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# TOPOLOGICAL PROPERTIES IN $BCC$-ALGEBRAS

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Abstract. In this paper, we show how to associate certain topologies with special ideals of $BCC$-algebras on these $BCC$-algebras. We show that it is natural for $BCC$-algebras to be topological $BCC$-algebras with respect to theses topologies. Furthermore, we show how certain standard properties may arise. In addition we demonstrate that it is natural for these topologies to have many clopen sets and thus to be highly connected via the ideal theory of $BCC$-algebras.

## 1. Introduction

In 1966, Y. Imai and K. Iséki [5] defined a class of algebras of type (2,0) called *BCK-algebras* which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra (see [6]). The class of all $BCK$-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [12]) whether the class of $BCK$-algebras is a variety. In connection with this problem, Y. Komori [9] introduced a notion of $BCC$-algebras, and W. A. Dudek [1, 2] redefined the notion of $BCC$-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [4], J. Hao introduced the concept of ideals in a $BCC$-algebra and studied some related properties. In this paper, we address the issue of attaching topologies to $BCC$-algebras in as natural a manner as possible. It turns out that we may use the class of $BCC$-ideals of a $BCC$-algebra as the underlying structure whence a certain uniformity and thence a topology is derived which provides a natural connection between the notion of a $BCC$-algebra and the notion of a topology in that we are able to conclude that in this setting a $BCC$-algebra becomes a topological $BCC$-algebra. Other properties are also identified both in the $BCC$-algebra and in the topology, such as $\{0\}$ is closed if and only if the topology is Hausdorff and $\{0\}$ is open if and only if the topology is discrete among others.

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## 2. Preliminaries

By a *BCC-algebra* ([9]) we mean a non-empty set $X$ with a constant 0 and a binary operation " $*$ " satisfying the following axioms: for all $x, y, z \in X$,

(I) $((x * y) * (z * y)) * (x * z) = 0$,

(II) $0 * x = 0$,

(III) $x * 0 = x$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For brevity, we also call $X$ a $BCC$-algebra. In $X$ we can define a binary operation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$, is called a $BCC$-*order* on $X$. Then it is easy to show that $\leq$ is a partial order on $X$.

In a $BCC$-algebra, the following hold: for any $x, y, z \in X$,

(2.1) $x * x = 0$,

(2.2) $(x * y) * x = 0$,

(2.3) $x \leq y \Rightarrow x * z \leq y * z$,

(2.4) $x \leq y \Rightarrow z * y \leq z * x$.

Any $BCK$-algebra is a $BCC$-algebra, but there are $BCC$-algebras which are not $BCK$-algebras (see [2]). Note that a $BCC$-algebra is a $BCK$-algebra if and only if it satisfies: (2.5) $(x * y) * z = (x * z) * y$.

A non-empty subset $S$ of a $BCC$-algebra $X$ is called a *subalgebra* of $X$ if it is closed under the $BCC$-operation.

**Definition 2.1.** Let $X$ be a $BCC$-algebra. An equivalence relation $\sim$ on $X$ is called a *left congruence* if $x \sim y$ implies $u * x \sim u * y$, where $x, y, u \in X$. An equivalence relation $\sim$ on $X$ is called a *right congruence* if $x \sim y$ implies $x * u \sim y * u$, where $x, y, u \in X$.

**Definition 2.2.** Let $X$ be a $BCC$-algebra. An equivalence relation $\sim$ on $X$ is called a *congruence* if $x \sim y,\ u \sim v$ imply $x * u \sim y * v$, where $x, y, u, v \in X$.

**Proposition 2.3.** *Let $X$ be a $BCC$-algebra and $\sim$ be an equivalence relation on $X$. Then $\sim$ is a congruence if and only if it is both a left congruence and a right congruence.*

**Definition 2.4** ([4]). Let $X$ be a $BCC$-algebra and $\emptyset \neq I \subseteq X$. $I$ is called an *ideal* of $X$ if it satisfies the following conditions: (i) $0 \in I$; (ii) $x * y, y \in I$ imply $x \in I$.

**Theorem 2.5** ([4]). *In a $BCC$-algebra $X$, any ideal of $X$ is a subalgebra of $X$.*

**Definition 2.6** ([3]). Let $X$ be a $BCC$-algebra and $\emptyset \neq I \subseteq X$. $I$ is called a $BCC$-*ideal* of $X$ if it satisfies the following conditions:

(i) $0 \in I$;

(ii) $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$.

**Lemma 2.7** ([3]). *In a $BCC$-algebra any $BCC$-ideal is an ideal.*

**Corollary 2.8** ([3]). *Any $BCC$-ideal of a $BCC$-algebra is a subalgebra.*

*Remark.* In a $BCC$-algebra, a subalgebra need not be an ideal and an ideal need not be a $BCC$-ideal, in general (see [3, 4]).

**Theorem 2.9** ([3]). *If $I$ is a $BCC$-ideal of a $BCC$-algebra $X$, then the relation defined on $X$ by*

$$x \sim y \text{ if and only if } x * y, y * x \in I$$

*is a congruence relation on $X$.*

**Theorem 2.10.** *Let $I$ be a $BCC$-ideal of a $BCC$-algebra $X$. If we define a binary operation on the quotient set $X/I := \{I_x | x \in X\}$ by $I_x * I_y = I_{x*y}$, then $(X/I; *, I_0)$ is a $BCC$-algebra, called the quotient algebra of $X$ relative to $I$.*

*Proof.* If $I_x = I_{x'}$ and $I_y = I_{y'}$, then $x \sim x'$ and $y \sim y'$. Hence $x * y \sim x' * y'$ since $\sim$ is a congruence relation. Therefore $I_x * I_y = I_{x*y} = I_{x'*y'} = I_{x'} * I_{y'}$. Thus "$*$" is well-defined on $X/I$. It is easy to show (I), (II), and (III). Assume that $I_x * I_y = I_y * I_x = I_0$. Then $I_{x*y} = I_{y*x} = I_0$. Hence $x * y \sim 0$ and $y * x \sim 0$. Therefore $x * y \in I$ and $y * x \in I$ and so $I_x = I_y$. Thus $(X/I; *, I_0)$ is a $BCC$-algebra. □

## 3. Uniformity in $BCC$-algebras

From now on, $X$ is a $BCC$-algebra, unless otherwise is stated.

Let $X$ be a non-empty set, and $U$ and $V$ be any subsets of $X \times X$. Define

$$U \circ V := \{(x, y) \in X \times X | \text{ for some } z \in X, (x, z) \in U \text{ and } (z, y) \in V\},$$
$$U^{-1} := \{(x, y) \in X \times X | (y, x) \in U\},$$
$$\Delta := \{(x, x) | x \in X\}.$$

**Definition 3.1.** By a *uniformity* on $X$, we mean a non-empty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:

($U_1$) $\Delta \subseteq U$ for any $U \in \mathcal{K}$,
($U_2$) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$,
($U_3$) if $U \in \mathcal{K}$, then there exists a $V \in \mathcal{K}$ such that $V \circ V \subseteq U$,
($U_4$) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$,
($U_5$) if $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{K}$.

The pair $(X, \mathcal{K})$ is called a *uniform structure.*

**Theorem 3.2.** *Let $A$ be a $BCC$-ideal of a $BCC$-algebra $X$. If we define*

$$U_A := \{(x, y) \in X \times X | x * y \in A \text{ and } y * x \in A\}$$

*and let*

$$\mathcal{K}^* := \{U_A | A \text{ is a } BCC\text{-ideal of } X\}.$$

*Then $\mathcal{K}^*$ satisfies the conditions ($U_1$)~($U_4$).*

*Proof.* ($U_1$): If $(x, x) \in \Delta$, then $(x, x) \in U_A$ since $x * x = 0 \in A$. Hence $\Delta \subseteq U_A$ for any $U_A \in \mathcal{K}^*$.

$(U_2)$: For any $U_A \in \mathcal{K}^*$,

$$\begin{aligned}(x,y) \in U_A &\Leftrightarrow x * y \in A \text{ and } y * x \in A \\ &\Leftrightarrow y \sim_A x \\ &\Leftrightarrow (y,x) \in U_A \\ &\Leftrightarrow (x,y) \in U_A^{-1}.\end{aligned}$$

Hence $U_A^{-1} = U_A \in \mathcal{K}^*$.
$(U_3)$: For any $U_A \in \mathcal{K}^*$, the transitivity of $\sim_A$ implies that $U_A \circ U_A \subseteq U_A$.
$(U_4)$: For any $U_I$ and $U_J$ in $\mathcal{K}^*$, we claim that $U_I \cap U_J \in \mathcal{K}^*$.

$$\begin{aligned}(x,y) \in U_I \cap U_J &\Leftrightarrow (x,y) \in U_I \text{ and } (x,y) \in U_J \\ &\Leftrightarrow x * y, y * x \in I \cap J \\ &\Leftrightarrow x \sim_{I \cap J} y \\ &\Leftrightarrow (x,y) \in U_{I \cap J}.\end{aligned}$$

Since $I \cap J$ is a $BCC$-ideal of $X$, $U_I \cap U_J = U_{I \cap J} \in \mathcal{K}^*$. This proves the theorem. □

**Theorem 3.3.** *Let $\mathcal{K} := \{U \subseteq X \times X | U_A \subseteq U$ for some $U_A \in \mathcal{K}^*\}$. Then $\mathcal{K}$ satisfies the conditions for a uniformity on $X$ and hence the pair $(X, \mathcal{K})$ is a uniform structure.*

*Proof.* By Theorem 3.2, the collection $\mathcal{K}$ satisfies the conditions $(U_1)$~$(U_4)$. It suffices to show that $\mathcal{K}$ satisfies $(U_5)$. Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$. Then there exists a $U_A \subseteq U \subseteq V$, which means that $V \in \mathcal{K}$. This proves the theorem. □

Let $x \in X$ and $U \in \mathcal{K}$. Define

$$U[x] := \{y \in X | (x,y) \in U\}.$$

**Theorem 3.4.** *Let $X$ be a $BCC$-algebra. Then*

$$\mathcal{T} := \{G \subseteq X | \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\}$$

*is a topology on $X$.*

*Proof.* It is clear that $\emptyset$ and the set $X$ belong to $\mathcal{T}$. Also from the definition, it is clear that $\mathcal{T}$ is closed under arbitrary unions. Finally to show that $\mathcal{T}$ is closed under finite intersection, let $G, H \in \mathcal{T}$ and suppose $x \in G \cap H$. Then there exist $U$ and $V \in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq H$. Let $W := U \cap V$. Then $W \in \mathcal{K}$. Also $W[x] \subseteq U[x] \cap V[x]$ and so $W[x] \subseteq G \cap H$. Therefore $G \cap H \in \mathcal{T}$. Thus $\mathcal{T}$ is a topology on $X$. □

Note that for any $x \in X$, $U[x]$ is an open neighborhood of $x$.

**Definition 3.5.** Let $(X, \mathcal{K})$ be a uniform structure. Then the topology $\mathcal{T}$ is called the *uniform topology* on $X$ induced by $\mathcal{K}$.

**Proposition 3.6.** *Topological space $(X, \mathcal{T})$ is completely regular.*

*Proof.* See [11]. □

**Example 3.7.** Let $X := \{0,1,2,3\}$ be a $BCC$-algebra which is not a $BCK$-algebra with the following table ([4]):

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | 1 |
| 3 | 3 | 3 | 3 | 0 |

Then it is easy to show that $A := \{0,1,2\}, \{0\}$, and $X$ are the only $BCC$-ideals of $X$. We can see that $U_{\{0\}} = \Delta$, $U_A = \Delta \cup \{(0,1), (1,0), (0,2), (2,0), (1,2), (2,1)\}$ and $U_X = X \times X$. Therefore $\mathcal{K}^* = \{U_{\{0\}}, U_A, U_X\}$ and $\mathcal{K} = \{U \subseteq X \times X | U_A \subseteq U$ for some $U_A \in \mathcal{K}^*\}$. If we take $U := U_A$, then $U[0] = U[1] = U[2] = \{0,1,2\}$ and $U[3] = \{3\}$. Therefore $\mathcal{T} = \{G \subseteq X | \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\} \supseteq \{X, \emptyset, \{3\}, \{0,1,2\}\}$. Since $\{X, \emptyset, \{3\}, \{0,1,2\}\}$ is a topology on $X$, the topology $\mathcal{T}$ on $X$ induced by the $BCC$-ideal $A = \{0,1,2\}$ relative to $U_A$ is a finer topology than $\{X, \emptyset, \{3\}, \{0,1,2\}\}$. Let $A := \{0\}$. Then $U_A = \Delta$. If we take $U := U_A$, then $U[x] = \{x\}, \forall x \in X$ and we obtain $\mathcal{T} = 2^X$, the discrete topology. Moreover, if we take $X$ as a $BCC$-ideal of $X$, then $U[x] = X, \forall x \in X$ and obtain $\mathcal{T} = \{\emptyset, X\}$, the indiscrete topology.

## 4. Topological property of space $(X, \mathcal{T})$

Let $X$ be a $BCC$-algebra and $C, D$ be subsets of $X$. We define a set $C * D$ as follows:

$$C * D := \{x * y | x \in C, y \in D\}.$$

Let $X$ be a $BCC$-algebra and $\mathcal{T}$ be a topology on the set $X$. Then we say that the pair $(X, \mathcal{T})$ is a topological $BCC$-algebra if the operation "$*$" is continuous with respect to $\mathcal{T}$. The continuity of the operation "$*$" is equivalent to the following property:

(C): If $O$ is an open set and $a, b \in X$ such that $a * b \in O$, then there exist open sets $O_1$ and $O_2$ such that $a \in O_1$, $b \in O_2$ and $O_1 * O_2 \subseteq O$.

**Theorem 4.1.** *The pair $(X, \mathcal{T})$ is a topological $BCC$-algebra.*

*Proof.* Let us first prove (C). Indeed, assume that $x * y \in G$, with $x, y \in X$ and $G$ an open subset of $X$. Then there exist $U \in \mathcal{K}$, $U[x*y] \subseteq G$ and a $BCC$-ideal $I$ of $X$ such that $U_I \subseteq U$. We claim that the following relation holds:

$$U_I[x] * U_I[y] \subseteq U[x * y].$$

Indeed, for any $h \in U_I[x]$ and $k \in U_I[y]$ we have that $x \sim_I h$ and $y \sim_I k$. Since $\sim_I$ is a congruence relation, it follows that $x * y \sim_I h * k$. From that fact we have $(x * y, h * k) \in U_I \subseteq U$. Hence $h * k \in U_I[x * y] \subseteq U[x * y]$. Then $h * k \in G$. Thus condition (C) is verified. □

**Theorem 4.2** ([11]). *Let $X$ be a set and $\mathcal{S} \subseteq \mathcal{P}(X \times X)$ be a family such that for every $U \in \mathcal{S}$ the following conditions hold:*

(a) $\Delta \subseteq U$,

(b) $U^{-1}$ *contains a member of* $\mathcal{S}$,

(c) *there exists an* $V \in \mathcal{S}$ *such that* $V \circ V \subseteq U$.

*Then there exists a unique uniformity $\mathcal{U}$, for which $S$ is a subbase.*

**Theorem 4.3.** *If we set $\mathcal{B} := \{U_I | I$ is a BCC-ideal of a BCC-algebra $X\}$, then $\mathcal{B}$ is a subbase for a uniformity of $X$. We denote its associated topology by $\mathcal{S}$.*

*Proof.* Since $\sim_I$ is an equivalence relation, it is clear that $\mathcal{B}$ satisfies the axioms of Theorem 4.2. □

In Example 3.7, we can see that $\mathcal{B} = \{U_{\{0\}} = \Delta,\ U_A = \Delta \cup \{(0,1), (1,0), (0,2), (2,0), (1,2), (2,1)\}, U_X = X \times X\}$.

We say that the topology $\sigma$ is *finer* than $\tau$ if $\tau \subseteq \sigma$ as subsets of the power set. Then we have the following:

**Corollary 4.4.** *The topology $\mathcal{S}$ is finer than $\mathcal{T}$.*

**Theorem 4.5.** *Let $\Lambda$ be an arbitrary family of BCC-ideals of a BCC-algebra $X$ which is closed under intersection. Then any BCC-ideal is a clopen subset of $X$.*

*Proof.* Let $I$ be a $BCC$-ideal of $X$ in $\Lambda$ and $y \in I^c$. Then $y \in U_I[y]$ and we obtain that $I^c \subseteq \cup\{U_I[y] | y \in I^c\}$. We claim that $U_I[y] \subseteq I^c$ for all $y \in I^c$. Let $z \in U_I[y]$, then $y \sim_I z$. Hence $y * z \in I$. If $z \in I$, then $y \in I$, since $I$ is a $BCC$-ideal of $X$, which is a contradiction. So $z \in I^c$ and we obtain

$$\cup\{U_I[y] | y \in I^c\} \subseteq I^c.$$

Hence $I^c = \cup\{U_I[y] | y \in I^c\}$. Since $U_I[y]$ is open for any $y \in X$, $I$ is a closed subset of $X$. We show that $I = \cup\{U_I[y] | y \in I\}$. If $y \in I$ then $y \in U_I[y]$ and hence $I \subseteq \cup\{U_I[y] | y \in I\}$. Given $y \in I$, if $z \in U_I[y]$, then $y \sim_I z$ and so $z * y \in I$. Since $y \in I$ and $I$ is a $BCC$-ideal of $X$, we have $z \in I$. Hence we get that $\cup\{U_I[y] | y \in I\} \subseteq I$, i.e., $I$ is also an open subset of $X$. □

In Example 3,7, the $BCC$-ideals $A, \{0\}$, and $X$ are clopen subsets of $X$.

**Theorem 4.6.** *For any $x \in X$ and $I \in \Lambda$, $U_I[x]$ is a clopen subset of a BCC-algebra $X$.*

*Proof.* We show that $(U_I[x])^c$ is open. If $y \in (U_I[x])^c$, then $x * y \in I^c$ or $y * x \in I^c$. We assume that $y * x \in I^c$. By applying Theorems 4.1 and 4.2, we obtain $(U_I[y] * U_I[x]) \subseteq U_I[y * x] \subseteq I^c$. We claim that $U_I[y] \subseteq (U_I[x])^c$. If $z \in U_I[y]$, then $z * x \in (U_I[z] * U_I[x])$. Hence $z * x \in I^c$ then we get $z \in (U_I[x])^c$, proving that $(U_I[x])^c$ is open. Hence $U_I[x]$ is closed. It is clear that $U_I[x]$ is open. So $U_I[x]$ is a clopen subset of $X$. □

A topological space $X$ is connected if and only if it has only $X$ and $\emptyset$ as clopen subsets of $X$. Therefore we have the following:

**Corollary 4.7.** *The space $(X, \mathcal{T})$ is not a connected space.*

We denote by $\mathcal{T}_\Lambda$ the uniform topology by an arbitrary family $\Lambda$. Especially, if $\Lambda = \{I\}$, we denote it by $\mathcal{T}_I$.

**Theorem 4.8.** *$\mathcal{T}_\Lambda = \mathcal{T}_J$, where $J = \cap\{I|I \in \Lambda\}$.*

*Proof.* Let $\mathcal{K}$ and $\mathcal{K}^*$ be as Theorems 3.2 and 3.3, respectively. Now consider $\Lambda_0 = \{J\}$. Define $(\mathcal{K}_0)^* := \{U_J\}$ and $\mathcal{K}_0 := \{U|U_J \subseteq U\}$. Let $G \in \mathcal{T}_\Lambda$. Given an $x \in G$, there exists $U \in \mathcal{K}$ such that $U[x] \subseteq G$. From $J \subseteq I$, we obtain that $U_J \subseteq U_I$, for any $BCC$-ideal $I$ of $X$. Since $U \in \mathcal{K}$, there exists $I \in \Lambda$ such that $U_I \subseteq U$. Hence $U_J[x] \subseteq U_I[x] \subseteq G$. Since $U_J \in \mathcal{K}_0$, $G \in \mathcal{T}_J$. Hence $\mathcal{T}_\Lambda \subseteq \mathcal{T}_J$.

Conversely, if $H \in \mathcal{T}_J$, then for any $x \in H$, there exists $U \in \mathcal{K}_0$ such that $U[x] \subseteq H$. So $U_J[x] \subseteq H$ and hence $\Lambda$ is closed under intersection, $J \in \Lambda$. Then we get $U_J \in \mathcal{K}$ and so $H \in \mathcal{T}_\Lambda$. Thus $\mathcal{T}_J \subseteq \mathcal{T}_\Lambda$. □

**Theorem 4.9.** *Let $I$ and $J$ be $BCC$-ideals of a $BCC$-algebra $X$ and $I \subseteq J$. Then $J$ is clopen in the topological space $(X, \mathcal{T}_I)$.*

*Proof.* Consider $\Lambda = \{I, J\}$. Then by Theorem 4.8, $\mathcal{T}_\Lambda = \mathcal{T}_I$ and therefore $J$ is clopen in the topological space $(X, \mathcal{T}_I)$. □

**Theorem 4.10.** *Let $I$ and $J$ be $BCC$-ideals of a $BCC$-algebra $X$. Then $\mathcal{T}_I \subseteq \mathcal{T}_J$ if $J \subseteq I$.*

*Proof.* Let $J \subseteq I$. Consider: $\Lambda_1 := \{I\}, \mathcal{K}_1^* := \{U_I\}, \mathcal{K}_1 := \{U|U_I \subseteq U\}$ and $\Lambda_2 := \{J\}, \mathcal{K}_2^* := \{U_J\}, \mathcal{K}_2 := \{U|U_J \subseteq U\}$. Let $G \in \mathcal{T}_I$. Then for any $x \in G$, there exists $U \in \mathcal{K}_1$ such that $U[x] \subseteq G$. Since $J \subseteq I$, we have $U_J \subseteq U_I$. $U_I[x] \subseteq G$ implies $U_J[x] \subseteq G$. This proves that $U_J \in \mathcal{K}_2$ and so $G \in \mathcal{T}_J$. Thus $\mathcal{T}_I \subseteq \mathcal{T}_J$. □

Recall that a uniform space $(X, \mathcal{K})$ is said to be *totally bounded* if for each $U \in \mathcal{K}$, there exist $x_1, \ldots, x_n \in X$ such that $X = \cup_{i=1}^n U[x_i]$, and $(X, \mathcal{K})$ is said to be *compact* if any open cover of $X$ has its finite subcover.

**Theorem 4.11.** *Let $I$ be a $BCC$-ideal of a $BCC$-algebra $X$. Then the following conditions are equivalent:*

(1) *the topological space $(X, \mathcal{T}_I)$ is compact,*

(2) *the topological space $(X, \mathcal{T}_I)$ is totally bounded,*

(3) *there exists $P = \{x_1, \ldots, x_n\} \subseteq X$ such that for all $a \in X$ there exist $x_i \in P$ $(i = 1, \ldots, n)$ with $a * x_i \in I$ and $x_i * a \in I$.*

*Proof.* (1)$\Rightarrow$(2): It is clear by [11].

(2)$\Rightarrow$(3): Let $U_I \in \mathcal{K}$. Since $(X, \mathcal{T}_I)$ is totally bounded, there exist $x_1, \ldots, x_n \in I$ such that $X = \cup_{i=1}^n U_I[x_i]$. If $a \in X$, then there exists $x_i$ such that $a \in U_I[x_i]$, therefore $a * x_i \in I$ and $x_i * a \in I$.

(3)⇒(1): For any $a \in X$, by hypothesis, there exists $x_i \in P$ with $a_i * x_i \in I$ and $x_i * a \in I$. Hence $a \in U_I[x_i]$. Thus $X = \cup_{i=1}^n U_I[x_i]$. Now let $X = \cup_{\alpha\in\Omega} O_\alpha$, where each $O_\alpha$ is an open set of $X$. Then for any $x_i \in X$ there exists $\alpha_i \in \Omega$ such that $x_i \in O_{\alpha_i}$, since $O_{\alpha_i}$ is an open set. Hence $U_I[x_i] \subseteq O_{\alpha_i}$. Hence $X = \cup_{i=1}^n U_I[x_i] \subseteq \cup_{i=1}^n O_{\alpha_i}$, i.e., $X = \cup_{i=1}^n O_{\alpha_i}$, which means that $(X, \mathcal{T}_I)$ is compact. □

**Theorem 4.12.** *If $I$ is a BCC-ideal of a BCC-algebra $X$, then $U_I[x]$ is a compact set in a topological space $(X, \mathcal{T}_I)$, for any $x \in X$.*

*Proof.* Let $U_I[x] \subseteq \cup_{\alpha\in\Omega} O_\alpha$, where each $O_\alpha$ is an open set of $X$. Since $x \in U_I[x]$, then there exists $\alpha \in \Omega$ such that $x \in O_\alpha$. Hence $U_I[x] \subseteq O_\alpha$, proving that $U_I[x]$ is compact. □

**Definition 4.13.** A topological $BCC$-algebra $X$ is said to be *discrete* if every element admits a neighborhood consisting of that element only.

**Proposition 4.14.** *If $\{0\}$ is an open set in a topological BCC-algebra $X$, then $X$ is discrete.*

*Proof.* Since $x * x = 0 \in \{0\}$ for any $x \in X$ and $\{0\}$ is open, there exist neighborhoods $U$ and $V$ of $x$ such that $U * V = \{0\}$. Let $W := U \cap V$. Then $W * W \subseteq U * V = \{0\}$ and so $W * W = \{0\}$. It follows from $x \in W$ that $W = \{x\}$, which means that $X$ is discrete. □

**Proposition 4.15.** *Let $X$ be a topological BCC-algebra. Then $\{0\}$ is closed in $X$ if and only if $X$ is Hausdorff.*

*Proof.* Assume that $\{0\}$ is closed and let $x, y \in X$ with $x \neq y$. Then either $x * y \neq 0$ or $y * x \neq 0$. We may assume that $x * y \neq 0$ without loss of generality. Then there exist neighborhoods $U$ and $V$ of $x$ and $y$ respectively such that $U * V \subseteq X - \{0\}$. It follows that $U \cap V = \emptyset$ and hence $X$ is Hausdorff.

Conversely, let $X$ be Hausdorff. We claim that $X - \{0\}$ is open. If $x \in X - \{0\}$, then $x \neq 0$ and so there exist disjoint neighborhoods $U$ and $V$ of $x$ and 0 respectively. Therefore $0 \notin U$ and hence $U \subseteq X - \{0\}$, which means that $X - \{0\}$ is open. This completes the proof. □

**Proposition 4.16.** *Let $A$ be a BCC-ideal of a topological BCC-algebra $X$. If 0 is an interior point of $A$, then $A$ is open.*

*Proof.* Let $x \in A$. Since $x * x = 0 \in A$ and 0 is an interior point of $A$, there exists a neighborhood $U$ of 0 which is contained in $A$. Then there exist neighborhoods $G$ and $H$ of $x$ such that $G * H \subseteq U \subseteq A$. On the other hand for every $y \in G$, $y * x \in G * H \subseteq A$. Since $A$ is a $BCC$-ideal and $x \in A$, it follows that $y \in A$ so that $x \in G \subseteq A$. Hence $A$ is open, proving the proof. □

**Proposition 4.17.** *Let $X$ be a topological BCC-algebra. If $A$ is an open set in $X$ which is also a BCC-ideal of $X$, then it is a closed set in $X$.*

*Proof.* Let $A$ be a $BCC$-ideal which is an open set in $X$ and let $x \in X - A$. Then there exists a neighborhood $U$ of $x$ such that $U*U \subseteq A$, since $x*x = 0 \in A$ and $A$ is open. We claim that $U \subseteq X - A$. If $U \nsubseteq X - A$, then there exists $y \in U \cap A$. Note that $z * y \in U * U \subseteq A$ for all $z \in U$. Since $y \in A$ and $A$ is a $BCC$-ideal, it follows that $z \in A$ which shows that $U \subseteq A$, a contradiction. Hence $A$ is closed. □

**Proposition 4.18.** *Let $X$ be a topological $BCC$-algebra and $\{0\}$ be closed. Then $\cap\aleph_0 = \{0\}$ where $\aleph_0$ is the neighborhood system of* $0$.

*Proof.* Since $\{0\}$ is closed, by Proposition 4.15 $X$ is Hausdorff. Given an element $x \in X \setminus \{0\}$, $0$ has a neighborhood $U$ such that $x \notin U$ and so $x \notin \cap\aleph_0$. Hence $\cap\aleph_0 = \{0\}$. □

Let $X$ be a $BCC$-algebra. For an arbitrary element $a \in X$ and any nonempty subset $V$ of $X$, denote

$$V(a) := \{x \in X | x * a \in V \text{ and } a * x \in V\}.$$

Note that $V(a) \subseteq U(a)$ whenever $V \subseteq U \subseteq X$.

**Theorem 4.19.** *Let $\Omega$ be a filter base on a $BCC$-algebra $X$ such that for every $p, q \in V \in \Omega$*

(1) $0 * p \in V$,

(2) $(x * p) * q = 0$ *implies* $x \in V$.

*Then $\mathcal{T} := \{O \subseteq X | \forall a \in O, \exists V \in \Omega : V(a) \subseteq O\}$ is a topology on $X$ and $\Omega$ is a local base at* $0$.

*Proof.* Let $\mathcal{T} := \{O \subseteq X | \forall a \in O, \exists V \in \Omega : V(a) \subseteq O\}$. Clearly, $\emptyset, X \in \mathcal{T}$. Let $\{O_\alpha\}$ be a family of members of $\mathcal{T}$ and let $a \in \cup O_\alpha$. Then $a \in O_\alpha$ for some $\alpha$. It follows that there exists $V \in \Omega$ such that $V(a) \subseteq O_\alpha \subseteq \cup O_\alpha$ so that $\cup O_\alpha \in \mathcal{T}$. Assume that $O_\alpha$ and $O_\beta$ belong to $\mathcal{T}$ and let $a \in O_\alpha \cap O_\beta$. Then there exist $V_\alpha \in \Omega$ and $V_\beta \in \Omega$ such that $V_\alpha(a) \subseteq O_\alpha$ and $V_\beta(a) \subseteq O_\beta$, respectively. Since $\Omega$ is a filter base, there exists $V \in \Omega$ such that $V \subseteq V_\alpha \cap V_\beta$. Thus we have

$$V(a) \subseteq (V_\alpha \cap V_\beta)(a) \subseteq V_\alpha(a) \cap V_\beta(a) \subseteq O_\alpha \cap O_\beta$$

and so $O_\alpha \cap O_\beta \in \mathcal{T}$. This proves that $\mathcal{T}$ is a topology on $X$ (In this case we call it a *topology induced by* $\Omega$, and is denoted by $\mathcal{T}_\Omega$).

Now we will show that $\Omega$ is the filter base of a neighborhood of $0$ with respect to the topology $\mathcal{T}$. Let $p \in V \in \Omega$. Then $0 * p \in V$ by (i), and since $(0 * p) * (0 * p) = 0$ it follows from (ii) that $0 \in V$, i.e., every element $V \in \Omega$ contains $0$. If $x \in V(p)$ then $x * p, p * x \in V$ and so $x * p = v$ for some $v \in V$. Hence $(x * p) * v = 0$ which implies that $x \in V$. Therefore $V(p) \subseteq V$ and $v \in \mathcal{T}$. Thus $V$ is a neighborhood of $0$. If we let $V$ be a neighborhood of $0$, then there exists a $U \in \Omega$ such that $U(0) \subseteq V$. Note that $0 * a \in U$ and $a * 0 \in U$ for some $a \in U$. Hence $a \in U(0)$ and $0 \in U \subseteq U(0) \subseteq V$. Thus $\Omega$ is a local base at $0$ with respect to the topology $\mathcal{T}$. □

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