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A SYSTEM OF NONLINEAR PROJECTION EQUATIONS WITH PERTURBATION IN HILBERT SPACES

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Abstract. In this paper, we introduce and studied a system of nonlinear projection equations with perturbation in Hilbert spaces. By using the fixed point theorem, we prove an existence of solution for this system of nonlinear projection equations. We construct an algorithm for approximating the solution of the system of nonlinear projection equations with perturbation and show that the iterative sequence generated by the algorithm converges to the solution of the system of nonlinear projection equations with perturbation under some suitable conditions.

1. Introduction

In recent years, the variational inequalities and complementarity problems have been become effective and useful tools for a wide class of problems arising in a lot of different fields of pure and applied subject, such as optimization theory, mathematical programming, elasticity theory, structural mechanics, engineering science, economics equilibrium, free boundary valued problems and so on. For more details, we refer to [2, 3, 4, 6, 7, 8, 9, 10, 11, 13, 17] and the references therein.

It is well known that the projection method plays an important role in solving the variational inequalities involved generalized monotonicity in Hilbert spaces (see, for example, [12, 14, 15, 18, 19] and the references therein). In 2001, Zhao and Sun [20] introduced and studied the solvability problems for a class of nonlinear projection equations in finite dimensional spaces. They proved some alternative theorems for the nonlinear projection equations and obtained some applications to generalized complementarity problems.

Throughout this paper, we assume that X is a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Let K_1 and K_2 be two nonempty subsets

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of X, $\rho_1, \rho_2 > 0$ be two constants, $g,\ h : X \to X$ and $S,\ T,\ C_i,\ f_i : X\times X \to X$ be nonlinear mappings for $i = 1, 2$.

In this paper, we consider the following nonlinear system of projection equations with perturbation: find $x, y \in X$ such that

$$\begin{cases} P_{K_1}[f_1(x,y) - \rho_1 T(x,\ y)] + C_1(x,\ y) = g(x), \\ P_{K_2}[f_2(x,y) - \rho_2 S(x,\ y)] + C_2(x,\ y) = h(y), \end{cases} \tag{1.1}$$

where P_{K_i} is the projection of X onto K_i $(i = 1, 2)$.

It is easy to see the system (1.1) includes many know nonlinear projection equations, variational inequalities and complementarity problems as special cases.

(I) If $C_i = 0$ for $i = 1, 2$, then the problem (1.1) reduces to the following problem: find $x, y \in X$ such that

$$\begin{cases} P_{K_1}[f_1(x,y) - \rho_1 T(x,\ y)] = g(x), \\ P_{K_2}[f_2(x,y) - \rho_2 S(x,\ y)] = h(y), \end{cases} \tag{1.2}$$

(II) If $K_1 = K_2 = K$, $T(x,y) = S(y,x)$, g and h are identity mappings, $f_1(x,y) = y$ and $f_2(x,y) = x$ for all $x, y \in X$, then the system (1.2) reduces to the following problem: find $x, y \in K$ such that

$$\begin{cases} P_K[y - \rho_1 S(y,x)] = x, \\ P_K[x - \rho_2 S(x,y)] = y, \end{cases} \tag{1.3}$$

which was studied by Chang, Joseph Lee and Chan in [1].

(III) If $f_1(x,y) = g(x)$ and $f_2(x,y) = h(y)$ for all $x, y \in X$, then the problem (1.1) reduces to the following problem: find $x, y \in X$ such that

$$\begin{cases} P_{K_1}[g(x) - \rho_1 T(x,\ y)] = g(x), \\ P_{K_2}[h(y) - \rho_2 S(x,\ y)] = h(y). \end{cases} \tag{1.4}$$

It is easy to see that the problem (1.4) is equivalent to the following system of variational inequalities: find $x, y \in X$ such that $g(x) \in K_1$, $h(y) \in K_2$ and

$$\begin{cases} \langle T(x,y), u - g(x)\rangle \geq 0, \quad \forall u \in K_1, \\ \langle S(x,y), v - h(y)\rangle \geq 0, \quad \forall v \in K_2. \end{cases} \tag{1.5}$$

Moreover, if K_1 and K_2 are cones, then the system of variational inequalities (1.5) is equivalent to the following system of complementarity problems: find $x, y \in X$ such that $g(x) \in K_1$, $h(y) \in K_2$, $T(x,y) \in K_1^*$, $S(x,y) \in K_2^*$ and

$$\begin{cases} \langle T(x,y), g(x)\rangle = 0, \\ \langle S(x,y), h(y)\rangle = 0, \end{cases} \tag{1.6}$$

where

$$K_1^* = \{u \in X : \langle u, v\rangle \geq 0,\ \forall v \in K_1\},$$
$$K_2^* = \{u \in X : \langle u, v\rangle \geq 0,\ \forall v \in K_2\}.$$

(IV) If $K_1 = K_2 = K$, $g(x) = h(x)$ for all $x \in X$, $f_1 = f_2 = f$ and $S = T$ are univariate mappings, then the problem (1.2) reduces to the following problem: find $x \in X$ such that

$$g(x) = P_K\big[f(x) - T(x)\big] \tag{1.7}$$

which was introduced and studied by Zhao and Sun in [20].

2. Preliminaries

Definition 2.1. Let $K \subset X$ be nonempty, closed and convex. For a given point $x \in X$, $u = P_K x$ is said to be a projection of x onto K if

$$\|x - u\| \leq \|x - v\|, \quad \forall v \in K.$$

Definition 2.2. A mapping $f : X \to X$ is said to be α-Lipschitz continuous if there exists a constant $\alpha > 0$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in X.$$

Definition 2.3. A mapping $f : X \to X$ is said to be γ-strongly monotone if there exists a constant $\gamma > 0$ such that

$$\langle f(x) - f(y), x - y\rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in X.$$

Definition 2.4. A mapping $f : X \to X$ is said to be β-strongly monotone with respect to a mapping $T : X \to X$ if there exists a constant $\beta > 0$ such that

$$\langle f(x) - f(y), Tx - Ty\rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in X.$$

Lemma 2.1 ([5]). *Let $F : X \times X \to X \times X$ be a mapping such that*

$$\begin{cases} \|F(x_1, y) - F(x_2, y)\| \leq m_1 \|x_1 - x_2\|, \\ \|F(x, y_1) - F(x, y_2)\| \leq m_2 \|y_1 - y_2\| \end{cases} \tag{2.1}$$

for all $x_1,\ x_2,\ y_1,\ y_2 \in X$, where $0 < m_1 < 1$ and $0 < m_2 < 1$ are constants. Then there exists a unique point $(x^*, y^*) \in X \times X$ such that $F(x^*, y^*) = (x^*, y^*)$.

Lemma 2.2 ([9]). *The projection mapping $P_K : X \to K$ is a nonexpansive mapping, that is,*

$$\|P_K x - P_K y\| \le \|x - y\|, \quad \forall x, y \in X$$

Lemma 2.3 ([9]). *For a given $x \in X$, $u = P_K x$ if and only if $u \in K$ satisfies the inequality*

$$\langle x - u, u - v\rangle \ge 0, \quad \forall v \in K.$$

3. Main Results

Now, we are ready to give our main results in this paper.

Theorem 3.1. *Let $g : X \to X$ be α_1-Lipschitz continuous and β_1-strongly monotone. Let $h : X \to X$ be α_2-Lipschitz continuous and β_2-strongly monotone. Suppose that $T, S, C_1, C_2 : X \times X \to X$ are four nonlinear mappings such that T is γ_{11}-Lipschitz continuous with respect to the first argument and γ_{12}-Lipschitz continuous with respect to the second argument, S is γ_{21}-Lipschitz continuous with respect to the second argument and γ_{22}-Lipschitz continuous with respect to the first argument, C_1 is γ_{31}-Lipschitz continuous with respect to the first argument and γ_{32}-Lipschitz continuous with respect to the second argument, C_2 is γ_{41}-Lipschitz continuous with respect to the first argument and γ_{42}-Lipschitz continuous with respect to the second argument. Let $f_1 : X \times X \to X$ be α_{31}-Lipschitz continuous and β_{31}-strongly monotone with respect to T for the first argument, and α_{32}-Lipschitz continuous and β_{32}-strongly monotone with respect to T for the second argument. Let $f_2 : X \times X \to X$ be α_{41}-Lipschitz continuous and β_{41}-strongly monotone with respect to S for the second argument, and α_{42}-Lipschitz continuous and β_{42}-strongly monotone with respect to S for the first argument. If*

$$0 < m_{11} + m_{12} < 1, \quad 0 < m_{21} + m_{22} < 1, \tag{3.1}$$

where

$$\begin{aligned} m_{11} &= \sqrt{\alpha_{31}^2 - 2\rho_1\beta_{31} + \rho_1^2\gamma_{11}^2} + \sqrt{1 - 2\beta_1 + \alpha_1^2} + \gamma_{31}, \\ m_{12} &= \sqrt{\alpha_{42}^2 - 2\rho_2\beta_{42} + \rho_2^2\gamma_{22}^2} + \gamma_{41}, \\ m_{21} &= \sqrt{\alpha_{41}^2 - 2\rho_2\beta_{41} + \rho_2^2\gamma_{21}^2} + \sqrt{1 - 2\beta_2 + \alpha_2^2} + \gamma_{42}, \\ m_{22} &= \sqrt{\alpha_{32}^2 - 2\rho_1\beta_{32} + \rho_1^2\gamma_{12}^2} + \gamma_{32}, \end{aligned}$$

then the system of nonlinear projection equations with perturbation problem (1.1) has a unique solution.

Proof. Let $F : X \times X \to X \times X$ be a mapping defined by

$$F(x, y) = (F_1(x, y), F_2(x, y)), \quad \forall (x, y) \in X \times X,$$

where

$$
\begin{aligned}
F_1(x,y) &= P_{K_1}[f_1(x,y)-\rho_1 T(x,y)]+C_1(x,y)+x-g(x),\\
F_2(x,y) &= P_{K_2}[f_2(x,y)-\rho_2 S(x,y)]+C_2(x,y)+y-h(y).
\end{aligned}
$$

We now show that F satisfies the condition (2.1). In fact, for any $x_1,\ x_2,\ y\in X$, it follows from the assumptions and Lemma 2.2 that

$$
\begin{aligned}
&\Big\|F_1(x_1,\ y)-F_1(x_2,\ y)\Big\|\\
&=\Big\|P_{K_1}[f_1(x_1,y)-\rho_1 T(x_1,\ y)]+C_1(x_1,y)+x_1-g(x_1)\\
&\quad -P_{K_1}[f_1(x_2,y)-\rho_1 T(x_2,y)]-C_1(x_2,y)-x_2+g(x_2)\Big\|\\
&\le \Big\|P_{K_1}[f_1(x_1,y)-\rho_1 T(x_1,y)]-P_{K_1}[f_1(x_2,y)-\rho_1 T(x_2,y)]\Big\|\\
&\quad +\Big\|(x_1-g(x_1))-(x_2-g(x_2))\Big\|+\Big\|C_1(x_1,y)-C_1(x_2,y)\Big\|\\
&\le \Big\|(f_1(x_1,y)-f_1(x_2,y))-\big(\rho_1 T(x_1,y)-\rho_1 T(x_2,y)\big)\Big\|\\
&\quad +\Big\|(x_1-x_2)-(g(x_1)-g(x_2))\Big\|+\Big\|C_1(x_1,y)-C_1(x_2,y)\Big\|\\
&=\Big(\big\|f_1(x_1,y)-f_1(x_2,y)\big\|^2-2\rho_1\big\langle f_1(x_1,y)-f_1(x_2,y),\ T(x_1,y)-T(x_2,y)\big\rangle\\
&\quad +\rho_1^2\big\|T(x_1,y)-T(x_2,y)\big\|^2\Big)^{1/2}\\
&\quad +\Big(\|x_1-x_2\|^2-2\big\langle x_1-x_2,g(x_1)-g(x_2)\big\rangle\\
&\quad +\big\|g(x_1)-g(x_2)\big\|^2\Big)^{1/2}+\Big\|C_1(x_1,y)-C_1(x_2,y)\Big\|\\
&\le \Big(\alpha_{31}^2\|x_1-x_2\|^2-2\rho_1\beta_{31}\|x_1-x_2\|^2+\rho_1^2\gamma_{11}^2\|x_1-x_2\|^2\Big)^{1/2}\\
&\quad +\Big(\|x_1-x_2\|^2-2\beta_1\|x_1-x_2\|^2+\alpha_1^2\|x_1-x_2\|^2\Big)^{1/2}+\gamma_{31}\|x_1-x_2\|\\
&=\Big(\sqrt{\alpha_{31}^2-2\rho_1\beta_{31}+\rho_1^2\gamma_{11}^2}+\sqrt{1-2\beta_1+\alpha_1^2}+\gamma_{31}\Big)\Big\|x_1-x_2\Big\|.
\end{aligned}
$$

This implies that

$$
\Big\|F_1(x_1,\ y)-F_1(x_2,\ y)\Big\|\le m_{11}\|x_1-x_2\|. \tag{3.2}
$$

Again for any $x_1,\ x_2,\ y\in X$, from the assumptions and Lemma 2.2, we have

$$
\begin{aligned}
&\Big\|F_2(x_1,\ y) - F_2(x_2,\ y)\Big\| \\
&\quad = \Big\|P_{K_2}[f_2(x_1,y) - \rho_2 S(x_1,y)] + C_2(x_1,y) + y - h(y) \\
&\qquad - P_{K_2}[f_2(x_2,y) - \rho_2 S(x_2,y)] - C_2(x_2,y) - y + h(y)\Big\| \\
&\quad \le \Big\|P_{K_2}[f_2(x_1,y) - \rho_2 S(x_1,y)] - P_{K_2}[f_2(x_2,y) - \rho_2 S(x_2,y)]\Big\| \\
&\qquad + \Big\|C_2(x_1,y) - C_2(x_2,y)\Big\| \\
&\quad \le \Big\|(f_2(x_1,y) - f_2(x_2,y)) - \big(\rho_2 S(x_1,y) - \rho_2 S(x_2,y)\big)\Big\| \\
&\qquad + \Big\|C_2(x_1,y) - C_2(x_2,y)\Big\| \\
&\quad = \Big(\big\|f_2(x_1,y) - f_2(x_2,y)\big\|^2 - 2\rho_2\big\langle f_2(x_1,y) - f_2(x_2,y),\ S(x_1,y) - S(x_2,y)\big\rangle \\
&\qquad + \rho_2^2\big\|S(x_1,y) - S(x_2,y)\big\|^2\Big)^{1/2} + \Big\|C_2(x_1,y) - C_2(x_2,y)\Big\| \\
&\quad \le \Big(\alpha_{42}^2\|x_1 - x_2\|^2 - 2\rho_2\beta_{42}\|x_1 - x_2\|^2 + \rho_2^2\gamma_{22}^2\|x_1 - x_2\|^2\Big)^{1/2} + \gamma_{41}\|x_1 - x_2\| \\
&\quad = \Big(\sqrt{\alpha_{42}^2 - 2\rho_2\beta_{42} + \rho_2^2\gamma_{22}^2} + \gamma_{41}\Big)\Big\|x_1 - x_2\Big\|.
\end{aligned}
$$

It follows that

$$\Big\|F_2(x_1,\ y) - F_2(x_2,\ y)\Big\| \le m_{12}\|x_1 - x_2\|. \tag{3.3}$$

By (3.2) and (3.3), we have

$$
\begin{aligned}
&\Big\|F(x_1,y) - F(x_2,y)\Big\| \\
&= \Big\|F_1(x_1,y) - F_1(x_2,y)\Big\| + \Big\|F_2(x_1,y) - F_2(x_2,y)\Big\| \\
&\le (m_{11} + m_{12})\|x_1 - x_2\|.
\end{aligned} \tag{3.4}
$$

Similarly, we can prove

$$\Big\|F_2(x,y_1) - F_2(x,y_2)\Big\| \le m_{21}\|y_1 - y_2\| \tag{3.5}$$

and

$$\Big\|F_1(x,y_1) - F_1(x,y_2)\Big\| \le m_{22}\|y_1 - y_2\|. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\Big\|F(x,y_1) - F(x,y_2)\Big\| \le (m_{21} + m_{22})\|y_1 - y_2\|. \tag{3.7}$$

Since $0 < m_{11} + m_{12} < 1$ and $0 < m_{21} + m_{22} < 1$, by (3.4) and (3.7), we know that F satisfies condition (2.1). It follows from Lemma 2.1 that there exists a unique $(x^*,\ y^*) \in X \times X$ such that

$$(x^*, y^*) = F(x^*,\ y^*) = \big(F_1(x^*, y^*),\ F_2(x^*, y^*)\big).$$

This implies that

$$g(x^*) = P_{K_1}[f_1(x^*, y^*) - \rho_1 T(x^*, y^*)] + C_1(x^*, y^*),$$

$$h(y^*) = P_{K_2}[f_2(x^*, y^*) - \rho_2 S(x^*, y^*)] + C_2(x^*, y^*)$$

and so $(x^*,\ y^*)$ is the unique solution of the system of nonlinear projection equations with perturbation problem (1.1). □

Algorithm 3.1. For any given point $(x_0,\ y_0) \in X \times X$, compute the sequence $\{(x_n,\ y_n)\}$ as follows:

$$\begin{cases} x_{n+1} = P_{K_1}[f_1(x_n, y_n) - \rho_1 T(x_n, y_n)] + C_1(x_n, y_n) + x_n - g(x_n), \\ y_{n+1} = P_{K_2}[f_2(x_n, y_n) - \rho_2 S(x_n, y_n)] + C_2(x_n, y_n) + y_n - h(y_n), \end{cases}$$

where P_{K_i} is the projection of X onto K_i, $\rho_i > 0$ $(i = 1, 2)$ are constants.

Theorem 3.2. *Suppose that all the conditions of Theorem 3.1 are satisfied. Let $\{x_n, y_n\}$ be a sequence generated by Algorithm 3.1. Then*

$$x_n \to x^*, \quad y_n \to y^*\ (n \to \infty),$$

where $(x^*,\ y^*)$ is the unique solution of the system of nonlinear projection equations with perturbation problem (1.1).

Proof. From Algorithm, we know that $x_{n+1} = F_1(x_n, y_n)$ and $y_{n+1} = F_2(x_n, y_n)$. It follows from (3.2), (3.3), (3.5), (3.6) that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|F_1(x_n, y_n) - F_1(x^*, y^*)\| \\ &\le m_{11}\|x_n - x^*\| + m_{22}\|y_n - y^*\| \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \|y_{n+1} - y^*\| &= \|F_2(x_n, y_n) - F_2(x^*, y^*)\| \\ &\le m_{12}\|x_n - x^*\| + m_{21}\|y_n - y^*\|. \end{aligned} \tag{3.9}$$

Let $m = \max\{m_{11} + m_{12}, m_{21} + m_{22}\}$. It follows from (3.8) and (3.9) that

$$\begin{aligned} &\big\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\big\| \\ &\quad = \|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\ &\quad \le (m_{11} + m_{12})\|x_n - x^*\| + (m_{21} + m_{22})\|y_n - y^*\| \\ &\quad \le m\big(\|x_n - x^*\| + \|y_n - y^*\|\big) \\ &\quad = m\big\|(x_n, y_n) - (x^*, y^*)\big\|. \end{aligned}$$

It is easy to see that

$$\big\|(x_n, y_n) - (x^*, y^*)\big\| \le m^n\big\|(x_0, y_0) - (x^*, y^*)\big\|.$$

Since $0 < m < 1$, we have

$$\big\|(x_n, y_n) - (x^*, y^*)\big\| \to 0 \quad (n \to \infty),$$

that is, $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. This completes the proof. □

From Theorems 3.1 and 3.2, we have the following result.

Theorem 3.3. *Suppose that $T,\ S : X \times X \to X$ are nonlinear mappings such that T is γ_{11}-Lipschitz continuous with respect to the first argument and γ_{12}-Lipschitz continuous with respect to the second argument, S is γ_{21}-Lipschitz continuous with respect to the second argument and γ_{22}-Lipschitz continuous with respect to the first argument. Let $g : X \to X$ be α_1-Lipschitz continuous, β_1-strongly monotone and β_{11}-strongly monotone with respect to T for the first argument. Let $h : X \to X$ be α_2-Lipschitz continuous, β_2-strongly monotone and β_{21}-strongly monotone with respect to S for the second argument. If*

$$0 < m_{11} + m_{12} < 1, \quad 0 < m_{21} + m_{22} < 1,$$

where

$$m_{11} = \sqrt{\alpha_1^2 - 2\rho_1\beta_{11} + \rho_1^2\gamma_{11}^2} + \sqrt{1 - 2\beta_1 + \alpha_1^2} \quad m_{12} = \rho_2\gamma_{22},$$
$$m_{21} = \sqrt{\alpha_2^2 - 2\rho_2\beta_{21} + \rho_2^2\gamma_{21}^2} + \sqrt{1 - 2\beta_2 + \alpha_2^2} \quad m_{22} = \rho_1\gamma_{12},$$

then the system of variational inequalities (1.5) *has a unique solution $(x^*, y^*) \in X \times X$. Moreover, for any given point $(x_0,\ y_0) \in X \times X$, let*

$$\begin{cases} x_{n+1} = P_{K_1}[g(x_n) - \rho_1 T(x_n, y_n)] + x_n - g(x_n), \\ \\ y_{n+1} = P_{K_2}[h(y_n) - \rho_2 S(x_n, y_n)] + y_n - h(y_n). \end{cases}$$

Then $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$.

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