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Long Waves Generated by Wave Groups over Trapezoidally Varying Topography
사다리형태로 변화하는 지형 위를 통과하는 파군에 의한 장파의 생성

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Abstract : A possible source of resonant problems in a harbor is long waves generated by incident wave groups. The analytical solutions of the governing equations of second-order long waves derived using a multiple-scale perturbation method consist of the locked and free long waves. The locked long waves propagate at some group velocity, whereas the free long waves propagate at the shallow-water speed. To study the resonance of free long waves, a trapezoidally varying topography is employed. With certain combinations of incident angle, water depth, and ambient current velocity, free long waves can be trapped and resonated.

Keywords : second-order long waves, wave group, locked long waves, free long waves, perturbation method

요 지 : 입사파의 파군에 의해 생성되는 장파는 항내에서 부진동의 문제를 발생시킬 가능성이 있다. 다변수 섭동법을 사용하여 구속장파와 자유장파로 구성된 2차 장파에 관한 지배 방정식의 해석 해를 유도하였다. 자유 장파의 경우 천해역에서의 파속도로 진행하는 반면 구속 장파의 경우 군속도로 진행한다. 장파의 의한 공명을 조사하기 위하여 사다리형태로 변화하는 지형을 고려하였다. 파의 입사각, 수심, 또는 유속에 의하여 자유 장파의 갇힘과 공명현상이 발생하였다.

핵심용어 : 2차 장파, 파군, 구속 장파, 자유 장파, 섭동법

1. Introduction

When a train of modulated wave groups propagates over a slowly varying topography and ambient current field, two types of second-order long waves can be generated due to refraction and shoaling: locked long waves propagate with the wave envelopes at some group velocity of carrier (short) waves, and free long waves propagate at the shallow-water wave speed, $\sqrt{gh}$ (Liu et al., 1992). Because the governing equations of long waves generated by wave groups are represented using the second-order terms given in Stokes' wave theory, they are called second-order long waves: They are of order $O((ka)^2)$, where k is the wavenumber and a is the amplitude. These locked and free long waves, although second-order quantities, may play significant roles in many coastal engineering problems, such as harbor resonance and coastal processes, if they are trapped and resonated in a nearshore area (Liu et al., 1992).

It has been reported that the second-order long waves may be related to generation of nearshore bar (Symonds and Bowen, 1984; Roelvink and Stive, 1989; Liu and Cho, 1993) and the long-period oscillations in a harbor (Mei and Agnon, 1989; Liu et al., 1990). The second-order long waves are also related to the mean free surface oscillation, the mean transport velocity, and the slowly varying drift force (Agnon and Mei, 1985; Zhou and Liu, 1987). Liu et al. (1992) investigated the propagation and trapping of obliquely incident wave groups over a trench with ambient currents.

In this study, the possibility of resonance from free long waves propagating over a trapezoidally varying topography is investigated. In the following section, the derivation of

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the governing equations of second-order long waves is summarized and the analytical solutions of locked and free long waves are introduced. In section 3, two numerical examples of the variation of amplitude of free long waves are given, and the possibility of the resonance of trapped free long waves is discussed. Finally, concluding remarks are made in section 4.

2. Governing Equations

It is assumed that fluid is inviscid and that flow is irrotational and incompressible. The fluid motions are hence governed by the Laplace equation:

$$\nabla^2 \Phi = 0, -h(x,y) < z < \zeta(x,y,t) \quad (1)$$

Here, ∇^2 is the Laplacian operator defined by $\nabla^2 = (\partial^2/\partial x^2, \partial^2/\partial y^2, \partial^2/\partial z^2)$, Φ is the velocity potential, h is the depth, and ζ is the free surface displacement (see Figure 1). Figure 1 is a schematic of the domain and coordinate system.

The bottom boundary condition is given by:

$$\frac{\partial\Phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial h}{\partial y} = \frac{\partial\Phi}{\partial z} = 0,\ z = -h(x,y) \quad (2)$$

The kinematic and dynamic free surface boundary conditions can be applied to the free surface of fluid. By combining these boundary conditions, the combined free surface boundary condition can written as:

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} + \frac{\partial}{\partial t}|\vec{u}|^2 + \frac{1}{2}\vec{u}\cdot\nabla|\vec{u}|^2 = 0,\ z = \zeta(x,y,t) \quad (3)$$

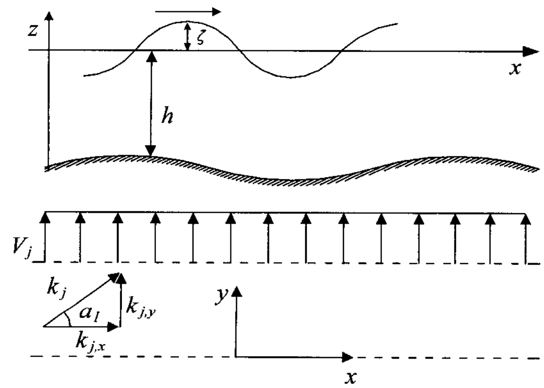

Fig. 1. A definitive sketch of the domain.

Here, g is the gravitational acceleration and $\vec{u}$ is the velocity vector, $\vec{u} = (\partial\Phi/\partial x, \partial\Phi/\partial y, \partial\Phi/\partial z)$.

The order of the free surface displacement is taken to be $O(ka) = O(\beta)$, where β is some small parameter. The combined free surface boundary condition given in Eq. (3) can then be expanded using the Taylor series (Mei, 1989):

$$\left(1+\zeta\frac{\partial}{\partial z}+\frac{1}{2}\zeta^2\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2\Phi}{\partial t^2}+g\frac{\partial\Phi}{\partial z}\right)$$
$$+\left(1+\zeta\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t}|\vec{u}|^2+\frac{1}{2}\vec{u}\cdot\nabla|\vec{u}|^2\right)=0 \quad (4)$$

The length and time scales of the second-order long waves are much longer than those of the short waves. Thus, the slow variables for the wave groups and long waves can be defined as:

$$x_1 = \beta x,\ y_1 = \beta y,\ t_1 = \beta t \quad (5)$$

in which x, y, and t vary quickly in space and time, whereas x_1, y_1, and t_1 vary slowly.

The velocity potential, Φ, and free surface displacement, ζ, are expanded in terms of the small parameter, β, by using the perturbation method. That is,

$$\Phi = \sum_{n=1}^{\infty}\beta^n\sum_{m=-n}^{n}\phi^{(n,m)}(x,y,z,x_1,y_1,t_1)\exp(-im\omega t) \quad (6)$$

$$\zeta = \sum_{n=1}^{\infty}\beta^n\sum_{m=-n}^{n}\zeta^{(n,m)}(x,y,z,x_1,y_1,t_1)\exp(-im\omega t) \quad (7)$$

The superscripts n and m are the number of orders and harmonics, respectively, in Eqs. (6) and (7). The governing equations and boundary conditions of each order (n = 1, 2, 3) can be derived by substituting the expansion (6) and (7) into Eqs. (1) and (4) (Cho et al., 1996). Detailed descriptions of the perturbation method can be found in Bender and Orszag (1987).

The mean free surface displacement, $\zeta^{(2,0)}$, is represented as (Liu et al., 1992):

$$\zeta^{(2,0)} = -\frac{1}{g}\left[\frac{\partial}{\partial t_1}\phi^{(1,0)} - \frac{\omega^2}{g}\left(\phi^{(1,-1)}\frac{\partial}{\partial z}\phi^{(1,1)} + *\right) + |\nabla\phi^{(1,1)}|^2\right] \quad (8)$$

in which * represents the complex conjugate.

The equation for the velocity potential is represented as:

$$\frac{\partial}{\partial x_1}\left(h\frac{\partial}{\partial x_1}\phi^{(1,0)}\right) + \frac{\partial}{\partial y_1}\left(h\frac{\partial}{\partial y_1}\phi^{(1,0)}\right) + \frac{\partial}{\partial t_1}\zeta^{(2,0)}$$

$$= -\frac{i\omega}{g}\left\{\frac{\partial}{\partial x_1}\left(\phi^{(1,1)}\frac{\partial}{\partial x}\phi^{(1,-1)}\right)+\frac{\partial}{\partial y_1}\left(\phi^{(1,1)}\frac{\partial}{\partial y}\phi^{(1,-1)}\right)+*\right\} \quad (9)$$

The governing equations of second-order long waves can be obtained by substituting $\partial_t\zeta^{(2,0)}$ into (9) and also by substituting the absolute frequency ω for $\sigma+\vec{k}\cdot\vec{V}$, where $\vec{V}(x_1, y_1, t) = (u, v)$ is the ambient current and the governing equations can be rewritten as (Cho et al., 1996) :

$$\left(\frac{\partial}{\partial t_1}+\vec{V}\cdot\nabla_1\right)\phi^{(1,0)}-g\nabla_1\cdot(h\nabla_1\phi^{(1,0)})=\left(\frac{\partial}{\partial t_1}+\vec{V}\cdot\nabla_1\right)$$
$$\left\{\left(\frac{\sigma^2}{g}\phi^{(1,1)}\frac{\partial}{\partial z}\phi^{(1,1)*}+*\right)-|\nabla\phi^{(1,1)}|^2\right\}$$
$$+\nabla_1\cdot(i\sigma\phi^{(1,1)}\Delta\phi^{(1,1)*}+*) \quad (10)$$

in which $\nabla_1 = (\partial/\partial x_1, \partial/\partial y_1)$ and the intrinsic angular frequency is represented as $\sigma = \omega-\vec{k}\cdot\vec{V}$.

By introducing the notation of $\phi^{(1,0)} = \Psi$, the dimensionless governing equations of second-order long waves are (Liu et al., 1992):

$$\left(\frac{\partial}{\partial t_1}+V\frac{\partial}{\partial y_1}\right)^2\psi-\left(\frac{\partial^2(\psi h)}{\partial x_1^2}+\frac{\partial^2(\psi h)}{\partial y_1^2}\right)=\frac{i}{4}k|A|^2(\Omega_0-K_yV)$$
$$\left(\frac{1}{\sinh 2kh}+\frac{1}{\sigma C_g}\right)\exp[2i(K_xx_1+K_yy_1-\Omega_0t_1)]+* \quad (11)$$

in which, Cg_j and K_j represent the velocity of wave groups and wave number of the wave envelope, respectively. The amplitude of the wave groups can be expressed as (Liu et al., 1992):

$$A = a_0\exp\{i(K_xx_1+K_yy_1-\Omega_0t_1)\}$$
$$+a_0b\exp\{-i(K_xx_1+K_yy_1-\Omega_0t_1)\} \quad (12)$$

where a_0 and a_0b represent the amplitudes of the short waves with slightly different frequencies.

If the long waves are unsteady, the velocity potential of long waves is:

$$\Psi=\frac{1}{2}\psi(x_1)\exp[2i(K_yy_1-\Omega_0t_1)]+* \quad (13)$$

Substituting Eq. (13) into Eq. (11) gives two classes of solutions. The first solutions obtained from the homogeneous equation are the free long waves, where as the second solutions obtained from the non-homogeneous equation are the locked long waves.

The reflected and transmitted velocity potentials of the locked long waves can be written as:

$$(\Psi^R)_L = \frac{ik(\Omega_0-K_yV)R^+R^{-*}}{8((\Omega_0-K_yV)^2-K^2h)}\left(\frac{1}{\sinh 2kh}+\frac{1}{\sigma Cg}\right)$$
$$\exp[-2i(K_xx_1)] \quad (14)$$

$$(\Psi^T)_L = \frac{ik(\Omega_0-K_yV)T^+T^{-*}}{8((\Omega_0-K_yV)-K^2h)}\left(\frac{1}{\sinh 2kh}+\frac{1}{\sigma Cg}\right)$$
$$\exp[2i(K_xx_1)] \quad (15)$$

where $R^\pm$ and $T^\pm$ are the reflection and transmission coefficients of short waves with slightly different frequencies, which can be determined by using the eigenfunction expansion method (Liu et al., 1992).

The reflected and transmitted velocity potentials of the free long waves can be written as:

$$(\Psi^T)_F = E_T\exp(2i\lambda x_1),\ (\Psi^R)_F = E_R\exp(-2i\lambda x_1) \quad (16)$$

$$\lambda = \left[\frac{(\Omega_0-K_yV)^2}{h}-K_y^2\right]^{\frac{1}{2}} \quad (17)$$

in which E_R and E_T are the reflected and transmitted amplitudes of the free long waves. When λ is real, $(\Psi)_F$ is a propagating wave. On the other hand, when λ is imaginary, $(\Psi)_F$ becomes an evanescent mode. When λ is zero, $(\Psi)_F$ is constant and free long waves are trapped.

The free surface displacements of the locked and free long waves can be written as:

$$(\xi^R)_L = \frac{1}{2}(\tilde{\xi}^R)_L\exp[2i(K_yy_1-\Omega_0t_1)]-\frac{(|R^+|+|R^-|^2)\sigma^2}{16\sinh^2 kh}+*$$

$$(\tilde{\xi}^R)_L = \frac{R^+R^{-*}}{4((\Omega_0-K_yV)^2-K^2h)}\left(\frac{K^2h\sigma^2}{2\sinh^2 kh}+\frac{(\Omega_0-K_yV)^2k}{\sigma Cg}\right)$$
$$\exp(-2iK_yx_1) \quad (18)$$

$$(\xi^T)_L = \frac{1}{2}(\tilde{\xi}^T)_L\exp[2i(K_yy_1-\Omega_0t_1)]-\frac{(|T^+|+|T^-|^2)\sigma^2}{16\sinh^2 kh}+*$$

$$(\tilde{\xi}^T)_L = \frac{T^+T^{-*}}{4((\Omega_0-K_yV)^2-K^2h)}\left(\frac{K^2h\sigma^2}{2\sinh^2 kh}+\frac{(\Omega_0-K_yV)^2k}{\sigma Cg}\right)$$
$$\exp(-2iK_yx_1) \quad (19)$$

$$(\tilde{\xi}^R)_F = 2i(\Omega_0-K_yV)(\psi^R)_F,\ (\tilde{\xi}^T)_F = 2i(\Omega_0-K_yV)(\psi^R)_F \quad (20)$$

The critical incident angle, α_{cr}, at which the free long waves can be generated, is represented as:

$$\sin\alpha_{cr}=\frac{Cg}{V+\sqrt{h}} \tag{21}$$

The velocity of currents, V, is defined by using the Froude number:

$$Fr=\frac{V}{\sqrt{h}} \tag{22}$$

The reflected and transmitted velocity potentials of the free long waves in j region can be written as:

$$(\Psi_j^T)_F=E_{T,j}\exp(2i\lambda_j x_1),\ (\Psi_j^R)_F=E_{R,j}\exp(2i\lambda_j x_1) \tag{23}$$

$$\lambda_j=\left[\frac{(\Omega_0-K_{j,y}V_j)^2}{h_j}-K_{j,y}^2\right]^{\frac{1}{2}} \tag{24}$$

To determine the values of the amplitudes of free long waves, two matching conditions are required. The first condition requires the continuity of the second-order free surface displacement. The second condition requires the continuity of the second-order flux (Liu et al., 1992).

3. Numerical examples

We discuss in this section numerical examples of free long waves over a trapezoidally varying topography (two rows, four rows). In Figure 2, the width, height, and distance of the trapezoid are assumed to be $B_1=0.5h_1$, $B_2=h_1$, $H=0.5h_1$, $W=h_1$. The symbols, $|E_{T1}|$, $|E_{T2}|$, $|E_{T3}|$, $|E_{T4}|$ denote the magnitude of the amplitude of transmitted

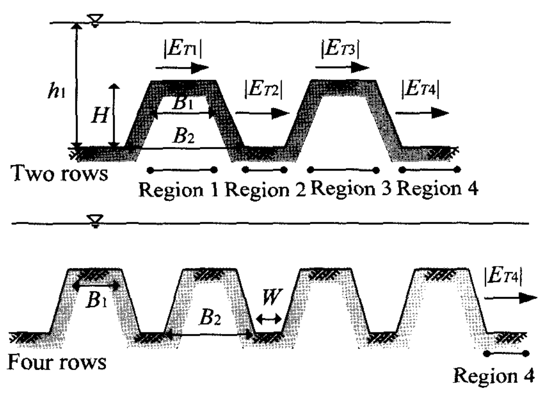

Fig. 2. A schematic sketch of a trapezoidally varying topography.

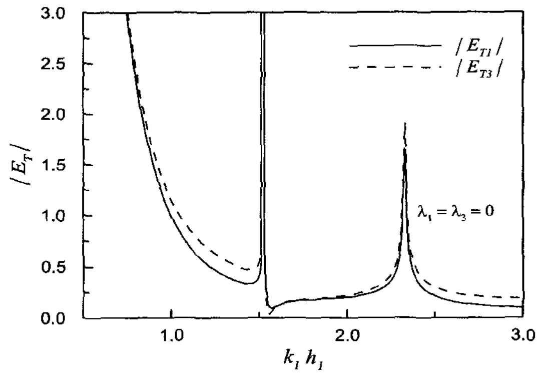

Fig. 3. The variation of the amplitudes of free long waves in regions 1 and 3 (two rows, $Fr=0$, $\alpha_I=30°$)

free long wave components in Eq. (16). In all calculations, $b=1$ is assumed in Eq. (12). Moreover, the small parameter, β, is chosen to be 0.1, and the number of evanescent modes employed in this study is fixed at $n=2$ (Cho and Lee, 2000).

Figures 3 and 4 show, in the case of $\alpha_I=30°$, the variation of amplitudes of free long wave components propagating over a trapezoidally varying topography which has two rows with a value equal to k_1h_1 because the ambient currents are absent ($Fr=0$). The amplitudes of free long waves propagating over the region with a constant water depth change similarly. In Figure 3, $|E_{T1}|$ and $|E_{T3}|$ show similar variations of amplitudes. $|E_{T2}|$ and $|E_{T4}|$ likewise show similar variations in Figure 4. When the conditions of water depth and ambient currents, the critical incident angles vary similarly in Figure 5. Thus, the conditions under which free long waves can be resonated depend on the water depth and ambient currents.

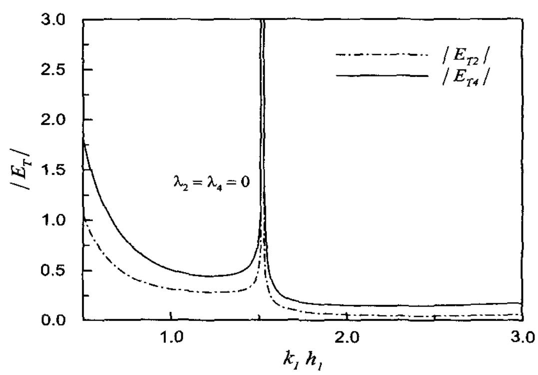

Fig. 4. The variation of the amplitudes of free long waves in regions 2 and 4 (two rows, $Fr=0$, $\alpha_I=30°$)

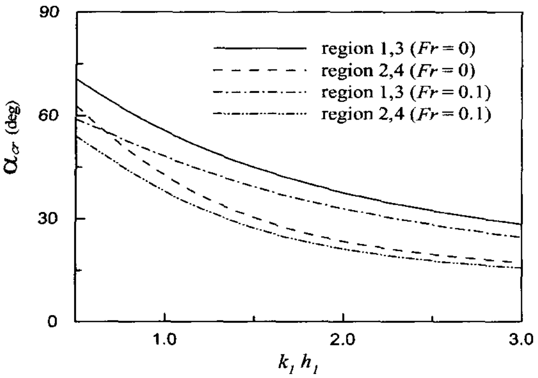

Fig. 5. The critical incident angles.

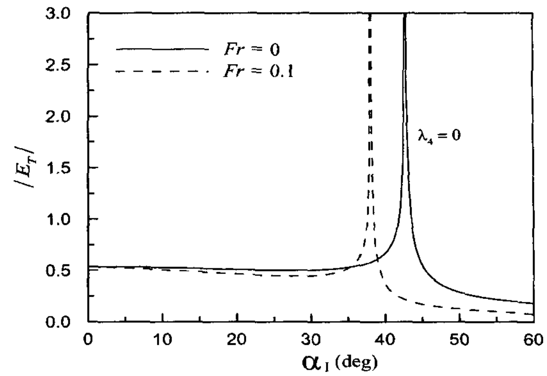

Fig. 7. The variation of the amplitudes of free long waves with incident angles (two rows, $k_1h_1 = 1.0$, region 4).

The free long waves propagating over regions 2 and 4, $|E_{T2}|$ and $|E_{T4}|$, are resonated a $k_1h_1 = 1.52$ (Figure 4). However, $|E_{T1}|$ and $|E_{T3}|$ are resonated at $k_1h_1 = 2.33$ (Figure 3). As the critical incident angle, α_{cr}, approaches the incident angle ($\alpha_I = 30°$) at $k_1h_1 = 1.52$ (Figure 5), the value of λ_2, λ_4 approaches zero and the trapped free long waves are resonated at the point $k_1h_1 = 1.52$ and decrease rapidly in regions 2 and 4. As the critical incident angle, α_{cr}, approaches the incident angle ($\alpha_I = 30°$) at the point of $k_1h_1 = 2.78$, however (Figure 5), the value of λ_1, λ_3 approaches zero and the trapped free long waves are resonated at the point $k_1h_1 = 2.33$, and decrease rapidly in regions 2 and 3.

Figure 6 shows, in the case of $\alpha_I = 30°$, the variation of the amplitudes of the free long wave components propagating over region 4 without ambient currents ($Fr = 0$) and with them ($Fr = 0.1$). In Figure 5, the value of k_1h_1, where the critical incident angle becomes the incident angle ($\alpha_I = 30°$), is 1.52 when $Fr = 0$ and 1.35 when $Fr = 0.1$. Thus, the value of λ_4 approaches zero and the trapped free long waves are resonated at $k_1h_1 = 1.52$ when $Fr = 0$ and at $k_1h_1 = 1.35$ when $Fr = 0.1$. So, ambient currents influence the conditions under which free long waves can be resonated. That is, the resonance of the free long wave components was shifted to smaller relative water depth region when the current existed.

Figure 7 shows, when $k_1h_1 = 1.0$, the variation of the amplitudes of free long wave components propagating over region 4 with incident angles α_I. The critical incident angle becomes 42.6° in the case $Fr = 0$ and 38° in the case $Fr = 0.1$ at the point of $k_1h_1 = 1.0$ in Figure 5. Thus, the value of λ_4 approaches zero and the trapped free long waves are resonated at the point $\alpha_I = 42.6°$ ($Fr_j = 0$) and $\alpha_I = 38°$ ($Fr_j = 0.1$) in Figure 7. Similar to Figure 6, the ambient currents influence the conditions under which free long waves can be resonated. For the smaller incident wave

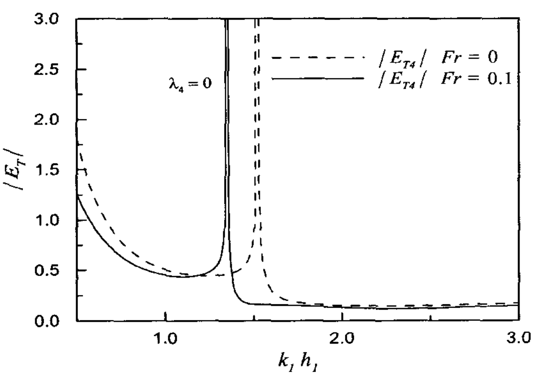

Fig. 6. The variation of the amplitudes of free long waves with ambient currents (two rows, $\alpha_I = 30°$, region 4).

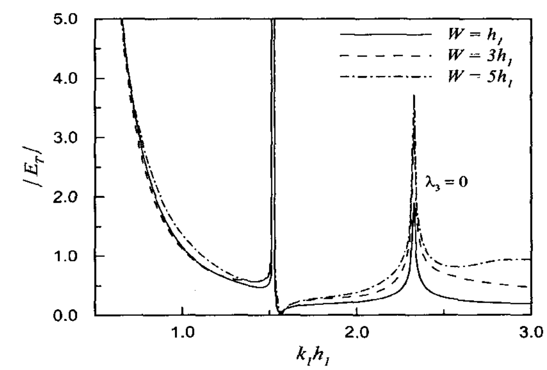

Fig. 8. The variation of the amplitudes of free long waves with the distance of two trapezoids (two rows, $\alpha_I = 30°$, region 3).

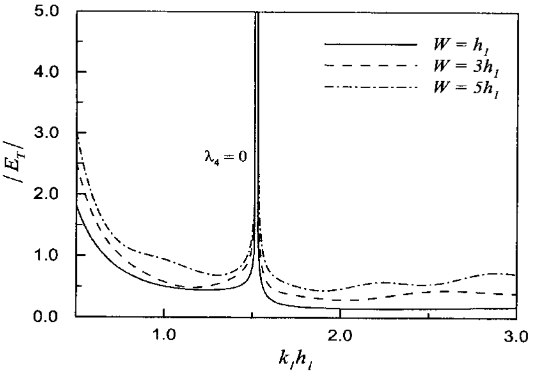

Fig. 9. The variation of the amplitudes of free long waves with the distance of two trapezoids (two rows, $\alpha_I = 30°$, region 4).

angle, the resonance of the long wave components was occurred as the current flowed. When the incident angle approaches the critical incident angle at which the free long waves can be generated, the free long waves are trapped and resonated.

Figures 8 and 9 show, under the same conditions of Figures 3 and 4, the variation of the amplitudes of free long wave components propagating over regions 3 and 4 ($W = h_I$, $W = 3h_I$, $W = 5h_I$). As the distances of the two trapezoids lengthens, the amplitudes of the free long wave components propagating over regions 3 and 4 increases slightly. However, the conditions of $k_I h_I$ under which the free long waves are resonated do not change. Thus, the conditions that free long waves can be resonated are not influenced by the distance of two trapezoids.

Finally, Figure 10 shows the variation of the amplitudes of free long wave components propagating over a trapezoidally varying topography which has four rows with and

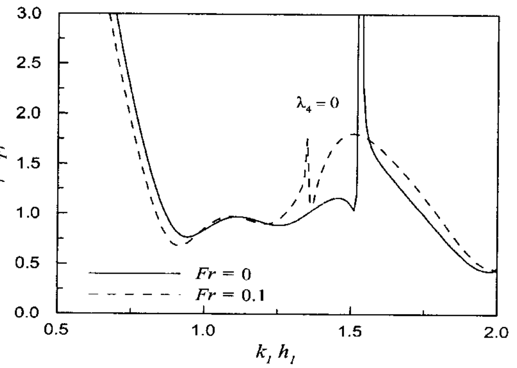

Fig. 10. The variation of the amplitudes of free long waves with ambient currents (four rows, $\alpha_I = 30°$, region 4).

without ambient currents. The trapped free long waves are resonated at the points $k_I h_I = 1.52$ ($Fr = 0$) and $k_I h_I = 1.35$ ($Fr = 0.1$), as seen in Figure 6.

4. Concluding remarks

The analytical solutions of the second-order long waves generated by the diffraction of short wave groups after abrupt changes in depth and current velocity are split into locked and free long waves. The possibility of resonance from trapped free long waves over a trapezoidally varying topography is confirmed in this study. The conditions under which the free long waves are resonated are related to the critical incident angles determined by the depth of water and the velocity of currents. As the incident angle of waves approaches the critical incident angle, free long waves can be trapped and resonated.

Acknowledgment

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