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PSEUDO-BCI ALGEBRAS

Wiesław A. Dudek and Young Bae Jun*

Abstract. As a generalization of BCI-algebras, the notion of pseudo-BCI algebras is introduced, and some of their properties are investigated. Characterizations of pseudo-BCI algebras are established. Some conditions for a pseudo-BCI algebra to be a pseudo-BCK algebra are given.

1. Introduction

In [1], G. Georgescu and A. Iorgulescu introduced the notion of pseudo-BCK algebras as an extension of BCK-algebras. In this paper, we introduce the notion of pseudo-BCI algebras as an extension of BCI-algebras, and investigate some properties.

2. Preliminaries

Recall that a *BCI-algebra* is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms: for every $x, y, z \in X$,

- $((x * y) * (x * z)) * (z * y) = 0$,
- $(x * (x * y)) * y = 0$,
- $x * x = 0$,
- $x * y = 0$ and $y * x = 0$ imply $x = y$.

For any BCI-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X.

3. Pseudo-BCI algebras

Definition 3.1. A *pseudo-BCI algebra* is a structure $\mathfrak{X} = (X, \leq, *, \diamond, 0)$, where "$\leq$" is a binary relation on a set X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, verifying the axioms: for all $x, y, z \in X$,

(a1) $(x * y) \diamond (x * z) \leq z * y$, $(x \diamond y) * (x \diamond z) \leq z \diamond y$,
(a2) $x * (x \diamond y) \leq y$, $x \diamond (x * y) \leq y$,
(a3) $x \leq x$,
(a4) $x \leq y$, $y \leq x \Longrightarrow x = y$,

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*Corresponding author.

(a5) $x \le y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0$.

Note that every pseudo-BCI algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$ is a BCI-algebra. Every pseudo-BCK algebra is a pseudo-BCI algebra.

Proposition 3.2. *In a pseudo-BCI algebra* $\mathfrak{X}$ *the following holds:*

(b1) $x \le 0 \Rightarrow x = 0$.
(b2) $x \le y \Rightarrow z * y \le z * x,\ z \diamond y \le z \diamond x$.
(b3) $x \le y,\, y \le z \Rightarrow x \le z$.
(b4) $(x * y) \diamond z = (x \diamond z) * y$.
(b5) $x * y \le z \Leftrightarrow x \diamond z \le y$.
(b6) $(x * y) * (z * y) \le x * z,\ (x \diamond y) \diamond (z \diamond y) \le x \diamond z$.
(b7) $x \le y \Rightarrow x * z \le y * z,\ x \diamond z \le y \diamond z$.
(b8) $x * 0 = x = x \diamond 0$.
(b9) $x * \big(x \diamond (x * y)\big) = x * y$ *and* $x \diamond \big(x * (x \diamond y)\big) = x \diamond y$.

Proof. (b1) If $x \le 0$, then $0 \diamond x = (x * 0) \diamond x = \big(x * (x \diamond x)\big) \diamond x = 0$, that is, $0 \le x$. Hence $x = 0$ by (a6).

(b2) Let $x, y \in X$ be such that $x \le y$. Then $(z * y) \diamond (z * x) \le x * y = 0$, and so $(z * y) \diamond (z * x) = 0$ by (b1). Therefore $z * y \le z * x$. Now

$$(z \diamond y) * (z \diamond x) \le x \diamond y = 0,$$

and thus $(z \diamond y) * (z \diamond x) = 0$ by (b1). This implies that $z \diamond y \le z \diamond x$.

(b3) Let $x, y, z \in X$ be such that $x \le y$ and $y \le z$. Then $x * z \le x * y = 0$, which implies that $x * z = 0$, that is, $x \le z$.

(b4) Since $x * (x \diamond z) \le z$ by (a2), it follows from (b2) and (a1) that

$$(x * y) \diamond z \le (x * y) \diamond \big(x * (x \diamond z)\big) \le (x \diamond z) * y.$$

Also since $x \diamond (x * y) \le y$, we have

$$(x \diamond z) * y \le (x \diamond z) * \big(x \diamond (x * y)\big) \le (x * y) \diamond z$$

by (b2) and (a1). Hence, by (a4), we get $(x * y) \diamond z = (x \diamond z) * y$.

(b5) If $x * y \le z$, then $0 = (x * y) \diamond z = (x \diamond z) * y$, and so $x \diamond z \le y$. Conversely, if $x \diamond z \le y$, then $0 = (x \diamond z) * y = (x * y) \diamond z$. Hence $x * y \le z$.

(b6) is by (a1) and (b5).

(b7) Let $x, y \in X$ be such that $x \le y$. Using (b6), we have

$$(x * z) * (y * z) \le x * y = 0 \ \text{ and } \ (x \diamond z) \diamond (y \diamond z) \le x \diamond y = 0.$$

It follows from (b1) that $(x * z) * (y * z) = 0$ and $(x \diamond z) \diamond (y \diamond z) = 0$, that is, $x * z \le y * z$ and $x \diamond z \le y \diamond z$.

(b8) Putting $y = 0$ in (a2), we have $x * (x \diamond 0) \le 0$ and $x \diamond (x * 0) \le 0$. It follows from (b1) that $x * (x \diamond 0) = 0$ and $x \diamond (x * 0) = 0$, so that $x \le x \diamond 0$ and $x \le x * 0$. On the other hand,

$$(x \diamond 0) * x = (x * x) \diamond 0 = 0 \diamond 0 = 0 \ \text{ and } \ (x * 0) \diamond x = (x \diamond x) * 0 = 0 * 0 = 0,$$

and so $x \diamond 0 \le x$ and $x * 0 \le x$. By (a4), $x * 0 = x = x \diamond 0$.

(b9) By (a2), $x * \big(x \diamond (x * y)\big) \leq x * y$ and $x \diamond \big(x * (x \diamond y)\big) \leq x \diamond y$. On the other hand,

$$(x * y) \diamond \big(x * (x \diamond (x * y))\big) \leq \big(x \diamond (x * y)\big) * y = (x * y) \diamond (x * y) = 0$$

and

$$(x \diamond y) * \big(x \diamond (x * (x \diamond y))\big) \leq \big(x * (x \diamond y)\big) \diamond y = (x \diamond y) * (x \diamond y) = 0.$$

It follows from (b1) that

$$(x * y) \diamond \big(x * (x \diamond (x * y))\big) = 0 \text{ and } (x \diamond y) * \big(x \diamond (x * (x \diamond y))\big) = 0,$$

that is, $x * y \leq x * \big(x \diamond (x * y)\big)$ and $x \diamond y \leq x \diamond \big(x * (x \diamond y)\big)$. Hence (b9) is valid by (a4). □

We now give a characterization of a pseudo-BCI algebra.

Theorem 3.3. *A structure $\mathfrak{X} = (X, \leq, *, \diamond, 0)$ is a pseudo-BCI algebra if and only if it satisfies* (a1), (a4), (a5) *and* (b8).

Proof. The necessity is obvious. Assume that $\mathfrak{X}$ satisfies (a1), (a4), (a5) and (b8). Substituting 0 for y and z in (a1) and using (b8), we have $x \diamond x \leq 0$ and $x * x \leq 0$. It follows from (b8) that

$$x \diamond x = (x \diamond x) * 0 = 0 \text{ and } x * x = (x * x) \diamond 0 = 0,$$

so that $x \leq x$. Putting $y = 0$ in (a1) and using (b8), we get

$$x \diamond (x * z) = (x * 0) \diamond (x * z) \leq z * 0 = z$$

and

$$x * (x \diamond z) = (x \diamond 0) * (x \diamond z) \leq z \diamond 0 = z.$$

This completes the proof. □

Definition 3.4. By a *pseudo-BCI subalgebra* of a pseudo-BCI algebra $\mathfrak{X}$, we mean a subset S of X which satisfies $x * y \in S$ and $x \diamond y \in S$ for all $x, y \in S$.

Theorem 3.5. *For any pseudo-BCI algebra $\mathfrak{X}$ the set*

$$K(\mathfrak{X}) := \{x \in X \mid 0 \leq x\}$$

is a pseudo-BCI subalgebra of $\mathfrak{X}$, and so a pseudo-BCK algebra.

Proof. Let $x, y \in K(\mathfrak{X})$. Then $0 \leq x$ and $0 \leq y$. It follows from (a5) and (b7) that $0 = 0 * y \leq x * y$ and $0 = 0 \diamond y \leq x \diamond y$ so that $x * y \in K(\mathfrak{X})$ and $x \diamond y \in K(\mathfrak{X})$. Hence $K(\mathfrak{X})$ is a pseudo-BCI subalgebra of $\mathfrak{X}$. □

Theorem 3.6. *If a pseudo-BCI algebra $\mathfrak{X}$ satisfies*

$$x \diamond (x * y) = y \diamond (y * x) \text{ and } x * (x \diamond y) = y * (y \diamond x)$$

for all $x, y \in X$, then $\mathfrak{X}$ is a pseudo-BCK algebra.

Proof. Let $\mathfrak{X}$ be a pseudo-BCI algebra such that

$$x \diamond (x * y) = y \diamond (y * x) \text{ and } x * (x \diamond y) = y * (y \diamond x)$$

for all $x, y \in X$. We first claim that $x * a \notin K(\mathfrak{X})$ and $y \diamond b \notin K(\mathfrak{X})$ for all $x, y \in K(\mathfrak{X})$ and $a, b \in X \setminus K(\mathfrak{X})$. Indeed, if $x * a \in K(\mathfrak{X})$ and $y \diamond b \in K(\mathfrak{X})$ for some $x, y \in K(\mathfrak{X})$ and $a, b \in X \setminus K(\mathfrak{X})$, then $x \diamond (x * a) \in K(\mathfrak{X})$ and $y * (y \diamond b) \in K(\mathfrak{X})$. Hence $0 \leq x \diamond (x * a) \leq a$ and $0 \leq y * (y \diamond b) \leq b$. It follows that $0 \leq a$ and $0 \leq b$ so that $a, b \in K(\mathfrak{X})$. This is a contradiction. Assume that $X \neq K(\mathfrak{X})$. Then there exists $a \in X \setminus K(\mathfrak{X})$, and so

$$0 * (0 \diamond a) = a * (a \diamond 0) = a * a = 0$$

and

$$0 \diamond (0 * a) = a \diamond (a * 0) = a \diamond a = 0.$$

Thus $0 \leq 0 * a$ and $0 \leq 0 \diamond a$, that is, $0 * a \in K(\mathfrak{X})$ and $0 \diamond a \in K(\mathfrak{X})$. This is a contradiction, and consequently $X = K(\mathfrak{X})$. Therefore $\mathfrak{X}$ is a pseudo-BCK algebra. □

Theorem 3.7. *If a pseudo-BCI algebra $\mathfrak{X}$ satisfies*

$$(x * y) \diamond y = x \diamond y \text{ and } (x \diamond y) * y = x * y \tag{1}$$

for all $x, y \in X$, then $\mathfrak{X}$ is a pseudo-BCK algebra.

Proof. Let $x = y$ in (1). Then $0 \diamond x = (x * x) \diamond x = x \diamond x = 0$ and $0 * x = (x \diamond x) * x = x * x = 0$, that is, $0 \leq x$ for all $x \in X$. Hence $X = K(\mathfrak{X})$, and so $\mathfrak{X}$ is a pseudo-BCK algebra. □

Theorem 3.8. *If a pseudo-BCI algebra $\mathfrak{X}$ satisfies*

$$x * (y \diamond x) = x \text{ and } x \diamond (y * x) = x \tag{2}$$

for all $x, y \in X$, then $\mathfrak{X}$ is a pseudo-BCK algebra.

Proof. Putting $x = 0$ in (2), then $0 = 0 * (y \diamond 0) = 0 * y$ and $0 = 0 \diamond (y * 0) = 0 \diamond y$ for any $y \in X$. It follows that $X = K(\mathfrak{X})$ so that $\mathfrak{X}$ is a pseudo-BCK algebra. □

References

[1] G. Georgescu and A. Iorgulescu, *Pseudo-BCK algebras: an extension of BCK algebras,* Combinatorics, computability and logic (Constanţa, 2001), 97–114, Springer Ser. Discrete Math. Theor. Comput. Sci. Springer, London, 2001.

W. A. Dudek
Institute of Mathematics, Technical University of Wrocław, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
E-mail address: dudek@im.pwr.wroc.pl

Y. B. Jun
Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Korea
E-mail address: skywine@gmail.com