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MINIMAL SETS OF PERIODS FOR MAPS ON THE KLEIN BOTTLE

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ABSTRACT. The main results concern with the self maps on the Klein bottle. We obtain the Reidemeister numbers and the Nielsen numbers for all self maps on the Klein bottle. In terms of the Nielsen numbers of their iterates, we totally determine the minimal sets of periods for all homotopy classes of self maps on the Klein bottle.

1. Introduction

In dynamical systems, topological information can be used to study qualitative and quantitative properties of the system. In this paper, we determined the minimal sets of periods (of the periodic orbits) of all homotopy classes of the maps on the Klein bottle.

In order to state our main results, let us fix some notations and terminologies. Let $f : X \to X$ be a continuous self-map of a topological space X, and n be a natural number. A fixed point of f is a point x in X such that $f(x) = x$. Denote the fixed point set of f by $\mathrm{Fix}(f)$, and the set of periodic points with least period n by $\mathrm{P}_n(f)$, i.e.,

$$\begin{aligned} \mathrm{Fix}(f) &:= \{x \in X \mid x = f(x)\}. \\ \mathrm{P}_n(f) &:= \{x \in X \mid x = f^n(x), \text{ but } \ x \neq f^k(x) \text{ for any } k < n\} \\ &= \mathrm{Fix}(f^n) \setminus \textstyle\bigcup_{k<n} \mathrm{Fix}(f^k). \end{aligned}$$

Denote by $\mathrm{Per}(f)$ the set of natural numbers corresponding to least periods of periodic points of f, i.e.,

$$\mathrm{Per}(f) := \{n \in \mathbb{N} \mid \mathrm{P}_n(f) \neq \emptyset\}.$$

When a map $g : X \to X$ is homotopic to f, we shall write $g \simeq f : X \to X$. Define the minimal set of periods of f to be the set

$$\mathrm{MPer}(f) := \bigcap_{g \simeq f} \mathrm{Per}(g).$$

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Such a subset of natural numbers was introduced in [1] and [4]. This has also been written as HPer(f) in the literature, such as [5], in order to emphases the homotopy invariance.

Nielsen fixed point theory [3] turns out to be a powerful tool in the study of the set MPer(f).

In 1911, Bieberbach proved that any automorphism of a crystallographic group is conjugation by an element of $\mathrm{Aff}(\mathbb{R}) = \mathbb{R} \rtimes \mathrm{GL}(n, \mathbb{R})$. This was generalized to an almost crystallographic group. In 1995, K. B. Lee generalized this result from isomorphisms to all homomorphisms ([8]). Topologically, this implies that every continuous map on an infra-nilmanifold is homotopic to a map induced by an affine endomorphism on the Lie group level. Here, we apply this result to the Klein bottle.

Let G be a connected, simply connected nilpotent Lie group. Let π be a subgroup of $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ with $\Gamma = \pi \cap G$ is a lattice of G. Then $M = \pi \backslash G$ is an infra-nilmanifold with a finite holonomy group $\Psi = \pi/\Gamma$. In this paper , we shall restrict our space to the Klein bottle K. $\pi_1(K)$ is denoted by π. In our case, $G = \mathbb{R}^2$, $\Gamma = \mathbb{Z}^2$, and the holonomy group $\Psi = \pi/\Gamma$ is $\mathbb{Z}_2$. We shall consider the continuous maps on the Klein bottle. By using the classification of homotopy classes of maps on the Klein bottle, we calculate their Reidemeister numbers and Nielsen numbers one by one. Using these calculations, we compute out the minimal sets of periods for all maps on the Klein bottle, and hence perfect the results of J. Llibre ([9]), where some estimations were given.

This paper is organized as follows: In Section 2, after reviewing the Nielsen fixed point theory, we state results for the minimal set of periods of a self map f by the behavior of the sequence $\{N(f^k)\}_{k\geq 1}$. In Section 3, we classify the homotopy classes of the self maps on the Klein bottle; typical self maps in each homotopy class are given according to the result in [8]. In Section 4, we compute the Nieslen numbers and Reidemeister numbers for all self maps on the Klein bottle. Our main results are given in the last section: the minimal set of periods for each map on the Klein bottle is obtained.

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2. Nielsen number and period points

In this section we give a brief account of Nielsen fixed point theory. See [3] for more details. We also obtain some sufficient conditions to determine if a given period occurs for a continuous map in terms of the Nielsen numbers of its iterates.

The Nielsen fixed point theory is defined for continuous self-maps of compact absolute neighborhood retracts (ANR). Here, we focus on more restricted spaces. Let $f : X \to X$ be a continuous map of a compact connected smooth manifold X. Consider the universal covering $p : \tilde{X} \to X$. Clearly, the fixed point set $\mathrm{Fix}\tilde{f}$ of a lifting $\tilde{f}$ will be projected down to a subset of the fixed point set of f. Moreover, the fixed point set of f is just the union $\cup_{\tilde{f}} p(\mathrm{Fix}(\tilde{f}))$, where $\tilde{f}$ ranges over all liftings of f. Two liftings $\tilde{f}$ and $\tilde{f}'$ of f are said to be conjugate if there is a deck transformation γ such that $\tilde{f}' = \gamma^{-1}\tilde{f}\gamma$. It is shown that (1) $p(\mathrm{Fix}(\tilde{f})) = p(\mathrm{Fix}(\tilde{f}'))$ if $\tilde{f}$ and $\tilde{f}'$ are conjugate; (2) $p(\mathrm{Fix}(\tilde{f})) \cap p(\mathrm{Fix}(\tilde{f}')) = \emptyset$ if $\tilde{f}$ and $\tilde{f}'$ are not conjugate. We call the subset $p(\mathrm{Fix}(\tilde{f}))$ of the fixed point set $\mathrm{Fix}(f)$ of f the fixed point class of f determined by the lifting conjugate class $[\tilde{f}]$. Thus, there is a one-to-one correspondence between the set of fixed point classes of f and the set of lifting conjugate classes of f with respect to the universal covering. In this sense, the fixed point classes determined by different lifting conjugate classes are considered to be different even if they are all empty sets. The *Reidemeister number* $R(f)$ of f is the number of fixed point classes of f.

Every fixed point class has a well-defined fixed point index. A fixed point class is said to be *essential* if it has non-zero index; to be inessential if it has zero index. The *Nielsen number* $N(f)$ of f is the number of essential fixed point classes of f. Each essential fixed point class is known to be non-empty. It can be proved that either the Nielsen number or the Reidemeister number is a homotopy invariant. Hence, every map homotopic to f has at least $N(f)$ fixed points. It is known that if X is a compact manifold which is not a surface with a negative Euler characteristic number, then the Nielsen number $N(f)$ can be realized as the number of the fixed points of a map homotopic to f.

For any positive integers k and r, a fixed point of f^k is certainly a fixed point of f^{rk}. On the fixed point class level, a fixed point class $F^{(k)}$ of f^k is said to be contained in a fixed point class $F^{(rk)}$ of f^{rk} if $F^{(k)}$ is determined by a lifting $\tilde{g}$ of f^k and $F^{(rk)}$ is determined by $\tilde{g}^r$.

A fixed point of f^n need not have least period n. It is natural to ask for some sufficient conditions for f to have a periodic point of period n. In [1], so-called "index assumption" is defined for this purpose.

Definition 2.1 (cf. [1, p.6]). A self map $f : X \to X$ is said to *satisfy the index assumption* if for any positive integers k and r, any fixed point class of f^k being contained in an essential fixed point class of f^{rk} is essential.

By definition, we obtain immediately that

Corollary 2.2. *A self map $f : X \to X$ satisfies the index assumption if and only if for any positive integers k and r, any inessential fixed point class of f^k is contained in an inessential fixed point class of f^{rk}.*

The next result gives a sufficient condition to assure that a self map has a periodic point of period n.

Proposition 2.3. *Let $f : X \to X$ be a self map satisfying the index assumption. If*

$$\sum_{\frac{n}{k}:prime} N(f^k) < N(f^n),$$

then any self map homotopic to f has a periodic point of period n, i.e., $n \in \mathrm{MPer}(f)$.

Proof. See the proof of [1, Proposition 2.2]. □

From this proposition, we have

Lemma 2.4. *Let $f : X \to X$ be a self map such that for integer n, $N(f^n) > 0$, $\frac{N(f^{n+1})}{N(f^n)} \geq 2$. If f satisfies the index assumption, then any self map homotopic to f has a periodic point of each period n, i.e., $\mathrm{MPer}(f) = \mathbb{N}$.*

Proof. Since $\frac{N(f^{n+1})}{N(f^n)} \geq 2$, we have that $N(f^k) \leq (\frac{1}{2})^{n-k} N(f^n)$ for all $k < n$,

$$\sum_{\frac{n}{k}:\text{prime}} N(f^k) \leq \sum_{k=1}^{n-1} N(f^k) \leq \sum_{k=1}^{n-1} (\frac{1}{2})^{n-k} N(f^n) = (1 - (\frac{1}{2})^n) N(f^n).$$

By Proposition 2.3, we are done. □

3. Homotopy classification of self maps the Klein bottle

It is known that the fundamental group of the Klein bottle is

$$\pi_1(K, *) = \{t_2, \alpha \mid t_2 \alpha t_2 = \alpha\}.$$

This group can be identified with the subgroup

$$\left\{ \begin{pmatrix} 1 & 0 & \frac{j}{2} \\ 0 & (-1)^j & i \\ 0 & 0 & 1 \end{pmatrix} \mid i, j \in \mathbb{Z} \right\}$$

of a 3×3-matrix group. It is easy to check that any element in $\pi_1(K, *)$ can be written uniquely as $t_2^i \alpha^j$, which is, of matrix form,

$$t_2^i \alpha^j = \begin{pmatrix} 1 & 0 & \frac{j}{2} \\ 0 & (-1)^j & i \\ 0 & 0 & 1 \end{pmatrix}.$$

Here, we follow the stand notations for the generators of a homomorphism of an almost crystallographic group, i.e.,

$$t_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $t_1 = \alpha^2$.

Since $\pi_1(K, *)$ acts freely on $R^2 \times \{1\} = \{\left(\begin{smallmatrix} x \\ y \\ 1 \end{smallmatrix}\right)\}$, the Klein bottle K can be identified as $(R^2 \times \{1\})/\pi_1(K, *)$. So, we have an expression of the Klein bottle as

$$K = \left\{ [x, y] = \left[\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right] \mid 0 \leq x < \frac{1}{2},\ 0 \leq y < 1 \right\}.$$

Thus, the base point $*$ of K is just $[0,0]$.

Next Lemma, which is not hard to prove, was obtained by Halpern [2]:

Lemma 3.1. *Any self homomorphism* $\psi : \pi_1(K,*) \to \pi_1(K,*)$ *of the fundamental group of the Klein bottle has the form:* $\psi(t_2) = t_2^{\frac{1-(-1)^w}{2}u}$, $\psi(\alpha) = t_2^v \alpha^w$, *i.e.,*

$$\psi(t_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1-(-1)^w}{2}u \\ 0 & 0 & 1 \end{pmatrix}, \quad \psi(\alpha) = \begin{pmatrix} 1 & 0 & \frac{w}{2} \\ 0 & (-1)^w & v \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 3.2. *Any self map on the Klein bottle is homotopic to a map covered by an affine endomorphism determined by one of two types of the following:*

$$\begin{pmatrix} w & 0 & d_1 \\ 2v & 0 & d_2 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } w \text{ is even;} \quad \begin{pmatrix} w & 0 & d_1 \\ 0 & u & \frac{v}{2} \\ 0 & 0 & 1 \end{pmatrix} \text{ if } w \text{ is odd,}$$

where u, v *and* w *are integers, and* d_1 *and* d_2 *are real numbers.*

Proof. Clearly, g is homotopic to a base point preserving map g'. By Lemma 3.1, the homomorphism induced by g' is given by $g'_\pi(t_2) = t_2^{\frac{1-(-1)^w}{2}u}$, $g'_\pi(\alpha) = t_2^v \alpha^w$.

Note that K itself is an infra-nilmanifold. By [8, Corollary 1.2], g' is homotopic to a map induced from an affine endomorphism determined by a 3×3 matrix:

$$D = \begin{pmatrix} d_{11} & d_{12} & d_1 \\ d_{21} & d_{22} & d_2 \\ 0 & 0 & 1 \end{pmatrix} : R^2 \times \{1\} \to R^2 \times \{1\}.$$

By semi-conjugate relation in [8, Theorem 1.1], we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1-(-1)^w}{2}u \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} d_{11} & d_{12} & d_1 \\ d_{21} & d_{22} & d_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_1 \\ d_{21} & d_{22} & d_2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & \frac{w}{2} \\ 0 & (-1)^w & v \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} d_{11} & d_{12} & d_1 \\ d_{21} & d_{22} & d_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_1 \\ d_{21} & d_{22} & d_2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So,

$$\begin{pmatrix} d_{11} & d_{12} & d_1 \\ d_{21} & d_{22} & d_2 + \frac{1-(-1)^w}{2}u \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_1 \\ d_{21} & d_{22} & d_2 + d_{22} \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} d_{11} & d_{12} & d_1 + \frac{w}{2} \\ (-1)^w d_{21} & (-1)^w d_{22} & (-1)^w d_2 + v \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d_{11} & -d_{12} & \frac{d_{11}}{2} + d_1 \\ d_{21} & -d_{22} & \frac{d_{21}}{2} + d_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The non-trivial restrictions are

$$d_2 + \frac{1-(-1)^w}{2}u = d_2 + d_{22}$$

$$\begin{aligned} d_{12} &= -d_{12} \\ d_1 + \frac{w}{2} &= d_1 + \frac{d_{11}}{2} \\ (-1)^w d_{21} &= d_{21} \\ (-1)^w d_{22} &= -d_{22} \\ (-1)^w d_2 + v &= d_2 + \frac{d_{21}}{2}. \end{aligned}$$

Thus, we are done. □

We shall write the restriction of affine endomorphisms on $R^2 \times \{1\}$ as $\tilde{f}_{(u,v,w)}$, where d_1 and d_2 are chosen as 0. The self map on K induced by $\tilde{f}_{(u,v,w)}$ is written by $f_{(u,v,w)}$. So, $\tilde{f}_{(u,v,w)}$ is a lifting of $f_{(u,v,w)}$.

Proposition 3.3. *When w is even, $f_{(u,v,w)} \simeq f_{(u',v',w')}$ if and only if $u' = u$, $v' = \pm v$ and $w' = w$; when w is odd, $f_{(u,v,w)} \simeq f_{(u',v',w')}$ if and only if $u' = \pm u$, $v' - v$ is even and $w' = w$.*

Proof. See [2, 2.7]. □

Now, we obtain a homotopy classification:

Theorem 3.4. *The set of homotopy classes of self maps on the Klein bottle K is*

$$[K, K] = \{f_{(u,v,w)} \mid (u,v,w) \in S\},$$

where

$S = \{(0, v, w) \in \mathbb{Z}^3 \mid v \geq 0, w \text{ is even}\} \cup \{(u,v,w) \in \mathbb{Z}^3 \mid u \geq 0, v = 0, 1, w \text{ is odd}\}$.

Except for the case in which w is odd and v is non-zero, $f_{(u,v,w)}$ preserves the base point $*$ of the Klein bottle. One can check that $f_{(u,v,w)}$ is always homotopic to a base point preserving map $f' : K \to K$ such that the homomorphism $f'_\pi : \pi_1(K, *) \to \pi_1(K, *)$ induced by f' is given by $f'_\pi(t_2) = t_2^{\frac{1-(-1)^w}{2}u}$ and $f'_\pi(\alpha) = t_2^v \alpha^w$.

The homotopy classes $[f_{(u,v,w)}]$ of self map on Klein bottle can be classified according to the derivative $Df_{(u,v,w)}$, i.e., the rotation part of $\tilde{f}_{(u,v,w)}$:

$$\begin{pmatrix} w & 0 \\ 2v & 0 \end{pmatrix} \text{ if } w \text{ is even}; \quad \begin{pmatrix} w & 0 \\ 0 & u \end{pmatrix} \text{ if } w \text{ is odd}.$$

(1) Degenerate map: $Df_{(u,v,w)}$ has zero eigenvalue, i.e., w is even or $u = 0$.

(2) Homeomorphism: Two eigenvalues of $Df_{(u,v,w)}$ are ± 1. There are four classes: $f_{(1,0,1)}$, $f_{(1,1,1)}$, $f_{(1,0,-1)}$ and $f_{(1,1,-1)}$, where $f_{(1,0,1)}$ is the identity, $f_{(1,1,1)} f_{(1,0,-1)} = f_{(1,1,-1)}$, and $f^2_{(1,1,1)} = f^2_{(1,0,-1)} = f^2_{(1,1,-1)}$ is the identity $f_{(1,0,1)}$.

(3) Half-expanding maps: $Df_{(u,v,w)}$ has an eigenvalue with absolute value 1 and an eigenvalue with absolute value greater than 1, i.e., either w is odd with $|w| \geq 3$ and $u = 1$, or $w = \pm 1$ and $u \geq 2$.

(4) Expanding maps: $Df_{(u,v,w)}$ has no eigenvalue values with absolute value less or equal to 1, i.e., w is odd with $|w| \geq 3$ and $u \geq 2$.

It is obvious that

Proposition 3.5. *If $f_{(u,v,w)}$ is a degenerate map on K, then the image of $f_{(u,v,w)}$ lies in a circle C in K. Moreover, $f_{(u,v,w)}|_C$ has degree w.*

4. Computations for Nielsen numbers and Reidemeister numbers

In this section, we shall compute directly the Nielsen numbers and the Reidemeister numbers of any self maps on the Klein bottle. In fact, the result about Nielsen numbers is a special case of the averaging formula in [7], and is also included in [2], and [6] for some special cases. Our expressions are more precise.

Given a self map $f_{(u,v,w)} : K \to K$, it is known that the set of all liftings of $f^k_{(u,v,w)}$ is $\{(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)} \mid i,j \in \mathbb{Z}\}$. Thus, each fixed point class of $f_{(u,v,w)}$ has the form $p(\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{f}_{(u,v,w)}))$.

Lemma 4.1. *Let $F = p(\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}))$ be a fixed point class of $f^k_{(u,v,w)}$. The following three statements are equivalent:*

(i) $(w^k - 1)((-1)^j u^k - 1) \neq 0$,

(ii) *the fixed point class F is essential,*

(iii) *the fixed point class F is a single point.*

Proof. Note that u is zero if w is even. By Theorem 3.2, the matrix form of $f_{(u,v,w)}$ can be written uniformly as

$$\begin{pmatrix} w & 0 & 0 \\ (1+(-1)^w)v & u & \frac{(1-(-1)^w)v}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

So, for any integers i and j, and any positive integer k,

$$(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)} = \begin{pmatrix} w^k & 0 & -\frac{j}{2} \\ v_1 & (-1)^j u^k & v_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.1}$$

where

$$\begin{aligned} v_1 &= (-1)^j(1+(-1)^w)v\sum_{s=0}^{k-1} u^s w^{k-s-1}, \\ v_2 &= -(-1)^j i + (-1)^j\frac{(1-(-1)^w)v}{4}\sum_{s=0}^{k-1} u^s. \end{aligned}$$

Assume that $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ is a fixed point of $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}$. We have that

$$\begin{cases} (w^k - 1)x = \frac{j}{2} \\ v_1 x + ((-1)^j u^k - 1)y = -v_2. \end{cases} \tag{4.2}$$

(i) $\Longrightarrow$ (ii). Since $(w^k-1)((-1)^j u^k-1)\neq 0$, by equality (4.2), the fixed point class $p(\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}))$ has unique fixed point

$$\left[\frac{j}{2(w^k-1)}, \frac{-v_1\dfrac{j}{2(w^k-1)}-v_2}{(-1)^j u^k-1}\right]\in K.$$

Note that

$$\begin{aligned}\det(I_{2\times 2}-Df^k_{(u,v,w)}) &= \det(I_{2\times 2}-\begin{pmatrix} w^k & 0\\ v_1 & (-1)^j u^k\end{pmatrix})\\ &= (w^k-1)((-1)^j u^k-1)\\ &\neq 0.\end{aligned}$$

By definition of the fixed point index, $\mathrm{ind}(f^k_{(u,v,w)}, p(\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)})))=\mathrm{sgn}(w^k-1)((-1)^j u^k-1)\neq 0$, i.e., this fixed point class is essential.

(ii) $\Longrightarrow$ (iii). As a linear map in R^3, the fixed point set of $(t_2^i\alpha^j)^{-1}\tilde{f}_{(u,v,w)}$ is a linear sub-space of R^3. Thus, the fixed point set of $(t_2^i\alpha^j)^{-1}\tilde{f}_{(u,v,w)}: R^2\times\{1\}\to R^2\times\{1\}$ is either an empty set, a single point, a line, or the total plane. If we project down to K, we have that $p(\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{f}_{(u,v,w)}))$ is a connected sub-manifold of K. The set $p(\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}))$ will be either the empty set, a circle, or the whole Klein bottle if it is not a single point. Hence, its index will be zero.

(iii) $\Longrightarrow$ (i). If $(w^k-1)((-1)^j u^k-1)=0$, then either w^k-1 or $(-1)^j u^k-1$ is zero. From equality (4.2), $p(\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}))$ is either an empty set or contains a circle. □

From the proof of Lemma 4.1, we know that each fixed point class

$$p(\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{f}_{(u,v,w)}))$$

of $f_{(u,v,w)}$ is a connected sub-manifold of K. Its fixed point index is given by

Corollary 4.2.

$$\mathrm{ind}(f_{(u,v,w)}, p(\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{f}_{(u,v,w)})))=\mathrm{sgn}(w-1)((-1)^j u-1).$$

In the remainder of this section, we shall prove the following two theorems case by case:

Theorem 4.3.

$$N(f^k_{(u,v,w)})=\frac{1}{2}(|u^k+1|+|u^k-1|)|w^k-1|=\begin{cases}|u^k(w^k-1)| & \textit{if } u\neq 0,\\ |w^k-1| & \textit{if } u=0.\end{cases}$$

Theorem 4.4.

$$R(f^k_{(u,v,w)})=\begin{cases}+\infty & \textit{if } w^k=1 \textit{ or } u=1.\\ N(f^k_{(u,v,w)}) & \textit{otherwise.}\end{cases}$$

By homotopy invariance and the classification theorem (Theorem 3.4) of self maps on the Klein bottle, it suffices to consider the self maps $f_{(u,v,w)}$, where $(u, v, w) \in \{(0, v, w) \in \mathbb{Z}^3 \mid v \geq 0, w \text{ is even }\} \cup \{(u, v, w) \in \mathbb{Z}^3 \mid u \geq 0, v = 0, 1, w \text{ is odd }\}$.

4.1. Case: w is even

The set of all liftings of $f^k_{(0,v,w)}$ is:

$$\left\{(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(0,v,w)} \mid i,j \in \mathbb{Z}\right\} = \left\{\begin{pmatrix} w^k & 0 & -\frac{j}{2} \\ 2(-1)^j vw^{k-1} & 0 & -(-1)^j i \\ 0 & 0 & 1 \end{pmatrix} \mid i,j \in \mathbb{Z}\right\}.$$

The conjugate class of the lifting $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(0,v,w)}$ is

$$\{(t_2^m\alpha^n)^{-1}(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(0,v,w)}t_2^m\alpha^n \mid m,n \in Z\},$$

i.e.,

$$\left\{\begin{pmatrix} w^k & 0 & -\frac{j}{2} + \frac{n}{2}(w^k-1) \\ 2(-1)^{j+n}vw^{k-1} & 0 & -(-1)^{j+n}i - (-1)^n m + (-1)^{j+n}vw^{k-1} \\ 0 & 0 & 1 \end{pmatrix} \mid m,n \in \mathbb{Z}\right\}.$$

There are $|w^k - 1|$ conjugate classes, which are represented by

$$(t_2^0\alpha^j)^{-1}\tilde{f}^k_{(0,v,w)} = \begin{pmatrix} w^k & 0 & -\frac{j}{2} \\ 2(-1)^j vw^{k-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad j = 0, 1, \ldots, |w^k - 1| - 1.$$

Each lifting has a unique fixed point. By Lemma 4.1, each fixed point class is essential. Thus, we have that $R(f^k_{(0,v,w)}) = N(f^k_{(0,v,w)}) = |w^k - 1|$. This proves Theorem 4.3 and Theorem 4.4 in this case.

4.2. Case: w is odd

The set of all liftings of $f^k_{(u,v,w)}$ is

$$\begin{aligned} &\{(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)} \mid i,j \in \mathbb{Z}\} \\ &= \left\{\begin{pmatrix} w^k & 0 & -\frac{j}{2} \\ 0 & (-1)^j u^k & -(-1)^j i + (-1)^j \frac{v}{2}\sum_{s=0}^{k-1} u^s \\ 0 & 0 & 1 \end{pmatrix} \mid i,j \in \mathbb{Z}\right\}. \end{aligned}$$

The conjugate class containing $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}$ is

$$\{(t_2^m\alpha^n)^{-1}(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}t_2^m\alpha^n \mid m,n \in \mathbb{Z}\},$$

i.e.,

$$\left\{\begin{pmatrix} w^k & 0 & -\frac{j}{2}+\frac{n}{2}(w^k-1) \\ 0 & (-1)^j u^k & -(-1)^{j+n}i+(-1)^{j+n}\frac{v}{2}\sum_{s=0}^{k-1}u^s-(-1)^n m+(-1)^{j+n}mu^k \\ 0 & 0 & 1 \end{pmatrix}\right.$$
$$\left. \mid m,n\in\mathbb{Z}\right\}.$$

Thus, we obtain the following:

Proposition 4.5. *Suppose that w is odd. Then two liftings $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}$ and $(t_2^{i'}\alpha^{j'})^{-1}\tilde{f}^k_{(u,v,w)}$ of $f^k_{(u,v,w)}$ are conjugate if and only if there are two integers m and n such that*

$$\text{(4.3)}\quad \begin{cases} j' &= j-n(w^k-1), \\ i' &= (-1)^n i+(1-(-1)^n)\frac{v}{2}\sum_{s=0}^{k-1}u^s-(-1)^n m(u^k-(-1)^j). \end{cases}$$

We shall discuss lifting conjugate classes in several subcases.

4.2.1. Case: $w\neq\pm1$ and $u\neq 1$. By Proposition 4.5, the set of conjugate classes of $f^k_{(u,v,w)}$ is:

$$\{[(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}]\mid i=0,1,\ldots,u^k-2,\ j=0,2,\ldots,|w^k-1|-2\}$$

$$\cup\{[(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}]\mid i=0,1,\ldots,u^k,\ j=1,3,\ldots,|w^k-1|-1\}.$$

Here, if $i=0,1,\ldots,u^k-2$, $j=0,2,\ldots,|w^k-1|-2$,

$$\begin{aligned}&[(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}]\\ &=\{(t_2^{i+m(u^k-1)}\alpha^{j+2n|w^k-1|})^{-1}\tilde{f}^k_{(u,v,w)}\mid m,n\in\mathbb{N}\}\\ &\quad\cup\{(t_2^{v\sum_{s=0}^{k-1}u^s-i+m(u^k-1)}\alpha^{j+(2n+1)|w^k-1|})^{-1}\tilde{f}^k_{(u,v,w)}\mid m,n\in\mathbb{N}\};\end{aligned}$$

if $i=0,1,\ldots,u^k$, $j=1,3,\ldots,|w^k-1|-1$,

$$\begin{aligned}&[(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}]\\ &=\{(t_2^{i+m(u^k-1)}\alpha^{j+2n|w^k-1|})^{-1}\tilde{f}^k_{(u,v,w)}\mid m,n\in\mathbb{N}\}\\ &\quad\cup\{(t_2^{v\sum_{s=0}^{k-1}u^s-i+m(u^k+1)}\alpha^{j+(2n+1)|w^k-1|})^{-1}\tilde{f}^k_{(u,v,w)}\mid m,n\in\mathbb{N}\}.\end{aligned}$$

The number of conjugate classes is $\frac{1}{2}(|u^k+1|+|u^k-1|)|w^k-1|$. By Lemma 4.1, each fixed point class is essential. Thus, we have that $R(f^k_{(u,v,w)})=N(f^k_{(u,v,w)})$ $=\frac{1}{2}(|u^k+1|+|u^k-1|)|w^k-1|$. This proves Theorem 4.3 and Theorem 4.4 in this case.

4.2.2. Case: $w \neq \pm 1$ and $u = 1$. By Proposition 4.5, two liftings $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(1,v,w)}$ and $(t_2^{i'}\alpha^{j'})^{-1}\tilde{f}^k_{(1,v,w)}$ of $f^k_{(1,v,w)}$ are conjugate if and only if there are two integers m and n such that

$$\begin{cases} j' &= j - n(w^k - 1), \\ i' &= (-1)^n i + (1-(-1)^n)\frac{kv}{2} - (-1)^n m(1-(-1)^j). \end{cases}$$

Thus, the set of conjugate classes of $f^k_{(1,v,w)}$ is

$$\{[(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(1,v,w)}] \mid i \in \mathbb{N},\ j = 0, 2, \ldots, |w^k - 1| - 2\}$$

$$\cup\ \{[(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(1,v,w)}] \mid i = 0, 1,\ j = 1, 3, \ldots, |w^k - 1| - 1\}.$$

Here, if $j = 0, 2, \ldots, |w^k - 1| - 2$,

$$\begin{aligned} [(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}] &= \{(t_2^i\alpha^{j+2n|w^k-1|})^{-1}\tilde{f}^k_{(1,v,w)} \mid n \in \mathbb{N}\} \\ &\quad \cup\{(t_2^{kv-i}\alpha^{j+(2n+1)|w^k-1|})^{-1}\tilde{f}^k_{(1,v,w)} \mid n \in \mathbb{N}\}. \end{aligned}$$

These are infinitely many lifting classes. We have $R(f^k_{(1,v,w)}) = +\infty$. By Lemma 4.1, each of them is inessential.

If $i = 0, 1$, $j = 1, 3, \ldots, |w^k - 1| - 1$,

$$\begin{aligned} [(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(1,v,w)}] &= \{(t_2^{i+2m}\alpha^{j+2n|w^k-1|})^{-1}\tilde{f}^k_{(u,v,w)} \mid m, n \in \mathbb{N}\} \\ &\quad \cup\{(t_2^{kv-i+2m}\alpha^{j+(2n+1)|w^k-1|})^{-1}\tilde{f}^k_{(1,v,w)} \mid m, n \in \mathbb{N}\}. \end{aligned}$$

Note that $(w^k - 1)((-1)^j u^k - 1) = -2(w^k - 1) \neq 0$. By Lemma 4.1, each of these $|w^k - 1|$ fixed point classes is essential. Thus, $N(f^k_{(1,v,w)}) = |w^k - 1|$. This proved Theorem 4.3 in our sub-case.

4.2.3. Case: $w = 1$. Then two liftings $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,1)}$ and $(t_2^{i'}\alpha^{j'})^{-1}\tilde{f}^k_{(u,v,1)}$ of $f^k_{(u,v,1)}$ are conjugate if and only if there are two integers m and n such that

$$\begin{cases} j' &= j, \\ i' &= (-1)^n i + (1-(-1)^n)\frac{v}{2}\sum_{s=0}^{k-1} u^s - (-1)^n m(u^k - (-1)^j). \end{cases}$$

This implies that $f^k_{(u,v,1)}$ has infinitely many fixed point classes. Thus, $R(f^k_{(u,v,1)}) = +\infty$. Note that $(w^k - 1)((-1)^j u^k - 1)$ is always zero. By Lemma 4.1, each fixed point class is inessential. Thus, $N(f^k_{(u,v,1)}) = 0$.

4.2.4. Case: $w = -1$ and $u \neq 1$. By Proposition 4.5, two liftings $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,-1)}$ and $(t_2^{i'}\alpha^{j'})^{-1}\tilde{f}^k_{(u,v,-1)}$ of $f^k_{(u,v,-1)}$ are conjugate if and only if there are two integers m and n such that

$$\begin{cases} j' &= j - n((-1)^k - 1), \\ i' &= (-1)^n i + (1-(-1)^n)\frac{v}{2}\sum_{s=0}^{k-1} u^s - (-1)^n m(u^k - (-1)^j). \end{cases}$$

If k is even, the set of lifting conjugate classes of $f^k_{(u,v,-1)}$ is:

$$\{[(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,-1)}] \mid i=0,1,\ldots,|u^k-1|-1, j \text{ is even}\}$$
$$\cup\ \{[(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,-1)}] \mid i=0,1,\ldots,u^k, j \text{ is odd }\}.$$

For each of these infinitely many classes, we have that $(w^k-1)((-1)^ju^k-1)=0$. By Lemma 4.1, all these fixed point class of $f^k_{(u,v,-1)}$ are inessential. It follows that $N(f^k_{(u,v,-1)})=0$ and $R(f^k_{(u,v,-1)})=+\infty$ if k is even.

If k is odd, the set of lifting conjugate classes of $f^k_{(u,v,-1)}$ is:

$$\{[(t_2^i\alpha^0)^{-1}\tilde{f}^k_{(u,v,-1)}] \mid i=0,1,\ldots,|u^k-1|-1\}$$
$$\cup\ \{[(t_2^i\alpha^1)^{-1}\tilde{f}^k_{(u,v,-1)}] | i=0,1,\ldots,u^k\}.$$

In this case, we have that $(w^k-1)((-1)^ju^k-1)\neq 0$. By Lemma 4.1, these $|u^k-1|+|u^k+1|$ fixed point classes are all essential. Hence, $N(f^k_{(u,v,-1)}) = R(f^k_{(u,v,-1)}) = |u^k-1|+|u^k+1|$.

4.2.5. Case: $w=-1$ and $u=1$. By Proposition 4.5, two liftings $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(1,v,-1)}$ and $(t_2^{i'}\alpha^{j'})^{-1}\tilde{f}^k_{(1,v,-1)}$ of $f^k_{(1,v,-1)}$ are conjugate if and only if there are two integers m and n such that

$$\begin{cases} j' &= j-n((-1)^k-1), \\ i' &= (-1)^n i+(1-(-1)^n)\frac{kv}{2}-(-1)^n m(1-(-1)^j). \end{cases}$$

If k is even, the set of lifting conjugate classes of $f^k_{(1,v,-1)}$ is:

$$\{[(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(1,v,-1)}] \mid i\in\mathbb{N}, j \text{ is even}\}\cup\{[(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(1,v,-1)}] \mid i=0,1, j \text{ is odd}\}.$$

For any lifting $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(1,v,-1)}$, we have that $(w^k-1)((-1)^ju^k-1)=0$. By Lemma 4.1, all these infinitely many fixed point classes of $f^k_{(1,v,-1)}$ are inessential. So, $R(f^k_{(1,v,-1)})=+\infty$ and $N(f^k_{(1,v,-1)})=0$.

If k is odd, the set of lifting conjugate classes of $f^k_{(1,v,-1)}$ is:

$$\{[(t_2^0\alpha^1)^{-1}\tilde{f}^k_{(1,v,-1)}],[(t_2^1\alpha^1)^{-1}\tilde{f}^k_{(1,v,-1)}]\}\cup\{[(t_2^i\alpha^0)^{-1}\tilde{f}^k_{(1,v,-1)}] \mid i\in\mathbb{N}\}.$$

Here,

$$\begin{aligned}
&[(t_2^0\alpha^1)^{-1}\tilde{f}^k_{(1,v,-1)}]\\
&=\{(t_2^{2m}\alpha^{1+4n})^{-1}\tilde{f}^k_{(1,v,-1)} \mid m,n\in\mathbb{N}\}\cup\{(t_2^{kv+2m}\alpha^{3+4n})^{-1}\tilde{f}^k_{(1,v,-1)} \mid m,n\in\mathbb{N}\},\\
&[(t_2^1\alpha^1)^{-1}\tilde{f}^k_{(1,v,-1)}]\\
&=\{(t_2^{2m+1}\alpha^{1+4n})^{-1}\tilde{f}^k_{(1,v,-1)} \mid m,n\in\mathbb{N}\}\\
&\quad\cup\{(t_2^{kv+2m-1}\alpha^{3+4n})^{-1}\tilde{f}^k_{(1,v,-1)} \mid m,n\in\mathbb{N}\},\\
&[(t_2^i\alpha^0)^{-1}\tilde{f}^k_{(1,v,-1)}]\\
&=\{(t_2^i\alpha^{4n})^{-1}\tilde{f}^k_{(1,v,-1)} \mid n\in\mathbb{N}\}\cup\{(t_2^{kv-i}\alpha^{4n+2})^{-1}\tilde{f}^k_{(1,v,-1)} \mid n\in\mathbb{N}\}.
\end{aligned}$$

For any fixed point class $[(t_2^i\alpha^0)^{-1}\tilde{f}^k_{(1,v,-1)}]$, since $(w^k-1)((-1)^j u-1) = (-1-1)((-1)^0-1)=0$, by Lemma 4.1, these infinitely many classes are all inessential, and hence $R(f^k_{(1,v,-1)}) = +\infty$. By Lemma 4.1 again, either the class $[(t_2^0\alpha^1)^{-1}\tilde{f}^k_{(1,v,-1)}]$ or $[(t_2^1\alpha^1)^{-1}\tilde{f}^k_{(1,v,-1)}]$ is essential. Hence, $N(f^k_{(1,v,-1)}) = 2$.

5. Minimal set of periods

In this section, we shall determine the minimal set of periods for each map class on the Klein bottle.

Proposition 5.1. *For all positive integers k and r, and any self map f on the Klein bottle, each inessential fixed point class of f^k is contained in an inessential fixed point class of f^{rk}, i.e., any self map on the Klein bottle satisfies the index assumption.*

Proof. By homotopy invariance, it suffices to consider the maps of the form $f_{(u,v,w)}$. From Theorem 4.4, for any k, $f^k_{(u,v,w)}$ has no inessential fixed point class if w is even. This proposition holds automatically for even w.

Let w be odd. Note that for any integers k and r,

$$\begin{aligned}((t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)})^r &= \begin{pmatrix} w^k & 0 & -\frac{j}{2} \\ 0 & (-1)^j u^k & -(-1)^j i + (-1)^j \frac{v}{2}\sum_{s=0}^{k-1} u^s \\ 0 & 0 & 1 \end{pmatrix}^r \\ &= \begin{pmatrix} w^{rk} & 0 & -\frac{j}{2}(\sum_{s=0}^{r-1} w^{sk}) \\ 0 & (-1)^{rj}u^{rk} & * \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

Thus, by equality (4.1), the fixed point class $F^{(k)}$ of $f^k_{(u,v,w)}$ determined by $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}$ is contained in the fixed point class $F^{(rk)}$ of $(t_2^i\alpha^j)^{-1}f^{rk}_{(u,v,w)}$ determined by $(t_2^{i'}\alpha^{j(\sum_{s=0}^{r-1} u^{sk})})^{-1}\tilde{f}^{rk}_{(u,v,w)}$. Since w is odd, we have $j(\sum_{s=0}^{r-1} w^{sk}) \equiv rj \mod 2$. If the fixed point class of $f^k_{(u,v,w)}$ determined by $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,w)}$ is inessential, by Lemma 4.1, we have that $(w^k-1)((-1)^j u^k - 1) = 0$. So,

$$\begin{aligned}&(w^{rk}-1)((-1)^{j(\sum_{s=0}^{r-1} w^{sk})}u^{rk}-1) \\ &= (w^{rk}-1)((-1)^{rj}u^{rk}-1) \\ &= (w^k-1)((-1)^j u^k - 1)\sum_{s=0}^{r-1} w^{sk}\sum_{s=0}^{r-1}(-1)^{sj}u^{sk} = 0.\end{aligned}$$

By Lemma 4.1 again, the fixed point class $F^{(rk)}$ of $f^{rk}_{(u,v,w)}$, which is the fixed point class containing the fixed point class $F^{(k)}$ of $f^k_{(u,v,w)}$, is also inessential. □

Proposition 5.2. *If $|w| \geq 3$, then* $\mathrm{MPer}(f_{(u,v,w)}) = \mathbb{N}$.

Proof. By Proposition 5.1 and Lemma 2.4, it sufficient to show that

$$\frac{N(f^{n+1}_{(u,v,w)})}{N(f^{n}_{(u,v,w)})} \geq 2$$

for all n. In fact,

$$\frac{N(f^{n+1}_{(u,v,w)})}{N(f^{n}_{(u,v,w)})} = \frac{\frac{1}{2}(|u^{n+1}+1|+|u^{n+1}-1|)|w^{n+1}-1|}{\frac{1}{2}(|u^{n}+1|+|u^{n}-1|)|w^{n}-1|} \geq \frac{|w^{n+1}-1|}{|w^{n}-1|}.$$

If $w \leq -3$ and n is even, then

$$\frac{N(f^{n+1}_{(u,v,w)})}{N(f^{n}_{(u,v,w)})} \geq \frac{|w^{n+1}-1|}{|w^{n}-1|} = \frac{|w|^{n+1}+1}{|w|^{n}-1} > \frac{|w|^{n+1}}{|w|^{n}} = |w| > 2.$$

If $w \leq -3$ and n is odd, then

$$\frac{N(f^{n+1}_{(u,v,w)})}{N(f^{n}_{(u,v,w)})} \geq \frac{|w^{n+1}-1|}{|w^{n}-1|} = \frac{|w|^{n+1}-1}{|w|^{n}+1} = |w| - \frac{|w|+1}{|w|^{n}+1} \geq |w|-1 \geq 2.$$

If $w \geq 3$, then

$$\begin{aligned}\frac{N(f^{n+1}_{(u,v,w)})}{N(f^{n}_{(u,v,w)})} &= \frac{\frac{1}{2}(|u^{n+1}+1|+|u^{n+1}-1|)|w^{n+1}-1|}{\frac{1}{2}(|u^{n}+1|+|u^{n}-1|)|w^{n}-1|} \\ &\geq \frac{w^{n+1}-1}{w^{n}-1} = w + \frac{w-1}{w^{n}-1} > 2.\end{aligned}$$

□

Proposition 5.3. *The map $f_{(u,v,1)}$ is homotopic to a map g without any periodic point, and therefore* $\mathrm{MPer}(f_{(u,v,1)}) = \emptyset$.

Proof. By Theorem 3.2, $f_{(u,v,1)}$ is homotopic to a map g which is covered by the map $\tilde{g} : R^2 \times \{1\} \to R^2 \times \{1\}$ defined by

$$\tilde{g}(\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & u & \frac{v}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Clearly, g has no periodic point of any period. So, we have $\mathrm{MPer}(f_{(u,v,1)}) = \emptyset$. □

Corollary 5.4. *A map on K has the empty minimal set of period if it induces the identity on rational 1-dimensional homology.*

Theorem 5.5. *The minimal sets of periods for any self maps, $f_{(u,v,w)}$'s, on the Klein bottle are tabulated as follows*

degenerate	$u=0$	$w\geq 2,\ w\ \leq -3$	$\mathbb{N}$
		$w=1$	$\emptyset$
		$w=0,-1$	$\{1\}$
		$w=-2$	$\mathbb{N}-\{2\}$
homeomorphisms		$w=1,\ u=1$	$\emptyset$
		$w=-1,\ u=1$	$\{1\}$
half expanding	$w=\pm 1,\ u\geq 2$	$w=1$	$\emptyset$
		$w=-1$	$\mathbb{N}-2\mathbb{N}$
	w *odd*, $\lvert w\rvert\geq 3,\ u=1$		$\mathbb{N}$
expanding	w *odd*, $\lvert w\rvert\geq 3,\ u\geq 2$		$\mathbb{N}$

Proof. (1) Degenerate maps. By Proposition 3.5, there is a circle C in K such that $f_{(u,v,w))}(K)\in C$. We have that $\mathrm{MPer}(f_{(u,v,w)})=\mathrm{MPer}(f_{(u,v,w)}|_C)$. It is well-known that the minimal set of periods for a map ϕ on the circles is totally determined by its degree $d(\phi)$, i.e.,

$$\mathrm{MPer}(\phi)=\begin{cases}\mathbb{N} & \text{if } d(\phi)\geq 2 \text{ or } d(\phi)\leq -3,\\ \emptyset & \text{if } d(\phi)=1,\\ \{1\} & \text{if } d(\phi)=0,-1,\\ \mathbb{N}-\{2\} & \text{if } d(\phi)=-2.\end{cases}$$

Note that $f_{(u,v,w)}|_C$ has degree w. We finish our proof in this case.

(2) homeomorphisms. Since the minimal sets of periods for $f_{(1,0,1)}$ and $f_{(1,1,1)}$ have been given by Proposition 5.3, it sufficient to consider $f_{(1,0,-1)}$ and $f_{(1,1,-1)}$.

Note that $f^2_{(1,v,-1)}$ is the identity. Any periodic point of $f_{(1,v,-1)}$ has either period 1 or period 2. It follows that $\mathrm{MPer}(f_{(1,v,-1)})\subset\{1,2\}$. Since $N(f_{(1,v,-1)})=2$, we have that $1\in\mathrm{MPer}(f_{(1,v,-1)})$.

In order to prove $2\notin\mathrm{MPer}(f_{(1,v,-1)})$, we shall construct a map homotopic to $f_{(1,v,-1)}$ without period points with least period 2.

By Theorem 3.2, $f_{(1,v,-1)}$ is homotopic to a map g_ϵ which is covered by the map $\tilde g_\epsilon : R^2\times\{1\}\to R^2\times\{1\}$ defined by

$$\tilde g_\epsilon(\begin{pmatrix}x\\y\\1\end{pmatrix})=\begin{pmatrix}-x+\epsilon\sin 4\pi x\\ y+\lambda_\epsilon(x,y)+\dfrac{v}{2}\\ 1\end{pmatrix},$$

where

$$\lambda_\epsilon(x,y)=\epsilon\cos 2\pi x+\epsilon\sin(4\pi y+\pi v)|\sin 2\pi x|,$$

in which ϵ is a small positive number. All lifting of g_ϵ will have the form $(t_2^i\alpha^j)^{-1}\tilde g_\epsilon$. The fixed point set of $(t_2^i\alpha^j)^{-1}\tilde g_\epsilon$ is the solution of the equation:

$$\tilde g_\epsilon(\begin{pmatrix}x\\y\\1\end{pmatrix})=(t_2^i\alpha^j)\begin{pmatrix}x\\y\\1\end{pmatrix}$$

i.e.,

$$\begin{pmatrix} -x+\epsilon\sin 4\pi x \\ y+\lambda_\epsilon(x,y)+\dfrac{v}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} x+\dfrac{j}{2} \\ (-1)^j y+i \\ 1 \end{pmatrix}.$$

Thus, we have

$$\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{g}_\epsilon) = \begin{cases} \begin{pmatrix} -\dfrac{j}{4} \\ \dfrac{2i-v}{4} \\ 1 \end{pmatrix} & \text{if } j \text{ is odd,} \\ \emptyset & \text{if } j \text{ is even.} \end{cases}$$

We obtain the fixed point set of g_ϵ:

$$\mathrm{Fix}(g_\epsilon) = \cup_{i,j}\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{g}_\epsilon) = \left\{ \left[\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2}-\frac{v}{4} \\ 1 \end{pmatrix}\right], \left[\begin{pmatrix} \frac{1}{4} \\ 1-\frac{v}{4} \\ 1 \end{pmatrix}\right] \right\}$$

Similarly, any lifting of g_ϵ^2 can be written as $(t_2^i\alpha^j)^{-1}\tilde{g}_\epsilon^2$. Its fixed points is just the solution of the equation:

$$\begin{pmatrix} -(-x+\epsilon\sin 4\pi x)+\epsilon\sin 4\pi(-x+\epsilon\sin 4\pi x) \\ y+\lambda_\epsilon(x,y)+\dfrac{v}{2}+\lambda_\epsilon(-x+\epsilon\sin 4\pi x,\ y+\lambda_\epsilon(x,y)+\dfrac{v}{2})+\dfrac{v}{2} \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} x+\dfrac{j}{2} \\ (-1)^j y+i \\ 1 \end{pmatrix}.$$

We also have $\mathrm{Fix}(g_\epsilon^2) = \{[\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2}-\frac{v}{4} \\ 1 \end{pmatrix}], [\begin{pmatrix} \frac{1}{4} \\ 1-\frac{v}{4} \\ 1 \end{pmatrix}]\}$. This implies that g_ϵ has no periodic point of least period 2. Thus, $2 \notin \mathrm{MPer}(g_\epsilon^2) = \mathrm{MPer}(f_{(1,v,-1)})$.

(3) Half expanding maps. The case that w is odd with $|w| \geq 3$ and $u = 1$ is included in Proposition 5.2, and the case $w = 1$ and $u \geq 2$ is included in Proposition 5.3. Thus, we consider the remaining case $w = -1$ and $u \geq 2$.

For any odd positive number k, $N(f_{(u,v,-1)}^k) = u^k$. If n is odd, each of its factors is odd. So,

$$\sum_{\frac{n}{k}:\text{prime}} N(f_{(u,v,-1)}^k) \leq \sum_{k=1}^{\frac{n-1}{2}} N(f_{(u,v,-1)}^{2k-1}) = \sum_{k=1}^{\frac{n-1}{2}} u^{2k-1} = \frac{u^n-u}{u^2-1} = N(f_{(u,v,-1)}^n).$$

We obtain that $\mathbb{N}-2\mathbb{N} \subset \mathrm{MPer}(f_{(u,v,-1)})$ for $u \geq 2$.

We need to show that $\mathrm{MPer}(f_{(u,v,-1)})$ does not contain any even number. For any even number k, we shall construct a map homotopic to $f_{(u,v,-1)}$ without periodic points with least period k. Note that the fixed point set of an arbitrary lifting $(t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,-1)}$ of $f_{(u,v,-1)}$ is given by:

$$\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{f}^k_{(u,v,-1)})$$

$$= \begin{cases} \begin{pmatrix} x \\ \dfrac{i}{u^k-1} - \dfrac{v}{2(u-1)} \\ 1 \end{pmatrix}_{-\infty<x<+\infty} & \text{if } j=0 \text{ and } k \text{ is even} \\ \emptyset & \text{if } j\neq 0 \text{ and } k \text{ is even} \\ \begin{pmatrix} -\dfrac{j}{4} \\ \dfrac{i}{u^k-(-1)^j} - \dfrac{v(u^k-1)}{2(u-1)(u^k-(-1)^j)} \\ 1 \end{pmatrix} & \text{if } k \text{ is odd.} \end{cases}$$

Clear, $f_{(u,v,-1)}$ is homotopic to a map g_ϵ which is covered by the map $\tilde{g}_\epsilon : R^2\times\{1\}\to R^2\times\{1\}$ defined by

$$\tilde{g}_\epsilon(\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}) = \begin{pmatrix} -x+\epsilon\sin 4\pi x \\ uy+\dfrac{v}{2} \\ 1 \end{pmatrix}.$$

The fixed point set of an arbitrary lifting $(t_2^i\alpha^j)^{-1}\tilde{g}_\epsilon^k$ of g_ϵ^k is the solution of the equation:

$$(t_2^i\alpha^j)^{-1}\tilde{g}_\epsilon^k(\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}) = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} (-1)^k x + (-1)^{k-1}\epsilon\sin 4\pi x + (-1)^{k-2}\epsilon\sin 4\pi(-x+\epsilon\sin(4\pi x)) + \cdots \\ u^k y + \dfrac{v(u^k-1)}{2(u-1)} \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x+\dfrac{j}{2} \\ (-1)^j y + i \\ 1 \end{pmatrix}.$$

Since ϵ is a small positive number, the first coordinate x of a fixed point of $(t_2^i\alpha^j)^{-1}\tilde{g}_\epsilon^k$ must have the form $\dfrac{m}{4}$ if k is even. This also implies that it is also true for k is odd, because any fixed point of $(t_2^i\alpha^j)^{-1}\tilde{g}_\epsilon^k$ is a fixed point of

$(t_2^{i'}\alpha^{j'})^{-1}\tilde{g}_\epsilon^{2k}$. Thus, we have

$$\begin{aligned}&\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{g}_\epsilon^k)\\ &= \begin{cases} \begin{pmatrix} \frac{m}{4} \\ \frac{i}{u^k-1} - \frac{v}{2(u-1)} \\ 1 \end{pmatrix} & \text{if } j = 0 \text{ and } k \text{ is even} \\ \emptyset & \text{if } j \neq 0 \text{ and } k \text{ is even} \\ \begin{pmatrix} -\frac{j}{4} \\ \frac{i}{u^k-(-1)^j} - \frac{v(u^k-1)}{2(u-1)(u^k-(-1)^j)} \\ 1 \end{pmatrix} & \text{if } k \text{ is odd.} \end{cases}\end{aligned}$$

Thus, we obtain a map g_ϵ having finitely many period points in any period.

Now fixed an even number $k = 2l$, we shall remove out period points with least period k. On the Klein bottle, the fixed point set of g_ϵ^k is:

$$\bigcup_{i=0}^{u^k-2} \left\{ \left[\begin{pmatrix} 0 \\ \frac{i}{u^k-1} - \frac{v}{2(u-1)} \\ 1 \end{pmatrix} \right], \left[\begin{pmatrix} \frac{1}{4} \\ \frac{i}{u^k-1} - \frac{v}{2(u-1)} \\ 1 \end{pmatrix} \right] \right\}.$$

From the structure of fixed point set of $\mathrm{Fix}((t_2^i\alpha^j)^{-1}\tilde{g}_\epsilon^n)$ for any n, we know that two period points in a pair

$$\left\{ \left[\begin{pmatrix} 0 \\ \frac{i}{u^k-1} - \frac{v}{2(u-1)} \\ 1 \end{pmatrix} \right], \left[\begin{pmatrix} \frac{1}{4} \\ \frac{i}{u^k-1} - \frac{v}{2(u-1)} \\ 1 \end{pmatrix} \right] \right\}$$

have the same least period. For a pair containing period points with least period $k = 2l$, indicated by i, let L_i denote the horizontal line containing the two points

$$L_i = \left\{ \left[\begin{pmatrix} x \\ \frac{i}{u^k-1} - \frac{v}{2(u-1)} \\ 1 \end{pmatrix} \right] \middle| -\infty < x < +\infty \right\}.$$

Since two fixed points in $L_{i,\delta}$ has least period $k = 2l$, the orbits $L_{i,\delta}$, $g_\epsilon(L_{i,\delta})$, $g_\epsilon^2(L_{i,\delta}), \ldots, g_\epsilon^k(L_{i,\delta})$ of $L_{i,\delta}$ under g_ϵ does not meet any other fixed points of g_ϵ^k. We change homotopically, in a small neighborhood of L_i, $\tilde{g}_\epsilon$ into $\tilde{g}_{\epsilon,\delta,i}$, which is defined by

$$\tilde{g}_{\epsilon,\delta,i}(\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}) = \begin{pmatrix} -x + \lambda_\delta(x,y)\varepsilon \sin 4\pi x + (1 - \lambda_\delta(x,y))\delta \\ uy + \frac{v}{2} \\ 1 \end{pmatrix},$$

where

$$\lambda_\delta(x,y) = \max\left\{1, \frac{d((x,y), L_i)}{\delta}\right\} = \max\left\{1, \frac{|y - \frac{i}{u^k - 1} - \frac{v}{2(u-1)}|}{\delta}\right\}.$$

As the positive number δ can be chosen small so that the δ-neighborhood of L_i, the support set of our new changing, does not meet any fixed point of $(t_2^i\alpha^j)^{-1}\tilde{g}_\epsilon^k$ except for the two points in it.

Let us consider the fixed point set of $(t_2^i\alpha^j)^{-1}\tilde{g}_{\epsilon,\delta,i}^k$. Suppose that $[x_0, y_0]$ is a point in it. Since the second coordinate remains unchanged, we have $y_0 = \frac{i}{u^k - 1} - \frac{v}{2(u-1)}$. For the first coordinate, x_0 should be a solution of the equation:

$$\begin{aligned} &(-1)^k x + (-1)^{k-1}\delta + (-1)^{k-2}\epsilon \sin 4\pi(-x + \delta)) + \cdots \\ &\quad + \epsilon \sin 4\pi((-1)^k x + (-1)^{k-1}\delta + (-1)^{k-2}\epsilon \sin 4\pi(\cdots)) = x + \frac{j}{2}. \end{aligned}$$

Note that k is even number. The left one is always a negative small real number as long as we pick $0 < \epsilon << \delta << 1$. It follows that this equation has empty solution. Thus, the periodic points of $(t_2^i\alpha^j)^{-1}\tilde{g}_{\epsilon,\delta,i}^k$ in L_i have been removed out. It is easy to see that we can make such a change for $\tilde{g}_\epsilon$ into $\tilde{g}_{\epsilon,\delta}$ symmetrically so that $\tilde{g}_{\epsilon,\delta}$ may cover a self map on Klein bottle. In fact, all the periodic points in their period orbits are also removed out meanwhile.

After changing all the periodic points with least period k pairwisely, we will prove that any even number is not contained in $\mathrm{MPer}(f_{(u,v,-1)})$ for $u \geq 2$.

(4) Expanding maps. This case is totally included in Proposition 5.2. □

Finally, we restate our main theorem as follows.

Theorem 5.6. *For any* $(u,v,w) \in \{(0,v,w) \in \mathbb{Z}^3 \mid v \geq 0, w \text{ is even}\} \cup \{(u,v,w) \in \mathbb{Z}^3 \mid u \geq 0, v = 0, 1, w \text{ is odd}\}$, *we have*

$$\mathrm{MPer}(f_{(u,v,w)}) = \begin{cases} \mathbb{N} & \text{if } w \geq 2, \\ \emptyset & \text{if } w = 1, \\ \{1\} & \text{if } w = 0, \\ \mathbb{N} - 2\mathbb{N} & \text{if } w = -1 \text{ and } u \geq 2, \\ \{1\} & \text{if } w = -1 \text{ and } u = 0, 1, \\ \mathbb{N} - \{2\} & \text{if } w = -2, \\ \mathbb{N} & \text{if } w \leq -3. \end{cases}$$

This theorem covers the result [9, p.88, Theorem]. It should be noticed that there is a switching in the letters v and w. Here, we follow the notation of

Halpern in [2]. The minimal set of period of a map on the Klein bottle is almost determined by w, its action on the generator of 1-dimensional rational homology.

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