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GALOIS GROUP OF GENERALIZED INVERSE POLYNOMIAL MODULES

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Abstract. Given an injective envelope E of a left R-module M, there is an associative Galois group $Gal(\phi)$. Let R be a left noetherian ring and E be an injective envelope of M, then there is an injective envelope $E[x^{-1}]$ of an inverse polynomial module $M[x^{-1}]$ as a left $R[x]$-module and we can define an associative Galois group $Gal(\phi[x^{-1}])$. In this paper we extend the Galois group of inverse polynomial module and can get $Gal(\phi[x^{-s}])$, where S is a submonoid of $\mathbb{N}$ (the set of all natural numbers).

1. Introduction

Given an injective envelope $M \subset E$, by the Galois group of this envelope we mean all $f \in Hom_R(E, E)$ such that $f(x) = x$ for all $x \in M$ Any such f is an automorphism of E. So we easily see that the set of f form a group (using the composition of functions as operation). If $\phi : M \longrightarrow E$ denotes the canonical injection then the group is denoted $Gal(\phi)$.

Northcott([4]) defined inverse polynomial modules and used inverse polynomial modules to study the properties of injective modules and he studied $K[x^{-1}]$ as $K[x]$-module on field K. And McKerraw([2]) showed that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-1}]$ is an injective envelope of $M[x^{-1}]$ as $R[x]$-module. Inverse polynomial modules were studied in ([5]), ([6]) and recently in ([1]), ([7]), ([8]), ([9]).

Definition 1.1. ([8]) Let R be a ring and M be a left R-module, and $S = \{0, k_1, k_2, \cdots\}$ be a submonoid of $\mathbb{N}$ (the set of all natural numbers). Then $M[x^{-s}]$ is a left $R[x^s]$-module such that

$$\begin{aligned} & x^{k_i}(m_0 + m_1x^{-k_1} + m_2x^{-k_2} + \cdots + m_nx^{-k_n}) \\ & = m_1^{-k_1+k_i} + m_2x^{-k_2+k_i} + \cdots + m_nx^{-k_n+k_i} \end{aligned}$$

where

$$x^{-k_j+k_i} = \begin{cases} x^{-k_j+k_i} & \text{if } k_j - k_i \in S \\ 0 & \text{if } k_j - k_i \notin S. \end{cases}$$

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For example, if $S = \{0, 2, 3, \cdots\}$, then $m_0 + m_2x^{-2} + m_3x^{-3} + \cdots + m_ix^{-i} \in M[x^{-s}]$ and if $S = \{0, 1, 2, 3, \cdots\}$, then $M[x^{-s}] = M[x^{-1}]$.

Similarly, we define $M[[x^{-s}]]$, $M[x^s, x^{-s}]$, $M[[x^s, x^{-s}]]$, $M[x^s, x^{-s}]]$ and $M[[x^s, x^{-s}]$ as left $R[x^s]$-modules.

Definition 1.2. *Given any module M and $f \in End(E)$ we say f is locally nilpotent on M if for every $x \in M$, there exist $n \geq 1$ such that $f^n(x) = 0$.*

Theorem 1.3. (Matlis and Gabriel) If R is a left noetherian ring and E is an injective left R-module and $f \in End(E)$ is such that E is an essential extension of $ker(f)$ then f is locally nilpotent on E.

Theorem 1.4. *Let R be a commutative noetherian ring and S be a submonoid, and E be an injective left R-module. Then $E[x^{-s}]$ is an injective left $R[x^s]$-module.*

Proof. Let $S = \{0, k_1, k_2, \cdots\}$ be a submonoid. Then

$$Hom_R(R[x^s], E) \cong E[[x^{-s}]]$$

is an injective left $R[x^s]$-module. Define $\phi : E[[x^{-s}]] \longrightarrow E[[x^{-s}]]$ by $\phi(f) = x^{k_1}f$ for $f \in E[[x^{-s}]]$. Then ϕ is not locally nilpotent on $E[[x^{-s}]]$. So $E[[x^{-s}]]$ is not an essential extension of $ker(\phi)$. Let $\bar{E}$ be an injective envelope of $ker(\phi)$. Then

$$ker(\phi) \subset \bar{E} \subset E[[x^{-s}]].$$

Then $\phi : \bar{E} \longrightarrow \bar{E}$ defined by

$$\phi(f) = x^{k_1}f,$$

for $f \in \bar{E}$ is locally nilpotent on $\bar{E}$. So $\bar{E} \subset E[x^{-s}]$. But $E[x^{-s}]$ is an essential extension of $ker(\phi)$, so that $E[x^{-s}]$ is an essential extension of $\bar{E}$. Therefore, $\bar{E} = E[x^{-s}]$. Hence, $E[x^{-s}]$ is an injective left $R[x^s]$-module. □

Lemma 1.5. ([8]) Let M and N be left R-modules, then

$$Hom_{R[x^s]}(M[x^{-s}], N[x^{-s}]) \cong Hom_R(M, N)[[x^s]].$$

Theorem 1.6. *There is a ring isomorphism*

$$Hom_{R[x^s]}(M[x^{-s}], N[x^{-s}]) \cong Hom_R(M, N)[[x^s]].$$

Proof. By the Lemma 1.5., we know that two groups are isomorphic.
Let $\sigma, \tau \in Hom_{R[x^s]}(M[x^{-s}], N[x^{-s}])$, then σ corresponds to $f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots \in Hom_R(M, N)[[x^s]]$ and τ corresponds to $g_{k_0} + g_{k_1}x^{k_1} + g_{k_2}x^{k_2} + \cdots \in Hom_R(M, N)[[x^s]]$. Then $\sigma \circ \tau$ corresponds to

$$\sum_{k_n=0}^{\infty} (\sum_{k_i + k_j = k_n} f_{k_i} \circ g_{k_j}) x^{k_n}.$$

Hence, $Hom_{R[x^s]}(M[x^{-s}], N[x^{-s}]) \cong Hom_R(M, N)[[x^s]]$. □

If $\phi[x^{-s}] : M[x^{-s}] \longrightarrow E[x^{-s}]$ denotes the canonical injection, then the group is denoted $Gal(\phi[x^{-s}])$.

2. $Gal(\phi)$ and $Gal(\phi[x^{-s}])$

Theorem 2.1. *If R is a left noetherian ring and if $M \subset E$ is an injective envelope of R-module, then $f = f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + f_{k_3}x^{k_3} + \cdots \in End_R(E)[[x^s]]$ is in $Gal(\phi[x^{-s}])$ if and only if $f_{k_0} \in Gal(\phi)$ and $f_{k_i}(M) = 0$, $k_i \in S$, $k_i \neq 0$.*

Proof. Let $m \in M$ and $f \in Gal(\phi[x^{-s}])$, then

$$\begin{aligned}
&f(m + 0x^{-k_1} + 0x^{-k_2} + \cdots + 0x^{-k_i})\\
&= (f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots)(m + 0x^{-k_1} + 0x^{-k_2} + \cdots + 0x^{-k_i})\\
&= (f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots)(m)\\
&= f_{k_0}(m) + f_{k_1}(m)x^{k_1} + f_{k_2}(m)x^{k_2} + \cdots\\
&= m.
\end{aligned}$$

Thus $f_{k_0}(m) = m$ for all $m \in M$, so that $f_{k_0} \in Gal(\phi)$. And

$$\begin{aligned}
&f(m + mx^{-k_1})\\
&= (f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots)(m + mx^{-k_1})\\
&= f_{k_0}(m) + f_0(m)x^{-k_1} + f_{k_1}(m)x^{k_1} + f_{k_1}(m) + f_{k_2}(m)x^{k_2}\\
&\quad + f_{k_2}(m)x^{k_2-k_1} + \cdots\\
&= (f_{k_0}(m) + f_{k_1}(m)) + f_0(m)x^{-k_1} + (f_{k_1}(m)x^{k_1} + f_{k_2}(m))x^{k_2}\\
&\quad + f_{k_2}(m)x^{k_2-k_1} + \cdots\\
&= m + mx^{-k_1}.
\end{aligned}$$

Since $f_{k_0}(m) = m$, $m + f_{k_1}(m) = m$ implies $f_{k_1}(m) = 0$. So $f_{k_1}(m) = 0$. And

$$\begin{aligned}
&f(m + mx^{-k_1} + mx^{-k_2})\\
&= (f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots)(m + mx^{-k_1} + mx^{-k_2})\\
&= f_{k_0}(m) + f_{k_0}(m)x^{-k_1} + f_{k_0}(m)x^{-k_2} + f_{k_1}(m)x^{k_1} + f_{k_1}(m)\\
&\quad + f_{k_1}(m)x^{k_1-k_2} + f_{k_2}(m)x^{k_2} + f_{k_2}(m)x^{k_2-k_1} + f_{k_2}(m) + \cdots\\
&= (f_{k_0}(m) + f_{k_1}(m) + f_{k_2}(m)) + f_{k_0}(m)x^{-k_1} + f_{k_0}(m))x^{-k_2}\\
&\quad + f_{k_1}(m)x^{k_1-k_2} + \cdots\\
&= m + mx^{-k_1} + mx^{-k_2}.
\end{aligned}$$

Since $f_{k_0}(m) = m$, $f_{k_1}(m) = 0$, $f_{k_0}(m) + f_{k_1}(m) + f_{k_2}(m) = m$ implies $f_{k_2}(m) = 0$. Thus $f_{k_2}(m) = 0$. By the same process we can get $f_{k_i}(M) = 0$, $k_i \in S$, $k_i \neq 0$.

Conversely, let $f = f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + f_{k_3}x^{k_3} + \cdots$ with $f \in Gal(\phi[x^{-s}])$ and $f_{k_i}(M) = 0$, $k_i \in S$, $k_i \neq 0$. Let $m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2} + \cdots + m_{k_i}x^{-k_i} \in M[x^{-s}]$. We want to show $f(m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2} + \cdots + m_{k_i}x^{-k_i}) =$

$m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2} + \cdots + m_{k_i}x^{-k_i}$. Then

$$\begin{aligned}
&f(m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2} + \cdots + m_{k_i}x^{-k_i})\\
&= (f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots)(m_0 + m_{k_1}x^{-k_1} + \cdots + m_{k_i}x^{-k_i})\\
&= f_{k_0}(m_0) + f_0(m_{k_1})x^{-k_1} + f_{k_0}(m_{k_2})x^{-k_2} + \cdots + f_{k_0}(m_{k_i})x^{-k_i}\\
&+ f_{k_1}(m_0)x^{k_1} + f_{k_1}(m_{k_1}) + f_{k_1}(m_{k_2})x^{k_1-k_2} + \cdots\\
&+ f_1(m_{k_i})x^{k_1-k_i} + f_{k_2}(m_0)x^{k_2} + f_{k_2}(m_{k_1})x^{k_2-k_1}\\
&+ f_{k_2}(m_{k_2}) + f_{k_2}(m_{k_3})x^{k_2-k_3} + \cdots + f_{k_2}(m_{k_i})x^{k_2-k_i} + \cdots + f_{k_i}(m_{k_i})\\
&= m_0 + m_{k_1}x^{-k_1} + m_{k_2}x^{-k_2} + \cdots + m_{k_i}x^{-k_i},
\end{aligned}$$

since $f_{k_0} \in Gal(\phi)$ and $f_{k_i}(M) = 0$, $k_i \in S$, $k_i \neq 0$. Therefore, $f = f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + f_{k_3}x^{k_3} + \cdots \in Gal(\phi[x^{-s}])$. □

There are natural group homomorphisms $Gal(\phi) \to Gal(\phi[x^{-s}])$ by $g \longmapsto g + 0x^{k_1} + 0x^{k_2} + \cdots$ and $Gal(\phi[x^{-s}]) \to Gal(\phi)$ by $f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots \longmapsto f_{k_0}$. The composition $Gal(\phi) \to Gal(\phi[x^{-s}]) \to Gal(\phi)$ is the identity map on $Gal(\phi)$. The kernel of $Gal(\phi[x^{-s}]) \to Gal(\phi)$ consists of all $id_E + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots$, where $f_{k_i} \in Hom_R(E, E)$ and $f_{k_i}(M) = 0$, $k_i \in S$, $k_i \neq 0$.

Lemma 2.2. *Let $\psi : Gal(\phi) \longrightarrow Gal(\phi[x^{-s}])$ be defined by $\psi(f) = f + 0x^{k_1} + 0x^{k_2} + \cdots$. If $End(E)$ is a commutative ring, then $Im(\psi)$ is a normal subgroup of $Gal(\phi[x^{-s}])$.*

Proof. Let $f_{k_0} + 0x^{k_1} + 0x^{k_2} + \cdots \in Im(\psi)$, and $g_{k_0} + g_{k_1}x^{k_1} + g_{k_2}x^{k_2} + \cdots \in Gal(\phi[x^{-s}])$. Let $(g_{k_0} + g_{k_1}x^{k_1} + g_{k_2}x^{k_2} + \cdots)^{-1} = h_{k_0} + h_{k_1}x^{k_1} + h_{k_2}x^{k_2} + \cdots$. Then $(g_{k_0} + g_{k_1}x^{k_1} + g_{k_2}x^{k_2} + \cdots) \circ (h_{k_0} + h_{k_1}x^{k_1} + h_{k_2}x^{k_2} + \cdots) = id_E + 0x^{k_1} + 0x^{k_2} + \cdots$, implies $g_{k_0} \circ h_{k_0} = id_E$ so that $h_{k_0} = g_{k_0}^{-1}$ and $\sum_{k_i + k_j = k_n} g_{k_i} \circ h_{k_j} = 0, n \geq k_1$. Thus

$$\begin{aligned}
&(g_{k_0} + g_{k_1}x^{k_1} + g_{k_2}x^{k_2} + \cdots) \circ (f_{k_0} + 0x^{k_1} + 0x^{k_2} + \cdots)\\
&\circ (h_{k_0} + h_{k_1}x^{k_1} + h_{k_2}x^{k_2} + \cdots)\\
&= ((g_{k_0} \circ f_{k_0}) + (g_{k_1} \circ f_{k_0})x^{k_1} + (g_{k_2} \circ f_{k_0})x^{k_2} + \cdots)\\
&\circ (h_{k_0} + h_{k_1}x^{k_1} + h_{k_2}x^{k_2} + \cdots)\\
&= (g_{k_0} \circ f_{k_0} \circ h_{k_0}) + (g_{k_0} \circ f_{k_0} \circ h_{k_1} + g_{k_1} \circ f_{k_0} \circ h_{k_0})x^{k_1}\\
&+ (g_{k_0} \circ f_{k_0} \circ h_{k_2} + g_{k_2} \circ f_{k_0} \circ h_{k_0})x^{k_2}\\
&+ (g_{k_1} \circ f_{k_0} \circ h_{k_2} + g_{k_2} \circ f_{k_0} + \circ h_{k_1})x^{k_1+k_2} + \cdots\\
&= f_{k_0},
\end{aligned}$$

since $End(E)$ is a commutative ring. Hence, $Im(\psi)$ is a normal subgroup of $Gal(\phi[x^{-s}])$. □

We note that $Im(\psi)$ is not a normal subgroup of $Gal(\phi[x^{-s}])$, in general. So $Gal(\phi[x^{-s}])$ is the semidirect product of $Gal(\phi)$ and $K = ker(Gal(\phi[x^{-s}]) \to Gal(\phi))$.

Lemma 2.3. *$Gal(\phi)$ is commutative if and only if $g \circ g' = g' \circ g$ for all $g, g' \in Hom_R(E,E)$ with $g(M) = 0, g'(M) = 0$.*

Proof. If $f \in Gal(\phi)$, then $g = f - id_E \in Hom_R(E,E)$ with $g(M) = 0$. And given $g \in Gal(\phi[x^{-s}])$ with $g(M) = 0$, $f = g + id_E \in Gal(\phi)$. Therefore, there is one to one correspondence between $Gal(\phi)$ and the set of $g \in Hom_R(E,E)$ with $g(M) = 0$. So, given $f, f' \in Gal(\phi)$ choose $g = f - id_E, g' = f' - id_E \in Hom_R(E,E)$ with $g(M) = 0, g'(M) = 0$. Then $g \circ g' = g' \circ g$.

Conversely, given $g, g' \in Hom_R(E,E)$ with $g(M) = 0, g'(M) = 0$ choose $f = g + id_E, f' = g' + id_E \in Gal(\phi)$. Then $f \circ f' = f' \circ f$. Thus, $Gal(\phi)$ is commutative. □

Theorem 2.4. *$Gal(\phi[x^{-s}])$ is commutative if and only if $Gal(\phi)$ is commutative.*

Proof. Since $Gal(\phi)$ is a subgroup of $Gal(\phi[x^{-s}])$, $Gal(\phi)$ is commutative. Conversely, let $f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots, g_{k_0} + g_{k_1}x^{k_1} + g_{k_2}x^{k_2} + \cdots \in Gal(\phi[x^{-s}])$. Then by the Theorem 2.1., $f_{k_0}, g_{k_0} \in Gal(\phi), f_{k_i}(M) = 0, g_{k_j}(M) = 0, k_i \in S, k_j \in S, k_i \neq 0, k_j \neq 0$. And by the Lemma 2.3., $f_{k_i} \circ g_{k_j} = g_{k_j} \circ f_{k_i}, k_i, k_j \geq k_1$. Given $f_{k_i} \in Gal(\phi)$ choose $g_{k_i} = f_{k_i} - id_E \in Hom(E,E)$ with $g_{k_i}(M) = 0, k_i \in S$. Then

$$\begin{aligned} f_{k_0} \circ g_{k_i} &= f_{k_0} \circ (f_{k_i} - id_E) = f_{k_0} \circ f_{k_i} - f_{k_0} = f_{k_i} \circ f_{k_0} - f_0 \\ &= (f_{k_i} - id_E) \circ f_{k_0} = g_{k_i} \circ f_{k_0}. \end{aligned}$$

Thus

$$\begin{aligned} &(f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots) \circ (g_0 + g_{k_1}x^{k_1} + g_{k_2}x^{k_2} + \cdots) \\ &= (f_{k_0} \circ g_{k_0}) + (f_{k_0} \circ g_{k_1} + f_{k_1} \circ g_{k_0})x^{k_1} + (f_{k_0} \circ g_{k_2} + f_{k_2} \circ g_{k_0})x^{k_2} \\ &+ (f_{k_1} \circ g_{k_2} + f_{k_2} \circ g_{k_1})x^{k_1+k_2} \cdots \\ &= (g_{k_0} \circ f_{k_0}) + (g_{k_1} \circ f_{k_0} + g_{k_0} \circ f_{k_1})x^{k_1} + (g_{k_2} \circ f_{k_0} + g_{k_0} \circ f_{k_2})x^{k_2} \\ &+ \cdots \\ &= (g_{k_0} + g_{k_1}x^{k_1} + g_{k_2}x^{k_2} + \cdots) \circ (f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots). \end{aligned}$$

Therefore, $Gal(\phi[x^{-s}])$ is commutative. □

Theorem 2.5. *Let $\varphi : Gal(\phi[x^{-s}]) \longrightarrow Gal(\phi)$ be defined by $\varphi(f_{k_0} + f_{k_1}x^{k_1} + f_{k_2}x^{k_2} + \cdots) = f_0$. Then $Gal(\phi[x^{-s}])$ is the direct product of K and $Gal(\phi)$ if and only if $Gal(\phi)$ is commutative, where $K = ker(\varphi)$.*

Proof. Let $g, g' \in Gal(\phi)$. Then $id_E + g \in Gal(\phi)$ and $(id_E + g'x^{k_1})^{-1} \circ (id_E + g) \circ (id_E + g'x^{k_1}) \in Gal(\phi)$. So let $(id_E + g'x)^{-1} = id_E - g'x + \text{etc.}$, then

$$\begin{aligned}&(id_E + g'x^{k_1})^{-1} \circ (id_E + g) \circ (id_E + g'x^{k_1})\\&= (id_E - g'x^{k_1} + \text{etc}) \circ (id_E + g) \circ (id_E + g'x^{k_1})\\&= id_E + (-g' \circ g + g \circ g')x^{k_1} + \text{etc.} \in Gal(\phi)\end{aligned}$$

implies $-g' \circ g + g \circ g' = 0$ so that $g' \circ g = g \circ g'$.
Therefore, $Gal(\phi)$ is commutative.

Conversely, by the Theorem 2.4., if $Gal(\phi)$ is commutative then $Gal(\phi[x^{-s}])$ is commutative. Therefore, $Gal(\phi[x^{-s}])$ is the direct product of K and $Gal(\phi)$. □

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