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A STUDY ON WEAK BI-IDEALS OF NEAR-RINGS

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Abstract. From the notion of bi-ideals in near-rings, various generalizations of regularity conditions have been studied. In this paper, we generalize further the notion of bi-ideals and introduce the notion of weak bi-ideals in near-rings and obtain some characterizations using this concept in left self distributive near-rings.

1. Introduction

In this paper, by a near-ring we mean a right near-ring. For basic definitions and notations, we may refer to Pilz [3]. Tamizh Chelvam and Ganesan [4] introduced the notion of bi-ideals in near-rings. Further Tamizh Chelvam [5] introduced the concept of b-regular near-rings and obtained equivalent conditions for regularity in terms of bi-ideals. In this paper the notion weak bi-ideals has been introduced and studied to the extent possible.

Let N be a right near-ring. For two subsets A and B of N, $AB = \{ab \mid a \in A, b \in B\}$ and $A * B = \{a_1(a_2 + b)$ - $a_1a_2 \mid a_1, a_2 \in A$ and $b \in B\}$. A subgroup B of $(N, +)$ is said to be a *bi-ideal* of N if $BNB \cap (BN) * B \subseteq B$ [4]. In the case of a zero-symmetric near-ring, a subgroup B of $(N, +)$ is a bi-ideal if $BNB \subseteq B$. A subgroup Q of $(N, +)$ is called a *quasi-ideal* of N if $QN \cap NQ \cap N * Q \subseteq Q$ [4]. If N is zero-symmetric, a subgroup Q of $(N, +)$ is a quasi-ideal of N if $QN \cap NQ \subseteq Q$.

A near-ring N is said to be *left (right)-unital* if $a \in Na (a \in aN)$ for all $a \in N$. A near-ring N is said to be *unital* if it is both left as well as right unital. An element $a \in N$ is said to be *regular* if $a = aba$ for some $b \in N$. A near-ring N is said to be *regular* if every element in N is regular. It may be noted that a regular near-ring is a unital near-ring, but not the converse. An element $a \in N$ is said to be strongly regular if $a = ba^2$, for some $b \in N$. A near-ring N is called strongly regular if every element in N is strongly regular. N is said to satisfy IFP (Insertion of Factors Property) if $ab = 0$ implies $axb = 0$ for all $x \in N$.

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A near-ring is called left bi-potent if $Na = Na^2$ for $a \in N$. A subgroup M of $(N, +)$ is said to be a left (right) N-subgroup if $NM \subseteq M(MN \subseteq M)$. A near-ring N is said to be two sided if every left N-subgroup is a right N-subgroup and vice versa. A near-ring N is called b-regular near-ring if $a \in (a)_r N(a)_l$ for every $a \in N$ where $(a)_r$ $((a)_l$ is the right (left) N-subgroup generated by $a \in N$[6]. Note that every regular near-ring is b-regular.

A near-ring N is said to be left self distributive if $abc = abac$ for all $a, b, c \in N$. Let E be the set of all idempotents of N and L the set of all nilpotent elements of N.

2. A Study on Weak Bi-ideals

In this section, we introduce weak bi-ideals and obtain some of the properties of this concept.

Definition 2.1. A subgroup B of (N, +) is said to be a *weak bi-ideal* if $B^3 \subseteq B$.

Example 2.2. Every bi-ideal is a weak bi-ideal, but the converse is not true. For, consider the near-ring N constructed on the Klein's 4-group according to the scheme (0,0,2,1) (p. 408, Pilz [3]). In this near-ring, one can check that $\{0, b\}$ and $\{0, c\}$ are weak bi-ideals. Note that $\{0, b\}N\{0, b\} = \{0, c, b\}$ and hence $\{0, b\}$ is not a bi-ideal of N.

Proposition 2.3. *If B is a weak bi-ideal of a near-ring N and S is a sub near-ring of N, then $B \cap S$ is a weak bi-ideal of N.*

Proof. Let $C = B \cap S$. Now $C^3 = (B\cap S)((B\cap S)(B\cap S)) \subseteq (B\cap S)(BB\cap SS) \subseteq (B\cap S)BB \cap (B\cap S)SS \subseteq BBB \cap SSS = B^3 \cap SS \subseteq B \cap S = C$, i.e., $C^3 \subseteq C$. Therefore C is a weak bi-ideal of N. □

Proposition 2.4. *Let B be a weak bi-ideal of N. Then Bb and $b'B$ are the weak bi-ideals of N where $b, b' \in B$ and b' is a distributive element.*

Proof. Clearly Bb is a subgroup of $(N, +)$. Also $(Bb)^3 = BbBbBb \subseteq BBBb \subseteq B^3b \subseteq Bb$. Since b' is distributive, $b'B$ is a subgroup of $(N, +)$ and $(b'B)^3 = b'Bb'Bb'B \subseteq b'BBB = b'B^3 \subseteq b'B$. Hence Bb and $b'B$ are weak bi-ideals of N. □

Corollary 2.5. *Let B be a weak bi-ideal of N. For $b, c \in B$, if b is distributive, then bBc is a weak bi-ideal of N.*

Proposition 2.6. *Let N be a left self-distributive left-unital near-ring. Then $B^3 = B$ for every weak bi-ideal B of N if and only if N is strongly regular.*

Proof. Let B be a weak bi-ideal of N. If N is strongly regular, then N has no non-zero nilpotent elements. This further implies that N has IFP. Let $x \in N$ and $x = ax^2$ for $a \in N$. Now $(xax - x)x = 0$ and so $x(xax - x) = 0$ as N has IFP. Hence $(xax - x)^2 = 0$ and so $xax - x = 0$. i.e., x is regular and N is regular. Let $b \in B$. Since N is regular, $b = bab$ for some $a \in N$. By our assumption that N is left self-distributive, we have $bab = babb$. Thus $b = bab = babb = babb^2 = bb^2 = b^3 \subseteq B^3$, i.e., $B \subseteq B^3$. Hence $B = B^3$ for every weak bi-ideal B of N. Conversely let $a \in N$. Since Na is a weak bi-ideal of N and N is a left-unital near-ring, we get $a \in Na = (Na)^3 = NaNaNa \subseteq NaNa$, i.e., $a = n_1an_2a$. Since N is left self-distributive, $a = n_1an_2a^2$, i.e., N is strongly regular. □

Proposition 2.7. *Let N be a left self-distributive left unital near-ring. Then $B = NB^2$ for every strong bi-ideal B of N if and only if N is strongly regular.*

Proof. Assume that $B = NB^2$ for every strong bi-ideal B of N. Since Na is a strong bi-ideal of N and N is a left unital near-ring, we have $a \in Na = N(Na)^2 = NNaNa \subseteq NaNa$, i.e., $a = n_1an_2a$. Since N is a left self-distributive near-ring, $a = n_1an_2a = n_1an_2a^2 \in Na^2$, i.e., N is strongly regular. Conversely, let B be a strong bi-ideal of N. Since N is strongly regular, for $b \in B$, $b = nb^2 \in NB^2$, i.e., $B \subseteq NB^2$. Hence $NB^2 = B$ for every strong bi-ideal B of N □

Theorem 2.8. *Let N be a left self-distributive left unital near-ring. Then $B^3 = B$ for every weak bi-ideal B of N if and only if $NB^2 = B$ for every strong bi-ideal B of N.*

Proof. Follows from the Propositions 2.6 and 2.7. □

Proposition 2.9. *Let N be a left self-distributive left-unital near-ring. Then $B = BNB$ for every bi-ideal B of N if and only if N is regular.*

Proof. Let B be a bi-ideal of N. If N is regular, then $B = BNB$ for every bi-ideal B of N. Conversely, let $B = BNB$ for every bi-ideal B of N. Since Na is a bi-ideal of N and N is a left-unital near-ring, we have $a \in Na = NaNNa$, i.e., $a = n_1an_2a$ for some $n_1, n_2 \in N$. Since N is a left self-distributive near-ring, $a = n_1an_2a^2 \subseteq Na^2$, i.e., N is strongly regular and as in the proof of Propostion 2.6, N is regular. □

Proposition 2.10. *Let N be a left self-distributive left-unital near-ring. Then $B = B^3$ for every weak bi-ideal B of N if and only if $A \cap C = AC$ for any two left N-subgroups A and C of N.*

Proof. Assume that $B = B^3$ for every weak bi-ideal B of N. By the Proposition 2.6, N is strongly regular. Therefore N is regular. Let A and C be any two left N-subgroups of N. Let $x \in A \cap C$. Since N is regular, $x = xax$ for some $a \in N$. Therefore $(xa)x \in ANC \subseteq AC$ which implies that $A \cap C = AC$. On the other hand, let $x \in AC$. Since N is strongly regular, $L = 0$ and so $en = ene$ for all $e \in E$. Then $x = yz \in AC$ with $y \in A$ and $z \in C$. Now $x = yz = (yby)z$. Since by is an idempotent element $(by)z = (by)z(by)$. Thus $x = yz = y(by)z = y(by)z(by) \in NA \subseteq A$. Thus $x \in A \cap C$. From the two inclusions proved above, we get that $AC = A \cap C$.

Conversely let $a \in N$. Since Na is a left N-subgroup of N, from the assumption we get that $Na = Na \cap Na = NaNa$. But $Na = Na \cap N = NaN$ implies that $Naa = NaNa$. Therefore $Na = Na^2$. Since N is a left-unital near-ring, $a \in Na = Na^2$, i.e., N is strongly regular. By the Proposition 2.6, $B = B^3$ for every weak bi-ideal B of N. □

Theorem 2.11. *Let N be a left self-distributive left unital near-ring. Then the following conditions are equivalent.*

(i) $B = B^3$ *for every weak bi-ideal* B *of* N.
(ii) N *is regular and* $NxNy = NyNx$ *for all* $x, y \in N$.
(iii) $NxNy = Nxy$ *for all* $x, y \in N$.
(iv) N *is left bi-potent.*
(v) N *is Boolean.*

Proof. $(i) \Rightarrow (ii)$ Assume that $B = B^3$ for every weak bi-ideal B of N. By the Proposition 2.6, N is strongly regular and so N is regular. Again by the Proposition 2.10, $A \cap B = AB$ for two left N-sub groups A and B of N. Let $x, y \in N$. Since Nx and Ny are left N-sub groups of N, from the above fact we get that $NxNy = Nx \cap Ny = Ny \cap Nx = NyNx$.
$(ii) \Rightarrow (iii)$ Let $x, y \in N$. Let A be a left N-subgroup of N. Trivially, $A^2 \subseteq A$. Since N is regular, for any $a \in N, a = aba$ for some $b \in N$. Hence $a = a(ba) \in A(NA) \subseteq AA = A^2$. Thus $A = A^2$. Since $Nx \cap Ny$ is a left N-sub group of N, $Nx \cap Ny = (Nx \cap Ny)^2 \subseteq NxNy \subseteq Ny$. Again by the assumption, $NxNy = NyNx \subseteq Nx$. Therefore $Nx \cap Ny = NxNy$. Now $Nx = Nx \cap N = NxN$ and from this we get that $Nxy = NxNy$. Therefore $Nxy = Nx \cap Ny$ for all $x, y \in N$.
$(iii) \Rightarrow (iv)$Let $a \in N$. Then $Na = Na \cap Na = Naa = Na^2$. i.e., N is left bi-potent near-ring.
$(iv) \Rightarrow (v)$By the assumption that $a \in Na = Na^2, N$ is strongly regular and so N is regular. Let $x \in N$. Then $x = xyx = xyxx = x^2$. i.e.,N is Boolean.
$(v) \Rightarrow (i)$ Let B be a weak bi-ideal of N. Let $x \in B$. By the assumption, $x = x^2 = x^3 \in B^3$. Therefore $B \subseteq B^3$ and hence $B = B^3$. □

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