function  $\bar{r}(z)$  satisfying  $\bar{r}(0) = R$  such that  $\bar{q}(z)$  is positive definite and radially unbounded.

By Young's inequality one has, for  $p > 0$ ,

$$2\delta_n z_{[n-1]}^T \bar{\kappa}_{[n-1]} z_n \leq p \|z_{[n-1]}\|^2 + \frac{\delta_n^2}{p} \|\bar{\kappa}_{[n-1]}\|^2 z_n^2.$$

Then, the function  $\bar{q}(z)$  which is already reduced to the form (31) can be again modified as

$$\begin{aligned}
 \bar{q}(z) &= z^T (\bar{Q} + \Delta \bar{\Pi} + \bar{\Pi}^T \Delta) z - 2\delta_n z_{[n-1]}^T \bar{\kappa}_{[n-1]} z_n \\
 &\quad + 2\delta_n (-\bar{\kappa}_{nn}) z_n^2 + (\bar{r}^{-1}(z) - R^{-1}) \delta_n^2 z_n^2 \\
 &\geq k_n \|z\|^2 - p \|z_{[n-1]}\|^2 - \frac{\delta_n^2}{p} \|\bar{\kappa}_{[n-1]}\|^2 z_n^2 \\
 &\quad + 2\delta_n (-\bar{\kappa}_{nn}) z_n^2 + (\bar{r}^{-1}(z) - R^{-1}) \delta_n^2 z_n^2 \\
 &= (k_n - p) \|z\|^2 + (\bar{r}^{-1}(z) - R^{-1} - \chi(z)) \delta_n^2 z_n^2,
 \end{aligned}$$

where  $\chi(z) := \frac{2\bar{\kappa}_{nn}(z)}{\delta_n} + \frac{1}{p} \|\bar{\kappa}_{[n-1]}(z)\|^2$ .

It is rather obvious that  $\chi(0) = 0$ . Therefore,  $\bar{q}(z)$  is positive definite and radially unbounded provided that  $p < k_n$  and  $\bar{r}^{-1} \geq R^{-1} + \chi(z)$ , namely  $\bar{q}(z) \geq (k_n - p) \|z\|^2$ .

With  $p = \frac{k_n}{2}$ , one particular choice of a function  $\bar{r}(z)$  suggested in [3] is

$$\bar{r}(z) := \left( \sqrt{R^{-2} + \chi^2(z)} + \chi(z) \right)^{-1},$$

$$\chi(z) = \frac{2\bar{\kappa}_{mm}(z)}{\delta_n} + \frac{2}{k_n} \|\bar{\kappa}_{[n-1]}^{-T}(z)\|^2,$$

where  $\bar{r}(z)$  is clearly a continuous positive function with  $\bar{r}(0) = R$  and  $\bar{r}^{-1} \geq R^{-1} + \chi(z)$ ,  $\forall z \in R^n$ . Therefore, the controller (32) satisfies the local optimality (6), i.e.,

$$\frac{\partial \bar{\mu}}{\partial z}(0)z = -R^{-1}B_2^T \Delta z,$$

where  $-R^{-1}B_2^T \Delta z$  is the local optimal controller (6) in  $z$ -coordinates. Moreover, the global inverse optimality follows, since the condition (8) holds from (30), namely

$$\min_u \max_w \left[ \bar{q}(z) + \bar{r}(z)u^2 - \gamma^2 w^T w + \dot{\bar{V}}(z) \right] = 0,$$

which means that the *clf* (26) is, in fact, the value function of the dynamic game  $\min_u \max_w J(u, w)$  for the nonlinear system (25), where the cost functional is given by

$$J(u, w) := \int_0^\infty \left[ \bar{q}(z) + \bar{r}(z)u^2 - \gamma^2 w^T w \right] dt.$$

Another possible choices are also depicted in [3]. The controller (32) and the worst case disturbance  $w = \bar{v}(z)$  satisfy the dissipation inequality

$$\begin{aligned} \dot{\bar{V}}(z) &\leq -\bar{q}(z) - \bar{r}(z)u^2 + \gamma^2 w^T w \\ &\leq -(k_n - p) \|z\|^2 + \gamma^2 w^T w, \end{aligned}$$

which implies  $z(t) \rightarrow 0$  as  $t \rightarrow \infty$  for  $w(t) \in L_2$  and  $z(t) \in L_\infty$  for  $w(t) \in L_\infty$ . Moreover, the origin of the system is the globally exponentially stable equilibrium in the absence of a disturbance.  $\square$

**Remark 2:** Because the smooth virtual control  $\bar{\alpha}_i$ ,  $1 \leq i \leq n-1$ , approximately cancels the nonlinearities, some remaining nonlinearities are turned over to the next step after being differentiated. Consequently, all the undesirable nonlinearities are condensed in the terms  $\bar{q}_{n-1}^{-H}$  and  $\bar{\rho}_n^{-H} + \bar{h}_n^{-H}$  of (29) at the last step.

Note that the term  $\bar{q}_{n-1}^{-H}$  is identically zero in the scheme of [3]; the nonlinearities are exactly cancelled out. Thus, it is desirable to choose  $\bar{\sigma}_{ij}$  sufficiently close to  $\bar{\kappa}_{ij}^{-H}$  to reduce the magnitude of  $\bar{q}_{n-1}^{-H}$ , which is closely related to the magnitude of control effort.

## 4. EXAMPLES

**Example 1:** Consider the following nonlinear system

$$\dot{x}_1 = x_2 + f_1^H(x_1) + w, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u, \quad (33)$$

where  $w \in L_\infty$  or  $L_2$  and

$$f_1^H(x_1) = x_1^2, \text{ if } x_1 > 0, \quad f_1^H(x_1) = 0, \text{ otherwise.} \quad (34)$$

Note that the approach of [3] is not applicable since  $f_1^H$  is only  $C^1$ . Suppose a cost functional is given by  $J(u, w) = \int_0^\infty [x^T x + u^2 - 25w^2] dt$ . Considering the linear part of (33), it can be seen that GARE (5) admits a unique solution

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} = \begin{bmatrix} 2.64 & 2.70 & 1.13 \\ 2.70 & 5.20 & 2.59 \\ 1.13 & 2.59 & 2.50 \end{bmatrix},$$

which can be factorized as  $P = L^T \Delta L$  where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha_{11} & 1 & 0 \\ -\alpha_{21} & -\alpha_{22} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.61 & 1 & 0 \\ 0.45 & 1.04 & 1 \end{bmatrix}$$

$$\Delta = \text{diag}(\delta_1, \delta_2, \delta_3) = \text{diag}(1.20, 2.52, 2.50).$$

The optimal controller for the linearized system of (33) is  $u_l = -p_{13}x_1 - p_{23}x_2 - p_{33}x_3$ , which renders two equilibriums  $(0, 0, 0)$  and  $(0.44, -0.19, 0)$  of (33) locally stable and unstable, respectively. Note that this system is unstable in some region, and thus the controller is modified to guarantee local optimality and globally inverse optimality of the closed loop system. At first, compute

$$\begin{aligned} \bar{A} &= \begin{bmatrix} \bar{a}_{11} & 1 & 0 \\ \bar{a}_{21} & \bar{a}_{22} & 1 \\ \bar{a}_{31} & \bar{a}_{32} & \bar{a}_{33} \end{bmatrix} = \begin{bmatrix} -0.61 & 1 & 0 \\ -0.19 & -0.43 & 1 \\ -0.09 & -0.62 & 1.04 \end{bmatrix} \\ \bar{B}_1 &= \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.61 \\ 0.45 \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} 1.40 & -0.79 & 0.18 \\ -0.79 & 2.08 & -1.04 \\ 0.18 & -1.04 & 1.00 \end{bmatrix}, \end{aligned}$$

which yields  $\varepsilon^* = \frac{\lambda_{\min}(\bar{Q})}{4\delta} = 0.03$  for  $\delta = \delta_2 = 2.52$ .

Pick  $\varepsilon = 0.02 < \varepsilon^*$ , and  $k_3 = \lambda_{\min}(\bar{Q}) - 4\delta\varepsilon = 0.09$ .

**Step 1:** Let  $z_1 = x_1$ . Note that  $f_1$  can be expressed as  $f_1 = \bar{\kappa}_{11}z_1$ , where  $\bar{\kappa}_{11}$  is given by

$$\bar{\kappa}_{11}(z_{[1]}) = z_1, \text{ if } z_1 > 0, \quad \bar{\kappa}_{11}(z_{[1]}) = 0, \text{ otherwise.} \quad (35)$$

Choose  $\bar{\sigma}_{11}(z_{[1]}) = z_1 e^{-0.01/z_1}$  for  $z_1 > 0$ ,  $\bar{\sigma}_{11}(z_{[1]}) = 0$ , otherwise. Let  $\bar{\alpha}_1^H = -\bar{\sigma}_{11} z_1$  so that  $\bar{\alpha}_1 = \alpha_{11} z_1 - \bar{\sigma}_{11} z_1$ .

**Step 2:** Let  $z_2 = x_2 - \bar{\alpha}_1$  and define

$$\begin{aligned} \bar{q}_2^H &= \bar{q}_1^H - 2\delta_2 z_2 \left( \bar{\alpha}_2^H + \bar{\rho}_2^H + \bar{h}_2^H \right) \\ \bar{\rho}_2^H &= -\frac{\partial \bar{\alpha}_1^H}{\partial z_{[1]}} \left( a_{[1]} z_{[1]} + z_2 \right) - \frac{\partial \bar{\alpha}_1^H}{\partial z_{[1]}} \bar{f}_{[1]}^H \\ \bar{h}_2^H &= \left[ \frac{1}{\gamma^2} (\bar{g}_2 \bar{g}_1 - \bar{b}_2 \bar{b}_1) \quad \frac{1}{2\gamma^2} (\bar{g}_2^2 - \bar{b}_2^2) \right] \Delta_{[2]} z_{[2]}. \end{aligned}$$

Note that  $\bar{\rho}_2^H$  can be decomposed into  $\bar{\rho}_2^H = \bar{\rho}_{2S}^H - \alpha_{11} f_1$  where  $\bar{\rho}_{2S}^H$  is smooth, and the  $C^1$  function  $-\alpha_{11} f_1$  can be written by  $-\alpha_{11} f_1 = \bar{\kappa}_{21} z_1$  with  $\bar{\kappa}_{21} = -\alpha_{11} \bar{\kappa}_{11}$ . Choose  $\bar{\sigma}_{21} = -\alpha_{11} \bar{\sigma}_{11}$ , which is smooth and it holds that  $|\bar{\sigma}_{21} - \bar{\kappa}_{11}| \leq \varepsilon$ . Define  $\bar{\alpha}_2^H = \bar{\rho}_{2S}^H - \bar{h}_2^H - \bar{\sigma}_{21} z_1$ , so that  $\bar{\alpha}_2 = \bar{\alpha}_{[2]} z_{[2]} - \bar{\rho}_{2S}^H - \bar{h}_2^H - \bar{\sigma}_{21} z_1$ .

**Step 3:** Let  $z_3 = x_3 - \bar{\alpha}_2$  for the final step. Following the derivation of the paper, the functions  $\bar{\rho}_3^H$  of (24) and  $\bar{h}_3^H$  of (28) can be factorized, i.e.,  $\bar{\rho}_3^H = \bar{F}_1 z_1 + \bar{F}_2 z_2 + \bar{F}_3 z_3$  and  $\bar{h}_3^H = \bar{H}_1 z_1 + \bar{H}_2 z_2 + \bar{H}_3 z_3$  where

$$\begin{aligned} \bar{F}_1 &:= -\frac{\partial \bar{\alpha}_2^H}{\partial z_1} a_{11} - \frac{\partial \bar{\alpha}_2^H}{\partial z_2} a_{21} - \frac{\partial \bar{\alpha}_2^H}{\partial z_1} (-\bar{\sigma}_{11} + \bar{\kappa}_{11}) \\ &\quad - \frac{\partial \bar{\alpha}_2^H}{\partial z_2} \left( -\alpha_{11} \bar{\kappa}_{11} - \frac{1}{\gamma^2} (\bar{g}_2 - \bar{b}_2) \delta_1 + \alpha_{11} \bar{\sigma}_{11} \right), \\ \bar{F}_2 &:= -\frac{\partial \bar{\alpha}_2^H}{\partial z_1} a_{22} - \frac{\partial \bar{\alpha}_2^H}{\partial z_2} a_{22} - \frac{\partial \bar{\alpha}_2^H}{\partial z_2} \left( -\frac{1}{2\gamma^2} (\bar{g}_2^2 - \bar{b}_2^2) \delta_2 \right), \\ \bar{F}_3 &:= -\frac{\partial \bar{\alpha}_2^H}{\partial z_2}, \quad \bar{H}_1 := \frac{1}{\gamma^2} (\bar{g}_3 - \bar{b}_3) \delta_1, \\ \bar{H}_2 &:= \frac{1}{\gamma^2} (\bar{g}_3 \bar{g}_2 - \bar{b}_3 \bar{b}_2) \delta_2, \quad \bar{H}_3 := \frac{1}{2\gamma^2} (\bar{g}_3^2 - \bar{b}_3^2) \delta_3. \end{aligned}$$

Therefore,  $\bar{\rho}_3^H + \bar{h}_3^H = \bar{\kappa}_{31} z_1 + \bar{\kappa}_{32} z_2 + \bar{\kappa}_{33} z_3$  where  $\bar{\kappa}_{31} := \bar{F}_1 + \bar{H}_1$ ,  $\bar{\kappa}_{32} := \bar{F}_2 + \bar{H}_2$  and  $\bar{\kappa}_{33} := \bar{F}_3 + \bar{H}_3$ , and the resulting optimal controller becomes  $u = \bar{\mu}(z) = -\left( \sqrt{1 + \chi^2} + \chi \right) \delta_3 z_3$  where  $\chi = \frac{2\bar{\kappa}_{33}}{\delta_3} + \frac{2}{k_3} \left( \frac{-2}{\bar{\kappa}_{31} + \bar{\kappa}_{32}} \right)$ .

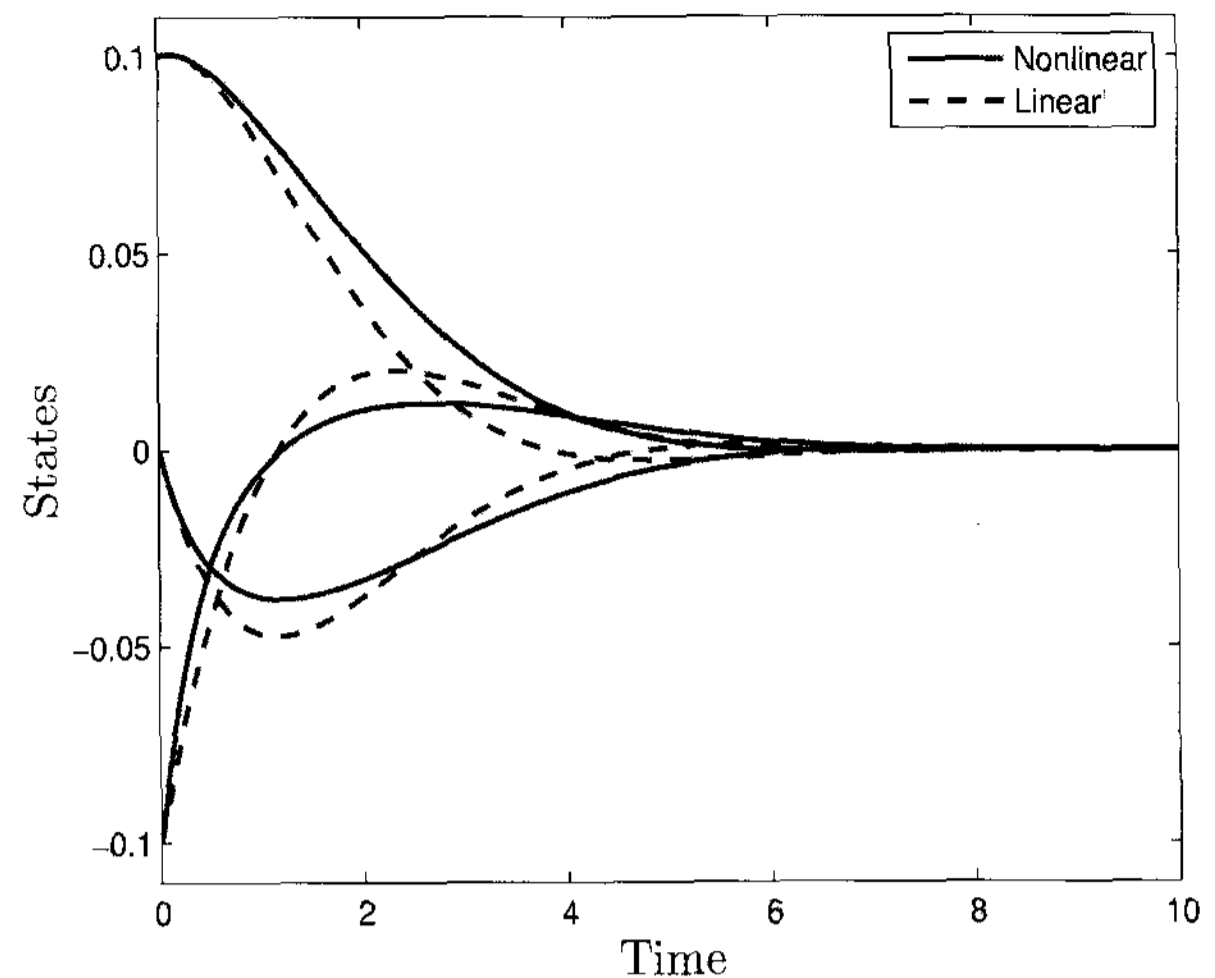


Fig. 1. Evolution of  $x$  without disturbance.

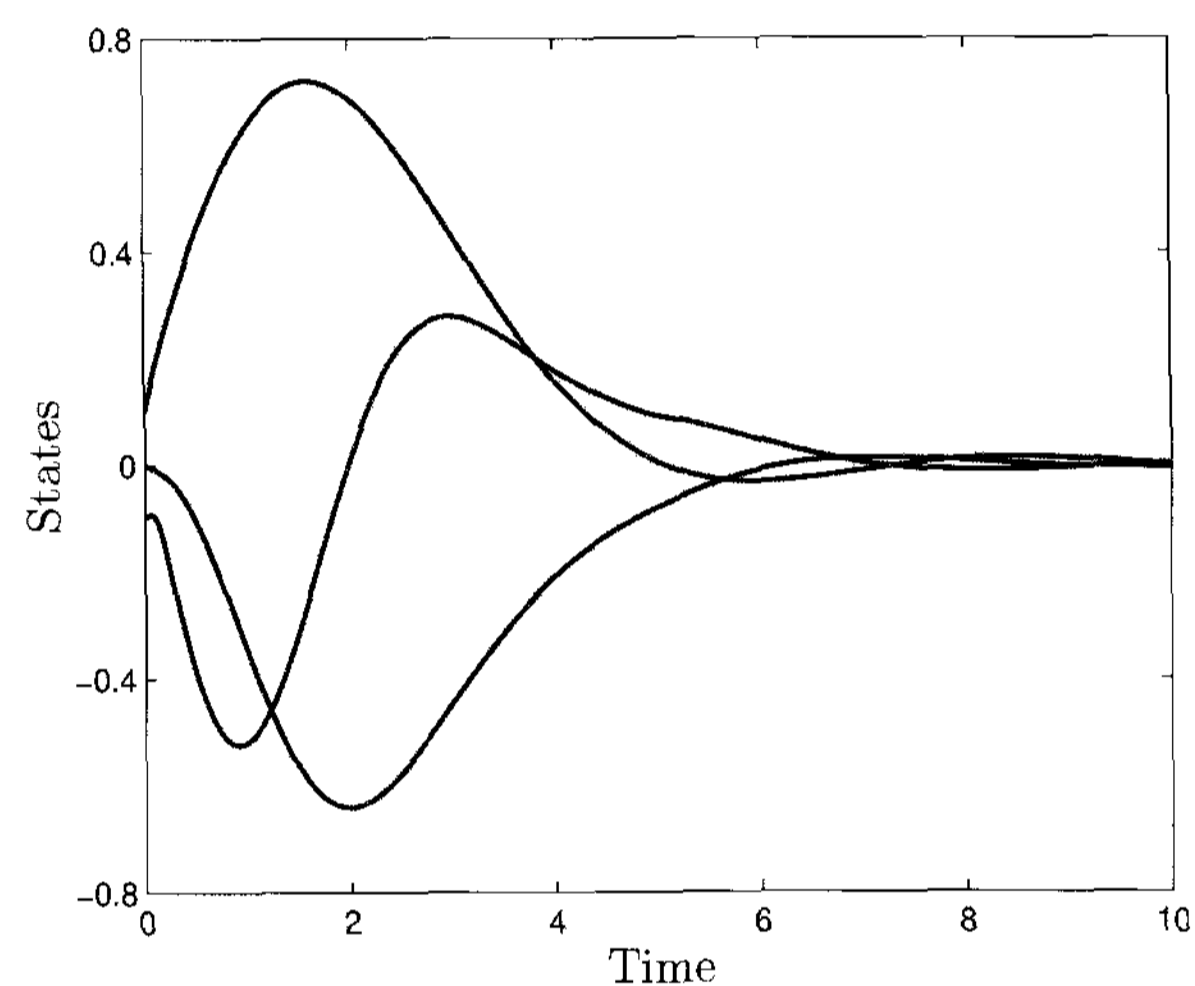


Fig. 2. Evolution of  $x$  with  $w(t) = \frac{\sin(t+1)}{(t+2)^2}$ .

Fig. 1 shows the state trajectories of the system driven by the nonlinear optimal controller and by the local controller. The initial state is  $(0.1, 0, -0.1)$  and  $w = 0$ . As shown in the figure, the qualitative behavior of two trajectories is similar since the initial state is located around the origin. In the presence of the disturbance  $w(t) = \frac{\sin(t+1)}{(t+2)^2} \in L_2$ , the nonlinear controller robustly stabilizes the system, while the linear controller cannot (for linear case,  $x(t)$  diverges). Fig. 2 illustrates this result.

**Example 2:** Consider the 2-dimensional system

$$\dot{x}_1 = f_1(x_1) + x_2 + w, \quad \dot{x}_2 = u$$

where  $f_1$  is already defined of the form (34), and suppose a cost is given by  $J(u, w) = \int_0^\infty [x_1^2 + x_2^2 + u^2 - 25w^2] dt$ . This example illustrates that even if the approach in [3] is applicable (for

two dimensional system,  $C^1$  is enough), the controller given by this paper differs from that of [3].

Through a series of calculations, we obtain

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 1.82 & 1.06 \\ 1.06 & 1.78 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.60 & 1 \end{bmatrix},$$

$$\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} = \begin{bmatrix} 1.18 & 0 \\ 0 & 1.78 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} \bar{a}_{11} & 1 \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} = \begin{bmatrix} -0.60 & 1 \\ -0.36 & 0.60 \end{bmatrix},$$

$$\bar{B}_1 = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.60 \end{bmatrix}$$

$$\bar{Q} = \begin{bmatrix} \bar{Q}_{[1]} & \bar{q}_1 \\ \bar{q}_1 & \bar{q}_2 \end{bmatrix} = \begin{bmatrix} 1.36 & -0.60 \\ -0.60 & 1 \end{bmatrix}, \quad \varepsilon^* = 0.135.$$

**Controller of [3]:** Let  $z_1 = x_1$ . Select  $\bar{\alpha}_1^H = -f_1$ , and it follows  $\bar{\alpha}_1 = \alpha z_1 - f_1$ . Let  $z_2 = x_2 - \bar{\alpha}_1$ . After some computation,  $\bar{\eta} = \bar{\eta}_1 z_1 + \bar{\eta}_2 z_2$ ,  $\chi = \frac{\bar{\eta}_1^2}{\bar{Q}_{[1]}} - \frac{2\bar{q}_1 \bar{\eta}_1}{\bar{Q}_{[1]} \delta_2} + \frac{2\bar{\eta}_2}{\delta_2}$  where  $\bar{\eta}_1(z_1) := -\frac{\partial \bar{\alpha}_1^H}{\partial z_1} \bar{a}_{11} + \frac{1}{\gamma} \delta_1 (\bar{g}_2 - \bar{b}_2)$  and  $\bar{\eta}_2(z) := \frac{1}{2\gamma^2} (\bar{g}_2^2 - \bar{b}_2^2) \delta_2$ . A possible choice of the controller is  $u_1 = -(\sqrt{1 + \chi^*} + \chi^*) \delta_2 z_2$ ,  $\chi^* := \chi + z_1^2$ .

**Proposed Controller:** Choose  $\varepsilon = 0.1$  and  $k_2 = 0.02$ . Pick  $\beta = 0.05$ , and define  $\bar{\sigma}_{11} = z_1 e^{-\beta/z_1}$ , for  $z_1 > 0$ , and vanishing otherwise. It holds that  $|\bar{\sigma}_{11} - \bar{\kappa}_{11}| \leq \varepsilon$  where  $\bar{\kappa}_{11}$  is of the form (35). Selecting  $\bar{\alpha}_1^H = -\bar{\sigma}_{11} z_1$  yields  $\bar{\alpha}_1 = \alpha z_1 - \bar{\sigma}_{11} z_1$ . For the second step, compute  $\bar{\rho}_2^H = \bar{F}_1 z_1 + \bar{F}_2 z_2$  and  $\bar{h}_2^H = \bar{H}_1 z_1 + \bar{H}_2 z_2$  where

$$\bar{F}_1 := -\frac{\partial \bar{\alpha}_1^H}{\partial z_1} \bar{a}_{11} - \frac{\partial \bar{\alpha}_1}{\partial z_1} (-\bar{\sigma}_{11} + \bar{\kappa}_{11}), \quad \bar{F}_2 := -\frac{\partial \bar{\alpha}_1^H}{\partial z_1},$$

$$\bar{H}_1 := \frac{1}{\gamma^2} (\bar{g}_2 \bar{g}_1 - \bar{b}_2 \bar{b}_1) \delta_1, \quad \bar{H}_2 := \frac{1}{2\gamma^2} (\bar{g}_2^2 - \bar{b}_2^2) \delta_2.$$

Therefore,  $\bar{\rho}_2^H + \bar{h}_2^H = \bar{\kappa}_{21} z_1 + \bar{\kappa}_{22} z_2$  in which  $\bar{\kappa}_{21} = \bar{F}_1 + \bar{H}_1$ ,  $\bar{\kappa}_{22} = \bar{F}_2 + \bar{H}_2$ , and the optimal controller is designed as  $u_2 = -(\sqrt{1 + \chi^2} + \chi) \delta_2 z_2$ ,  $\chi := \frac{2\bar{\kappa}_{22}}{\delta_2} + \frac{2}{k_2} \bar{\kappa}_{21}$ . The trajectories starting at several

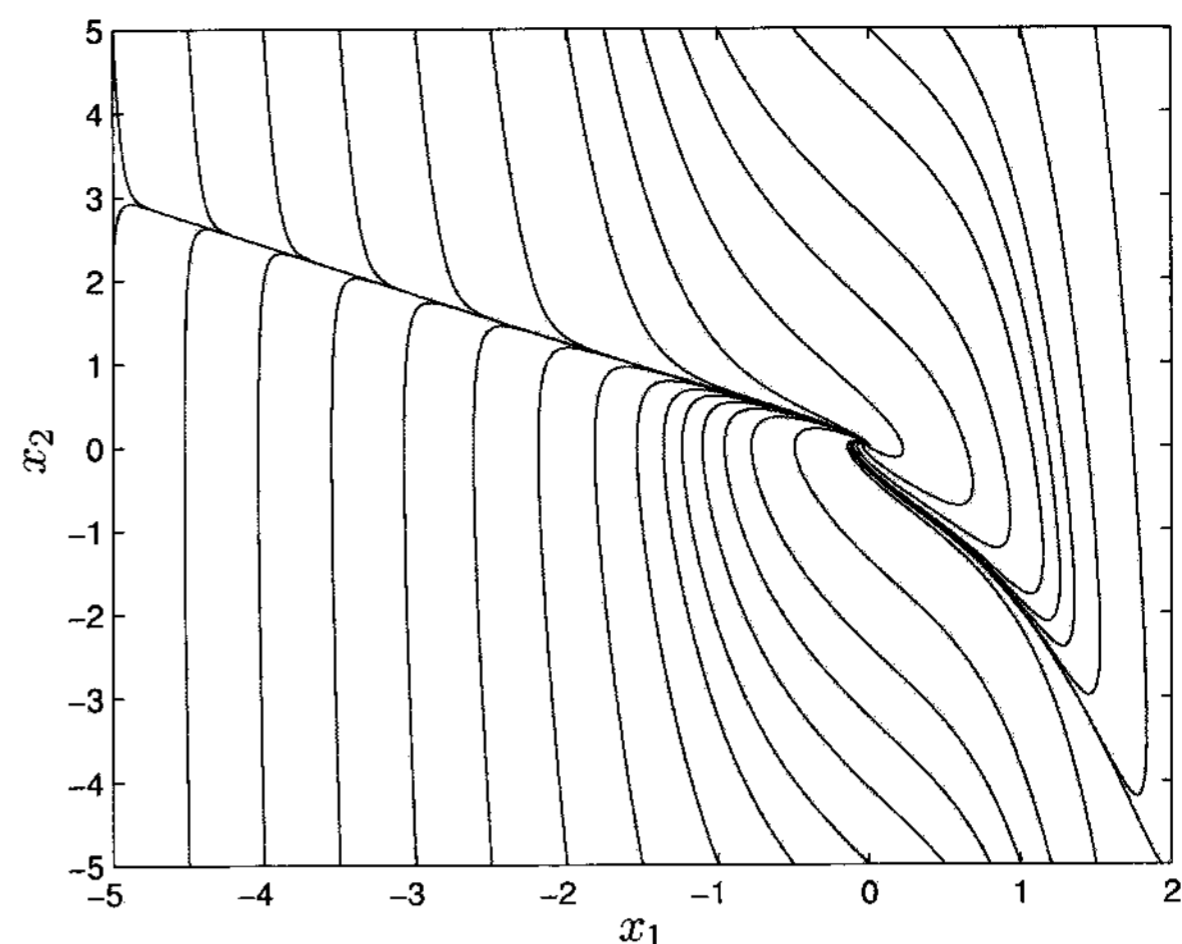


Fig. 3. State trajectories, controller of [3].

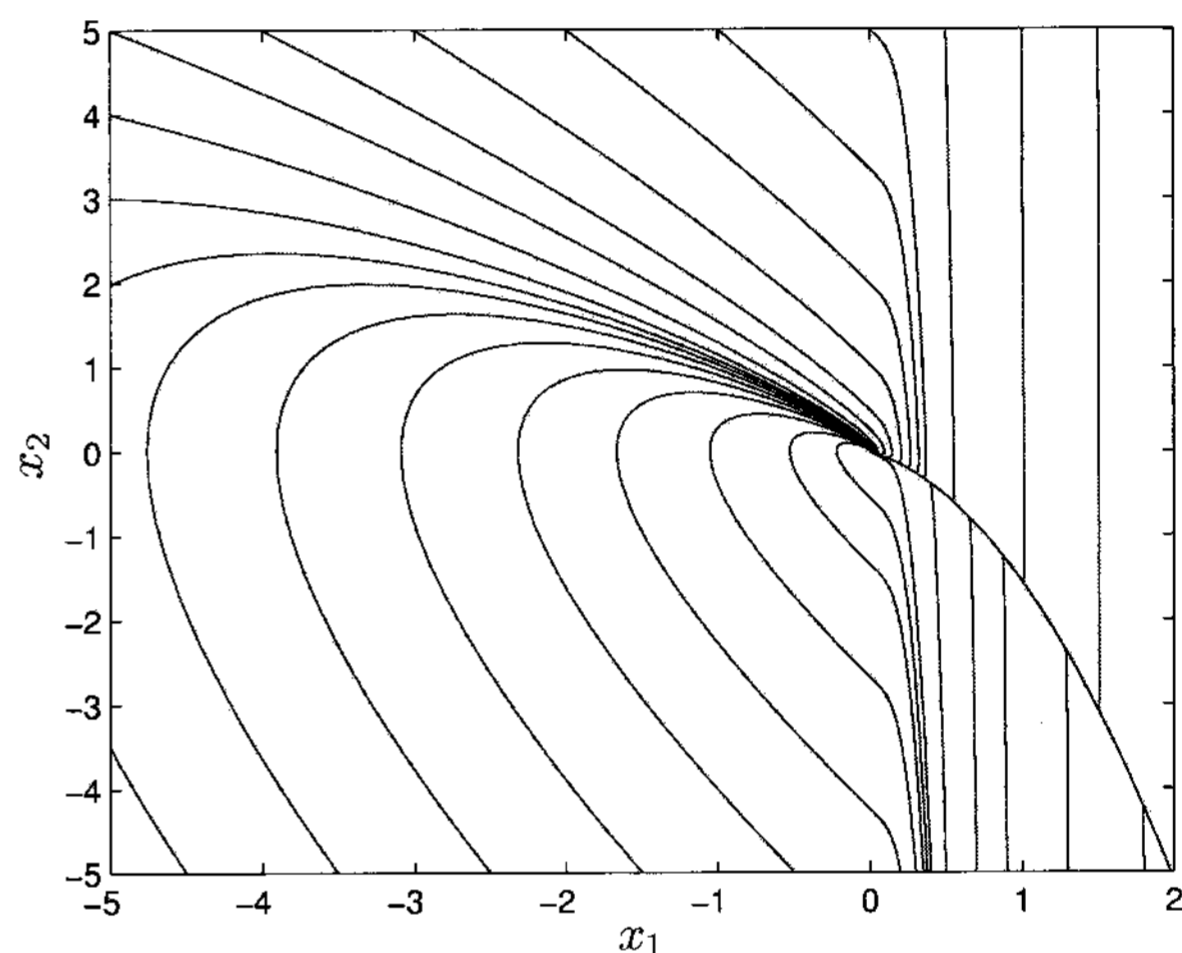


Fig. 4. State trajectories, proposed controller ( $\varepsilon = 0.1$ ).

initial states are illustrated in Figs. 3 and 4 with control inputs  $u_1$  and  $u_2$ , respectively. It is clearly seen from these figures that two approaches are different from each other although the parameters associated with linear optimal controllers are identical. This comes from different ways of designing high order terms in virtual controllers.

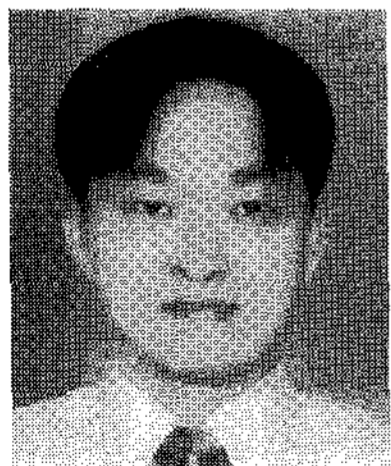
## 5. CONCLUSIONS

The virtual controls are chosen to approximately cancel the nonlinearities of a system with  $C^1$  vector fields at each recursive step, and a robust controller is designed at the last step such that it meets the dual goal, i.e., the local optimality and the global inverse optimality. Note that the nonlinearities cannot be exactly canceled out because it might contain the  $C^1$  functions as assumed, and this is why the approximation technique is used as an alternative strategy. While applying proposed method, it is desirable that the  $C^1$  functions are approximated by smooth functions as closely as possible, which can be

done by reducing the design variable  $\varepsilon$ . The main limitation of our approach is that the system needs to be known exactly. Consideration of plant uncertainty is a future research topic beyond the scope of this paper.

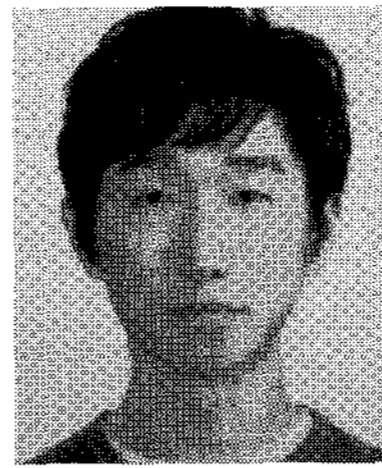
### REFERENCES

- [1] Z. Artstein, "Stabilization with relaxed controls," *Nonlinear Analysis*, vol. 7, no. 11, pp. 1163-1173, 1983.
- [2] K. Ezal, P. V. Kokotović, A. R. Teel, and T. Başar, "Disturbance attenuating output-feedback control of nonlinear systems with local optimality," *Automatica*, vol. 37, no. 6, pp. 805-817, 2001.
- [3] K. Ezal, Z. Pan, and P. V. Kokotović, "Locally optimal and robust backstepping design," *IEEE Trans. on Automatic Control*, vol. 45, no. 2, pp. 260-271, 2000.
- [4] H. K. Khalil, *Nonlinear Systems*, Third Ed., Prentice-Hall, 2002.
- [5] D. E. Kirk, *Optimal Control Theory: An Introduction*, Prentice-Hall, 1970.
- [6] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*, Wiley, New York, 1995.
- [7] R. Sepulchre, M. Jankovic, and P. V. Kokotović, *Constructive Nonlinear Control*, Springer-Verlag, 1997.
- [8] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, Second Ed., Springer-Verlag, 1997.
- [9] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. I, Third Ed., Publish or Perish Inc., 1999.
- [10] A. R. Teel, "Asymptotic convergence from  $L_p$  stability," *IEEE Trans. on Automatic Control*, vol. 44, no. 11, pp. 2169-2170, 1999.
- [11] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*, Prentice-Hall, 1996.



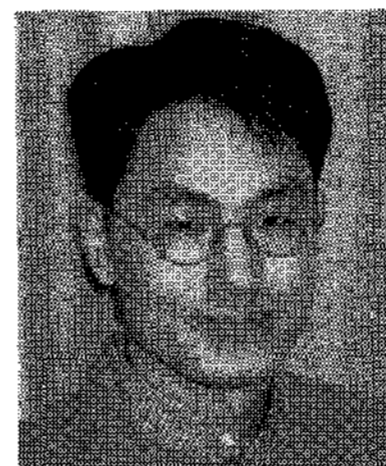
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