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ON j-INVARIANTS OF WEIERSTRASS EQUATIONS

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Abstract. A simple proof of the fact that the j-invariants for Weierstrass equations are invariant under birational transformations which keep the forms of Weierstrass equations is given by finding a non-trivial explicit birational transformation which sends a normalized Weierstrass equation to the same equation.

1. Introduction

For Legendre forms

$$y^2 = x(x-1)(x-\lambda) \tag{1}$$

the j-invariants ([4], p.55, [2], p.119)

$$j(\lambda) = 256 \cdot \frac{\lambda^2 - \lambda + 1}{\lambda^2 \cdot (\lambda - 1)^2} \tag{2}$$

are invariant under birational transformations which keep Legendre forms. This is proved in Hartshorne's book ([3], p.317) using Galois theory and linear series theory. The point is in showing that the group of birational transformations which keep Legendre forms is generated by

$$\text{(i)} \quad \begin{cases} x' = 1 - x \\ y' = y \end{cases} \qquad \text{(ii)} \quad \begin{cases} x' = \dfrac{1}{x} \\ y' = \dfrac{y}{x^2}, \end{cases} \tag{3}$$

which preserve the set $\{0, 1, \infty\}$, and map λ respectively to

$$\text{(i)} \quad 1 - \lambda, \quad \text{(ii)} \quad \frac{1}{\lambda}. \tag{4}$$

The purpose of this paper is to prove the following theorem.

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Theorem. *For non-singular Weierstrass equations*

$$y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0, \tag{5}$$

the j-invariants are defined by

$$j(a_1, a_2, a_3, a_4, a_6) = \frac{4({b_2}^2 - 24\,b_4)^3}{36\,b_2b_4b_6 - (32\,{b_4}^3 + 108\,{b_6}^2) - {b_2}^2(b_2b_6 - {b_4}^2)}, \tag{6}$$

where

$$\begin{aligned} b_2 &= {a_1}^2 + 4\,a_2 \\ b_4 &= a_1a_3 + 2\,a_4 \\ b_6 &= {a_3}^2 + 4\,a_6. \end{aligned} \tag{7}$$

Two elliptic curves defined by equations like (5) *are isomorphic if and only if they have the same j-invariant.*

In fact, the invariance of the j-invariants under birational transformations which keep the form of Weierstrass equations is proved by finding an explicit non-trivial birational transformation which sends a normalized Weierstrass equation to the same equation.

We are working over the field of characteristic not equal to two or three.

Remark. Using Legendre forms it is proved that the j-invariant is independent of the choice of the base points of elliptic curves ([4], pp.108–109).

2. Weierstrass equations

Let T be a Riemann surface of genus one. By the Riemann-Roch theorem for a fixed point P_∞ there are meromorphic functions, x of order two with a pole of order two at P_∞ and y of order three with a pole of order three at P_∞. Then there is a relation between x and y:

$$f(x, y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6 = 0 \tag{8}$$

called the Weierstrass equation ([4], p.46), where the coefficient of y^2 is normalized as 1 and those of x^3 is normalized as -1. In general, the Weierstrass equations (8) may be singular, but we assume that the equations are non-singular, i.e., they define elliptic curves, since our surfaces are of genus one. There are many choices of functions x, y so that the relation is not unique, and there are many birationally equivalent Weierstrass equations.

We choose such a point $P_0 \neq P_\infty$ on T as a zero of the discriminant of (8) w.r.t. x. Subtracting the constant values $x(P_0), y(P_0)$ from the functions x, y respectively, we may assume that $a_4 = a_6 = 0$. The Weierstrass equation then takes the form

$$y^2 + a_1xy + a_3y - x^3 - a_2x^2 = 0 \tag{9}$$

and the coordinates of the point P_0 is $(x, y) = (0, 0)$. We call the forms (9) normalized Weierstrass equations.

3. Proof of Theorem

It is known ([4], pp.46 55, 63 65) or checked by direct computation that the isomorphisms

$$\text{(10)} \qquad \begin{aligned} x &= u^2X + r \\ y &= u^3Y + su^2X + t \end{aligned}$$

fixing the point $P_\infty = (\infty, \infty)$ keep the form of Weierstrass equations and make the j-invariant (6) invariant.

The Weierstrass equation (5) can be transformed to a normalized Weierstrass equation

$$\text{(11)} \qquad y^2 + a_1xy + a_3y - x^3 - a_2x^2 = 0 \qquad (a_3 \neq 0)$$

by some isomorphisms (10), and it is transformed to completely the same form

$$\text{(12)} \qquad Y^2 + a_1XY + a_3Y - X^3 - a_2X^2 = 0$$

by the birational transformation

$$\text{(13)} \qquad \left\{ \begin{aligned} X &= \frac{a_3\,(x + a_2)}{y} \\ Y &= \frac{a_3\,x(x + a_2)^2}{y^2} - \frac{a_1a_3\,(x + a_2)}{y} - a_3 \end{aligned} \right.$$

which transforms the point P_∞ to the finite point $P_0 = (0, 0)$. The inverse of (13) is

$$\text{(14)} \qquad \left\{ \begin{aligned} x &= \frac{a_3(Y + a_1X + a_3)}{X^2} \\ y &= \frac{a_3(a_3Y + a_1a_3X + a_3{}^2 + a_2X^2)}{X^3}. \end{aligned} \right.$$

One can easily check these facts by using Maple as follows :

> $f := y^2 + a[1] * x * y + a[3] * y - x^3 - a[2] * x^2;$
> $X = a[3] * (x + a[2])/y; \quad Y = a[3] * x * (x + a[2])^2/y^2 - a[1] * a[3] * (x + a[2])/y - a[3];$
> solve($\{X = a[3] * (x + a[2])/y, \quad Y = a[3] * x * (x + a[2])^2/y^2 - a[1] * a[3] * (x + a[2])/y - a[3]\}, \{x, y\}$);
> factor(subs(%, f));

Of course, the original equation (11) and the transformed equation (12) have the same j-invariant (6).

The transformations (10) and (13) (cf. [1], p.374) generate all birational transformations which keep the form of Weierstrass equations. Consequently we have the result.

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