J. Korean Math. Soc. **45** (2008), No. 3, pp. 795–805

WEAK LAW OF LARGE NUMBERS FOR ADAPTED DOUBLE ARRAYS OF RANDOM VARIABLES

Nguyen Van Quang and Nguyen Ngoc Huy

Abstract. The aim of this paper is to extend the "classical degenerate convergence criterion" and the Feller weak law of large numbers to double adapted arrays of random variables.

1. Introduction

The celebrated Feller weak law of large numbers (WLLN) say that if X_1, X_2, ... is a sequence of independent and identically distributed (i.i.d.) random variables satisfying $nP(|X_1| > n) = o(1)$, then $\sum_{i=1}^{n} \left(X_i - \mathbf{E}X_1 I(|X_1| \leqslant n)\right)/n \to 0$ in probability as $n \to \infty$.

The basis for proving weak laws is the "classical degenerate convergence criterion":

Theorem 1.1 ([10], p. 290). *Let $X_1, X_2, \ldots$ be independent random variables with partial sums $\{S_n, n \geqslant 1\}$, and let $\{b_n, n \geqslant 1\}$ a sequence of reals, $b_n \uparrow \infty$ as $n \to \infty$. Then, writing $X_{ni} = X_i I\{|X_i| \leqslant b_n\}, 1 \leqslant i \leqslant n$, we have that*

$$b_n^{-1} S_n \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty \tag{1.1}$$

if and only if

$$\text{i)} \quad \sum_{i=1}^{n} P(|X_i| > b_n) \to 0 \tag{1.2}$$

$$\text{ii)} \quad b_n^{-1} \sum_{i=1}^{n} \mathbf{E}X_{ni} \xrightarrow{P} 0 \tag{1.3}$$

$$\text{iii)} \quad b_n^{-2} \sum_{i=1}^{n} Var X_{ni} \to 0. \tag{1.4}$$

This theorem was extended in [9].

Received October 13, 2006; Revised April 23, 2007.

2000 *Mathematics Subject Classification.* 60F05, 60G50, 60G42.

Key words and phrases. double adapted array of random variables, weak law of large numbers, convergence in probability, martingale difference, sum of i.i.d. random variables.

This work was supported in part by the National Science Council of Vietnam No 100706.

Theorem 1.2 ([9], pp. 29–30). *Let $\{S_n = \sum_{i=1}^n X_i, \mathcal{F}_n, n \geq 1\}$ be a martingale and let $\{b_n, n \geqslant 1\}$ be a sequence of positive constants with $b_n \uparrow \infty$ as $n \to \infty$. Then, writing $X_{ni} = X_i I\{|X_i| \leqslant b_n\}, 1 \leqslant i \leqslant n$, we have that*

$$b_n^{-1} S_n \xrightarrow{P} 0 \quad as \quad n \to \infty \tag{1.5}$$

if

$$\text{i)} \quad \sum_{i=1}^{n} P(|X_i| > b_n) \to 0 \tag{1.6}$$

$$\text{ii)} \quad b_n^{-1} \sum_{i=1}^{n} \mathbf{E}(X_{ni}|\mathcal{F}_{i-1}) \xrightarrow{P} 0 \tag{1.7}$$

$$\text{iii)} \quad b_n^{-2} \sum_{i=1}^{n} \{\mathbf{E}X_{ni}^2 - \mathbf{E}[\mathbf{E}(X_{ni}|\mathcal{F}_{i-1})]^2\} \to 0. \tag{1.8}$$

Note here that in the general case, when X_i are not independent, then the reverse is not true. (see [9] pp. 29–30).

The WLLN has been extended to the arrays of random variables or random elements (for random variables, see Hong and Lee [5], Hong and Oh [6], Sung [11] and Sung et al. [12], and for random elements, see Adler et al. [1], Ahmed et al. [2], Hong et al. [7] and Sung et al. [13]).

The aim of this paper is to extend the "classical degenerate convergence criterion" and the Feller weak law of large numbers to double adapted arrays of random variables.

2. Preliminaries

In this section, notation, technical definitions and lemmas needed in connection with the main results will be presented. Some of the lemmas may be of independent interest.

For $a, b \in \mathbb{R}$, $\max\{a, b\}$ will be denoted by $a \vee b$. Throughout this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

Let $\mathbb{N}$ denote the set of all positive integers. As in [8], we note $\prec$ the lexicographic order on $\mathbb{N} \times \mathbb{N}$, i.e., $(i, j) \prec (k, l)$ if and only if either $i < k$ or $i = k$ and $j < l$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Then, a double array $\{\mathcal{F}_{mn}, m \geqslant 1, n \geqslant 1\}$ of sub-σ-algebras of $\mathcal{F}$ with indices in $\mathbb{N} \times \mathbb{N}$ will be called a *stochastic basis* if it is increasing, i.e., $\mathcal{F}_{ij} \subset \mathcal{F}_{kl}$ for $(i, j) \prec (k, l)$. If $\{\mathcal{F}_{mn}, m \geqslant 1, n \geqslant 1\}$ is a stochastic basis and X_{mn} is an $\mathcal{F}_{mn}$-measurable random variable for each $(m, n) \in \mathbb{N} \times \mathbb{N}$, then $\{X_{mn}, \mathcal{F}_{mn}, m \geqslant 1, n \geqslant 1\}$ is called an *adapted double array*.

An adapted double array $\{X_{mn}, \mathcal{F}_{mn}, m \geqslant 1, n \geqslant 1\}$ is called *rowwise martingale difference* if it is martingale difference in each row, i.e., for each $m \in \mathbb{N}$

$$\mathbf{E}(X_{m,n+1}|\mathcal{F}_{mn}) = 0 \text{ almost surely (a.s.)}, \forall n \in \mathbb{N}.$$

Remark 1. It is easy to show that if $\{X_{mn}, \mathcal{F}_{mn}\}$ is a rowwise martingale difference, then for all $(i,j) \prec (r,s)$, we have $\mathbf{E}(X_{rs}|\mathcal{F}_{ij}) = 0$ a.s.

Random variables $\{X_{mn}, m \geqslant 1, n \geqslant 1\}$ are said to be *stochastically dominated* by a random variable X if for some constant $C < \infty$

$$\mathbf{P}\{|X_{mn}| > t\} \leqslant C\mathbf{P}\{|X| > t\}, \quad t \geqslant 0, m \geqslant 1, n \geqslant 1.$$

An array of positive numbers (b_{mn}) will be called *increasing to* $+\infty$ if $b_{ij} < b_{rs}$ if and only if $(i,j) \prec (r,s)$ and $b_{mn} \uparrow \infty$ as $m \vee n \to \infty$.

Lemma 2.1. *Let $\{X_{mn}, \mathcal{F}_{mn}, m \geqslant 1, n \geqslant 1\}$ be an adapted double array. Then $\{Y_{mn} = X_{mn} - \mathbf{E}(X_{mn}|\mathcal{F}_{m,n-1}), \mathcal{F}_{mn}\}$ is a rowwise martingale difference.*

Proof. Indeed, since $\mathbf{E}(X_{m,n+1}|\mathcal{F}_{mn})$ is an $\mathcal{F}_{mn}$-measurable, we have

$$\begin{aligned}\mathbf{E}(Y_{m,n+1}|\mathcal{F}_{mn}) &= \mathbf{E}(X_{m,n+1} - \mathbf{E}(X_{m,n+1}|\mathcal{F}_{mn})|\mathcal{F}_{mn}) \\ &= \mathbf{E}(X_{m,n+1}|\mathcal{F}_{mn}) - \mathbf{E}(X_{m,n+1}|\mathcal{F}_{mn}) = 0.\end{aligned}$$

□

Lemma 2.2. *Let $\{X_{mn}, \mathcal{F}_{mn}\}$ be a rowwise martingale difference and $\mathbf{E}X_{mn}^2 < \infty$ for all $(m,n) \in \mathbb{N} \times \mathbb{N}$. Then*

$$\mathbf{E}(\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij})^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbf{E}X_{ij}^2.$$

Proof. For all $(i,j), (k,l) \in \mathbb{N} \times \mathbb{N}, (i,j) \neq (k,l)$, we can assume that $(i,j) \prec (k,l)$. Then

$$\mathcal{F}_{ij} \subset \mathcal{F}_{kl}$$

and

$$\begin{aligned}\mathbf{E}(X_{ij}X_{kl}) &= \mathbf{E}\{\mathbf{E}(X_{ij}X_{kl}|\mathcal{F}_{ij})\} \\ &= \mathbf{E}\{X_{ij}\mathbf{E}(X_{kl}|\mathcal{F}_{ij})\} \\ &= \mathbf{E}(X_{ij} \cdot 0) = 0.\end{aligned}$$

From that point, we yield the conclusion. □

Lemma 2.3. *For all $k \in \mathbb{N}, k \geqslant 1$, the following inequalities hold.*

(1) $k^{2/\rho} \leqslant \frac{2}{\rho}\sum_{r=1}^{k} r^{\frac{2}{\rho}-1}$ *for* $\rho \in (0,2)$.

(2) $\sum_{r=r_0}^{k} r^{\frac{2}{\rho}-2} \leqslant \frac{\rho}{2-\rho}\{k^{\frac{2}{\rho}-1} - (r_0-1)^{\frac{2}{\rho}-1}\}$ *for all* $r_0 \in \mathbb{N}, \rho \in (1,2)$.

Proof. 1) For $\rho \in (0,2)$ then $\frac{2}{\rho} - 1 > 0$ and function $y = x^{\frac{2}{\rho}-1}$ is increasing on $(0,\infty)$. Hence

$$r^{\frac{2}{\rho}-1} \geqslant \int_{r-1}^{r} x^{\frac{2}{\rho}-1} dx \quad \text{for all } r = 1, 2, \ldots, k.$$

So

$$\frac{2}{\rho}\sum_{r=1}^{k} r^{\frac{2}{\rho}-1} \geqslant \frac{2}{\rho}\int_{0}^{k} x^{\frac{2}{\rho}-1} = k^{\frac{2}{\rho}}.$$

2) For $\rho \in (1,2)$ then $\frac{2}{\rho} - 2 < 0$. Hence, function $y = x^{\frac{2}{\rho}-2}$ is decreasing on $(0,\infty)$ and

$$r^{\frac{2}{\rho}-2} \leqslant \int_{r-1}^{r} x^{\frac{2}{\rho}-2} dx \quad \text{for all } r = r_0, r_0+1, \ldots, k; r_0 \in \mathbb{N}.$$

Eventually,

$$\sum_{r=r_0}^{k} r^{\frac{2}{\rho}-2} \leqslant \sum_{r=r_0}^{k}\int_{r-1}^{r} x^{\frac{2}{\rho}-2} dx = \frac{\rho}{2-\rho}\{k^{\frac{2}{\rho}-1} - (r_0-1)^{\frac{2}{\rho}-1}\}$$

for all $r_0 \in \mathbb{N}, \rho \in (1,2)$.

The proof is complete. □

3. Main results

With the notations and lemmas as above, the main results can now be established.

Theorem 3.1. *Let $\{X_{mn}, \mathcal{F}_{mn}, m \geqslant 1, n \geqslant 1\}$ be an adapted double array, (b_{mn}) be an array of positive numbers increasing to $+\infty$. Put $Y_{ij} = X_{ij}I\{|X_{ij}| \leqslant b_{mn}\}$. Then we have*

$$\frac{1}{b_{mn}}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij} \xrightarrow{\mathbf{P}} 0 \quad \text{as } m \vee n \to \infty, \tag{3.1}$$

if

$$\text{i)} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbf{P}\{|X_{ij}| > b_{mn}\} \to 0 \qquad \text{as } m \vee n \to \infty, \tag{3.2}$$

$$\text{ii)} \quad \frac{1}{b_{mn}}\sum_{i=1}^{m}\sum_{j=1}^{n} \mathbf{E}\{Y_{ij}|\mathcal{F}_{i,j-1}\} \xrightarrow{\mathbf{P}} 0 \qquad \text{as } m \vee n \to \infty, \tag{3.3}$$

(3.4) iii) $$\frac{1}{b_{mn}^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\{\mathbf{E}Y_{ij}^2-\mathbf{E}\big(\mathbf{E}(Y_{ij}|\mathcal{F}_{i,j-1})\big)^2\}\to 0 \qquad \textit{as } m\vee n\to\infty.$$

Proof. For $m\geqslant 1, n\geqslant 1$, we put

$$S_{mn}=\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij},$$

$$\widetilde{S}_{mn}=\sum_{i=1}^{m}\sum_{j=1}^{n}Y_{ij},$$

$$\mu_{mn}=\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{E}(Y_{ij}|\mathcal{F}_{i,j-1}).$$

On account of (i),

$$\begin{aligned}\mathbf{P}(S_{mn}/b_{mn}\neq\widetilde{S}_{mn}/b_{mn}) &= \mathbf{P}(S_{mn}\neq\widetilde{S}_{mn})\\ &\leqslant \mathbf{P}\{\bigcup_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} Y_{ij}\neq X_{ij}\}\\ &\leqslant \sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{P}\{Y_{ij}\neq X_{ij}\}\\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{P}\{|X_{ij}|>b_{mn}\}\end{aligned}$$

$\to 0$ as $m\vee n\to\infty$.

And so it suffices to prove that $\frac{1}{b_{mn}}\widetilde{S}_{mn}\xrightarrow{\mathbf{P}}0$. But on account of (ii),

$$\frac{1}{b_{mn}}\mu_{mn}\xrightarrow{\mathbf{P}}0 \text{ as } m\vee n\to\infty,$$

so that it suffices to prove that

$$\frac{1}{b_{mn}}(\widetilde{S}_{mn}-\mu_{mn})\xrightarrow{\mathbf{P}}0 \text{ as } m\vee n\to\infty.$$

For $\epsilon>0$, from Chebyshev's inequality together with Lemmas 2.1, 2.2, and (iii), we have

$$\begin{aligned}\mathbf{P}\{|\frac{1}{b_{mn}}(\widetilde{S}_{mn}-\mu_{mn})|>\epsilon\} &= \mathbf{P}\{|\frac{1}{b_{mn}}\sum_{i=1}^{m}\sum_{j=1}^{n}(Y_{ij}-\mathbf{E}(Y_{ij}|\mathcal{F}_{i,j-1}))|>\epsilon\}\\ &\leqslant \frac{1}{\epsilon^2}\mathbf{E}\{\frac{1}{b_{mn}}\sum_{i=1}^{m}\sum_{j=1}^{n}(Y_{ij}-\mathbf{E}(Y_{ij}|\mathcal{F}_{i,j-1}))\}^2\\ &= \frac{1}{b_{mn}^2\epsilon^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{E}\{Y_{ij}-\mathbf{E}(Y_{ij}|\mathcal{F}_{i,j-1})\}^2\end{aligned}$$

$$= \frac{1}{b_{mn}^2 \epsilon^2} \sum_{i=1}^{m} \sum_{j=1}^{n} \{\mathbf{E} Y_{ij}^2 - \mathbf{E}\big(\mathbf{E}(Y_{ij}|\mathcal{F}_{i,j-1})\big)^2\}$$

$\to 0$ as $m \vee n \to \infty$.

The proof is completed. □

It is easy to show that if $X_{mn} \xrightarrow{\mathbf{P}} X$ as $m \vee n \to \infty$, then $X_{1n} \xrightarrow{\mathbf{P}} X$ as $n \to \infty$ and we get

Corollary 3.2. *Let $\{S_n = \sum_{i=1}^n X_i, \mathcal{F}_n, n \geq 1\}$ be an adapted sequence and let $\{b_n, n \geqslant 1\}$ be a sequence of positive constants with $b_n \uparrow \infty$ as $n \to \infty$. Then, writing $X_{ni} = X_i I\{|X_i| \leqslant b_n\}, 1 \leqslant i \leqslant n$, we have that*

$$b_n^{-1} S_n \xrightarrow{P} 0 \quad as \quad n \to \infty \tag{3.5}$$

if

$$\text{i)} \quad \sum_{i=1}^{n} P(|X_i| > b_n) \to 0 \tag{3.6}$$

$$\text{ii)} \quad b_n^{-1} \sum_{i=1}^{n} \mathbf{E}(X_{ni}|\mathcal{F}_{i-1}) \xrightarrow{P} 0 \tag{3.7}$$

$$\text{iii)} \quad b_n^{-2} \sum_{i=1}^{n} \{\mathbf{E} X_{ni}^2 - \mathbf{E}[\mathbf{E}(X_{ni}|\mathcal{F}_{i-1})]^2\} \to 0. \tag{3.8}$$

Thus, Theorem 1.2 is also true if the martingale condition of ($S_n = \sum_{i=1}^n X_i,$; $\mathcal{F}_n$) is replaced by the weaker condition: ($S_n = \sum_{i=1}^n X_i,$; $\mathcal{F}_n$) is an adapted sequence. The below example shows that the above corollary is really stronger then the theorem 1.2.

Let (Y_i) be a sequence of independent and identically distributed random variables such that

$$P(Y_i = -1) = P(Y_i = 1) = \frac{1}{2}.$$

Then $EY_i = 0$ ($\forall i = 1, 2, \ldots$). Applying the Feller weak law of large numbers to (Y_i), we have

$$\frac{1}{n} \sum_{i=1}^{n} Y_i \xrightarrow{P} 0 \text{ as } n \to \infty.$$

(In fact $\frac{1}{n} \sum_{i=1}^n Y_i \xrightarrow{a.s} 0$ as $n \to \infty$.)

Put

$$X_i = Y_i + \frac{1}{i}\,.$$

Then $EX_i = \frac{1}{i}$ ($\forall i = 1, 2, \ldots$) and

$$\frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{n} \sum_{i=1}^{n} Y_i + \frac{1}{n} \sum_{i=1}^{n} \frac{1}{i} \xrightarrow{P} 0 + 0 = 0 \text{ as } n \to \infty.$$

Thus, $(S_n = \sum_{i=1}^n X_i)$ satisfies the condition (1.1) and by Theorem 1.1, it also satisfies the conditions (1.2), (1.3), (1.4) (with $b_n = n$).

Now, let $\mathcal{F}_n$ be the σ-field generated by $(X_i;\ 1 \leqslant i \leqslant n)$. By the independence of (X_i) the conditions (1.2), (1.3), (1.4) can be replaced by the conditions (3.6), (3.7), (3.8), respectively. Then $(S_n = \sum_{i=1}^n X_i,\ ;\ \mathcal{F}_n)$ satisfies all assumptions of Corollary 3.2. On the other hand, $(S_n = \sum_{i=1}^n X_i,\ ;\ \mathcal{F}_n)$ is not a martingale. This shows that the martingale condition of $(S_n = \sum_{i=1}^n X_i,\ ;\ \mathcal{F}_n)$ in Theorem 1.2 is too strong.

Corollary 3.3. *Let array of random variables $\{X_{mn}, m \geqslant 1, n \geqslant 1\}$ be independent, and let $\{b_{mn}, m \geqslant 1, n \geqslant 1\}$ be an array of positive numbers increasing to $+\infty$. Put $Y_{ij} = X_{ij} I\{|X_{ij}| \leqslant b_{mn}\}$. Then we have*

$$\frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} \xrightarrow{\mathbf{P}} 0 \quad as \quad m \vee n \to \infty \tag{3.9}$$

if

$$\text{i)} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{P}\{|X_{ij}| > b_{mn}\} \to 0 \qquad as \quad m \vee n \to \infty, \tag{3.10}$$

$$\text{ii)} \quad \frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{E} Y_{ij} \to 0 \qquad as \quad m \vee n \to \infty, \tag{3.11}$$

$$\text{iii)} \quad \frac{1}{b_{mn}^2} \sum_{i=1}^{m} \sum_{j=1}^{n} \operatorname{Var} Y_{ij} \to 0 \qquad as \quad m \vee n \to \infty. \tag{3.12}$$

Proof. For each $(m, n) \in \mathbb{N} \times \mathbb{N}$, let $\mathcal{F}_{mn}$ be the σ-algebra generated by all the elements X_{ij}, where $(i, j) \prec (m, n)$ or $(i, j) = (m, n)$. Then, array $\{\mathcal{F}_{mn}, m \geqslant 1, n \geqslant 1\}$ is a stochastic basis and X_{mn} is an $\mathcal{F}_{mn}$-measurable random variable for each $(m, n) \in \mathbb{N} \times \mathbb{N}$.

From the hypothesis, we have that array $\{X_{mn}, m \geqslant 1, n \geqslant 1\}$ is independent. For this reason, these conditions (3.2), (3.3), and (3.4) in Theorem 3.1, correspondingly, change to (3.10), (3.11) and (3.12) in Corollary 3.3. Hence, the proof is clear. □

We shall now prove the following extension of the well-known Feller theorem for adapted double arrays. This theorem also is extended by Gut in the case of sequences (see [4]).

Theorem 3.4. *Let $\{X_{mn}, \mathcal{F}_{mn}, m \geqslant 1, n \geqslant 1\}$ be an adapted double array. Suppose that $\{X_{mn}, m \geqslant 1, n \geqslant 1\}$ is stochastically dominated by a random variable X. Let real number $\rho \in (0, 2)$. Put $Y_{ij} = X_{ij} I\{|X_{ij}| \leqslant m^{\frac{1}{\rho}} n^{\frac{1}{\rho}}\}$.*

If

$$\lim_{r \to \infty} r\mathbf{P}\{|X| > r^{\frac{1}{\rho}}\} = 0,$$

then the WLLN

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\mathbb{E}\{Y_{ij}|\mathcal{F}_{i,j-1}\})}{m^{\frac{1}{\rho}}n^{\frac{1}{\rho}}} \xrightarrow{\mathbf{P}} 0 \quad as \quad m\vee n\to\infty$$

obtains.

Proof. We verify the conditions (3.2) and (3.4) in turn, where $b_{mn}=m^{\frac{1}{\rho}}n^{\frac{1}{\rho}}$.

We first verify the condition (3.2). By the assumption $\{X_{mn}, m\geqslant 1, n\geqslant 1\}$ is stochastically dominated by a random variable X, we have

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{P}(|X_{ij}|>m^{\frac{1}{\rho}}n^{\frac{1}{\rho}}) &\leqslant C\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{P}(|X|>m^{\frac{1}{\rho}}n^{\frac{1}{\rho}})\\ &= Cmn\mathbf{P}(|X|>m^{\frac{1}{\rho}}n^{\frac{1}{\rho}})\\ &\to 0 \text{ as } m\vee n\to\infty. \end{aligned}\tag{3.13}$$

Next, we verify the condition (3.4). We have

$$\begin{aligned}0 &\leqslant m^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}\sum_{i=1}^{m}\sum_{j=1}^{n}\{\mathbf{E}Y_{ij}^2-\mathbf{E}\big(\mathbf{E}(Y_{ij}|\mathcal{F}_{i,j-1})\big)^2\}\\
&\leqslant m^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{E}Y_{ij}^2\\
&= m^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{E}\{X_{ij}^2I(|X_{ij}|\leqslant m^{1/\rho}n^{1/\rho})\}\\
&= m^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{mn}\mathbf{E}\left\{X_{ij}^2I\{(k-1)^{1/\rho}<|X_{ij}|\leqslant k^{1/\rho}\}\right\}\\
&\leqslant m^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{mn}k^{2/\rho}\mathbf{P}\{(k-1)^{1/\rho}<|X_{ij}|\leqslant k^{1/\rho}\}.\\
&\leqslant Cm^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{mn}(\sum_{r=1}^{k}r^{\frac{2}{\rho}-1})\mathbf{P}\{(k-1)^{1/\rho}<|X_{ij}|\leqslant k^{1/\rho}\}\\
&= Cm^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{mn}r^{\frac{2}{\rho}-1}\{\sum_{k=r}^{mn}\mathbf{P}\{(k-1)^{1/\rho}<|X_{ij}|\leqslant k^{1/\rho}\}\}\\
&= Cm^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{mn}r^{\frac{2}{\rho}-1}\mathbf{P}\{(r-1)^{1/\rho}<|X_{ij}|\leqslant (mn)^{1/\rho}\}\\
&\leqslant Cm^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{mn}r^{\frac{2}{\rho}-1}\mathbf{P}\{|X_{ij}|>(r-1)^{1/\rho}\}\end{aligned}$$

$$
\begin{aligned}
&\leqslant Cm^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{r=1}^{mn} r^{\frac{2}{\rho}-1}\mathbf{P}\{|X|>(r-1)^{1/\rho}\}\\
&= Cm^{\frac{-2}{\rho}}n^{\frac{-2}{\rho}}mn\sum_{r=1}^{mn} r^{\frac{2}{\rho}-2}r\mathbf{P}\{|X|>(r-1)^{1/\rho}\}\\
&= C(mn)^{\frac{-2}{\rho}+1}\sum_{r=1}^{mn} r^{\frac{2}{\rho}-2}\Big\{r\mathbf{P}\big(|X|>(r-1)^{1/\rho}\big)\Big\}.
\end{aligned}
$$

It remains to prove the last term

$$
(mn)^{\frac{-2}{\rho}+1}\sum_{r=1}^{mn} r^{\frac{2}{\rho}-2}\Big\{r\mathbf{P}\big(|X|>(r-1)^{1/\rho}\big)\Big\}
$$

converges to 0 as $m \vee n \to \infty$.

In the case of $0<\rho\leqslant 1$, we have $r^{\frac{2}{\rho}-2}\leqslant (mn)^{\frac{2}{\rho}-2}$ for all $r=1,2,\ldots,mn$. So, by the fact $\lim_{r\to\infty} r\mathbf{P}\big(|X|>(r-1)^{1/\rho}\big)=0$ and Stolz's theorem, we have

$$
\begin{aligned}
&(mn)^{\frac{-2}{\rho}+1}\sum_{r=1}^{mn} r^{\frac{2}{\rho}-2}\Big\{r\mathbf{P}\Big(\big(|X|>(r-1)^{1/\rho}\Big)\Big\}\\
&\leqslant \frac{\sum_{r=1}^{mn} r\mathbf{P}\big((|X|>(r-1)^{1/\rho}\big)}{mn} \to 0 \ \text{ as } \ m\vee n\to\infty.
\end{aligned}
$$

Consider now the case $1<\rho<2$.

By the fact $\lim_{r\to\infty} r\mathbf{P}\big(|X|>(r-1)^{1/\rho}\big)=0$ again, for any $\epsilon>0$, there exists $r_0\in\mathbb{N}$ such that

$$
r\mathbf{P}\big(|X|>(r-1)^{1/\rho}\big)<\epsilon \text{ for all } r\geqslant r_0.
$$

Then, we have

$$
\begin{aligned}
&(mn)^{\frac{-2}{\rho}+1}\sum_{r=1}^{mn} r^{\frac{2}{\rho}-2}\{r\mathbf{P}\big(|X|>(r-1)^{1/\rho}\big)\}\\
&= (mn)^{\frac{-2}{\rho}+1}\Big\{\sum_{r=1}^{r_0-1} r^{\frac{2}{\rho}-2}\{r\mathbf{P}\big(|X|>(r-1)^{1/\rho}\big)\}\\
&\qquad + \sum_{r=r_0}^{mn} r^{\frac{2}{\rho}-2}\{r\mathbf{P}\big(|X|>(r-1)^{1/\rho}\big)\}\Big\}\\
&\leqslant C\frac{1}{(mn)^{\frac{2-\rho}{\rho}}} + (mn)^{\frac{-2}{\rho}+1}\sum_{r=r_0}^{mn} r^{\frac{2}{\rho}-2}\epsilon\\
&= C\frac{1}{(mn)^{\frac{2-\rho}{\rho}}} + \epsilon(mn)^{\frac{-2}{\rho}+1}\sum_{r=r_0}^{mn} r^{\frac{2}{\rho}-2}.
\end{aligned}
$$

Note that

$$\lim_{m\vee n\to\infty}\frac{1}{(mn)^{\frac{2-\rho}{\rho}}}=0$$

and by Lemma 2.3

$$\epsilon(mn)^{\frac{-2}{\rho}+1}\sum_{r=r_0}^{mn} r^{\frac{2}{\rho}-2}\leqslant \frac{\rho}{2-\rho}\epsilon(mn)^{\frac{-2}{\rho}+1}(mn)^{\frac{2}{\rho}-1}=\epsilon' \text{ for all } mn>r_0.$$

Thus

$$(mn)^{\frac{-2}{\rho}+1}\sum_{r=1}^{mn} r^{\frac{2}{\rho}-2}\left\{r\mathbf{P}\left(|X|>(r-1)^{1/\rho}\right)\right\}\to 0 \quad \text{as} \quad m\vee n\to\infty. \tag{3.14}$$

Combining (3.13) and (3.14) we complete the proof. □

Corollary 3.5. *Let $\{X_n,\mathcal{F}_n,n\geqslant 1\}$ be an adapted sequence. Suppose that $\{X_n,n\geqslant 1\}$ is stochastically dominated by a random variable X. Let real number $\rho\in(0,2)$. Put $Y_i=X_iI\{|X_i|\leqslant n^{\frac{1}{\rho}}\}$.*

If

$$\lim_{r\to\infty} r\mathbf{P}\{|X|>r^{\frac{1}{\rho}}\}=0,$$

then the WLLN

$$\frac{\sum_{i=1}^n (X_i-\mathbf{E}\{Y_i|\mathcal{F}_{i-1}\})}{n^{\frac{1}{\rho}}}\xrightarrow{\mathbf{P}}0 \quad \text{as} \quad n\to\infty$$

obtains.

Corollary 3.6. *Suppose that $X,X_1,X_2,\ldots$ are identically distributed, independent random variables, the real number $\rho\in(0,2)$.*

If

$$\lim_{r\to\infty} r\mathbf{P}\{|X|>r^{\frac{1}{\rho}}\}=0, \tag{3.15}$$

then the WLLN

$$\frac{\sum_{i=1}^n X_i-\mathbf{E}\{XI(|X|\leqslant n^{\frac{1}{\rho}})\}}{n^{\frac{1}{\rho}}}\xrightarrow{\mathbf{P}}0 \quad \text{as} \quad n\to\infty \tag{3.16}$$

obtains.

In the special case, when $\rho=1$ we get the Feller's weak law of large numbers.

References

[1] A. Adler, A. Rosalsky, and A. I. Volodin, *A mean convergence theorem and weak law for arrays of random elements in martingale type p Banach spaces*, Statist. Probab. Lett. **32** (1997), no. 2, 167–174.

[2] S. E. Ahmed, S. H. Sung, and A. I. Volodin, *Mean convergence theorem for arrays of random elements in martingale type p Banach spaces*, Bull. Inst. Math. Acad. Sinica **30** (2002), no. 2, 89–95.

[3] Y. S. Chow and H. Teicher, *Probability Theory*, Independence, interchangeability, martingales. Springer-Verlag, New York-Heidelberg, 1978.

[4] A. Gut, *An extension of Feller's weak law of large numbers*, http://www.math.uu.se/research/pub/Gut8.pdf.

[5] D. H. Hong and S. Lee, *A general weak law of large numbers for arrays*, Bull. Inst. Math. Acad. Sinica **24** (1996), no. 3, 205–209.

[6] D. H. Hong and K. S. Oh, *On the weak law of large numbers for arrays*, Statist. Probab. Lett. **22** (1995), no. 1, 55–57.

[7] D. H. Hong, M. Ordonez Cabrera, S. H. Sung, and A. I. Volodin, *On the weak law for randomly indexed partial sums for arrays of random elements in martingale type p Banach spaces*, Statist. Probab. Lett. **46** (2000), no. 2, 177–185.

[8] D. H. Hong and A. I. Volodin, *Marcinkiewicz-type law of large numbers for double arrays*, J. Korean Math. Soc. **36** (1999), no. 6, 1133–1143.

[9] P. Hall and C. C. Heyde, *Martingale Limit Theory and Its Application*, Probability and Mathematical Statistics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[10] M. Loève, *Probability Theory. I*, Fourth edition. Graduate Texts in Mathematics, Vol. 45. Springer-Verlag, New York-Heidelberg, 1977.

[11] S. H. Sung, *Weak law of large numbers for arrays*, Statist. Probab. Lett. **38** (1998), no. 2, 101–105.

[12] S. H. Sung, T. C. Hu, and A. I. Volodin, *On the weak laws for arrays of random variables*, Statist. Probab. Lett. **72** (2005), no. 4, 291–298.

[13] ———, *On the weak laws with random indices for partial sums for arrays of random elements in martingale type p Banach spaces*, Bull. Korean Math. Soc. **43** (2006), no. 3, 543–549.

Nguyen Van Quang
Department of Mathematics
Vinh University
182 Le Duan, Vietnam
E-mail address: nvquang@hotmail.com

Nguyen Ngoc Huy
Department of Mathematics
Vinh University
182 Le Duan, Vietnam
E-mail address: hu_nz@yahoo.com.vn