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From Counting to Mathematical Structure[1]

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The most important aim of mathematics education is to promote mathematical thinking. In the Hong Kong primary school, mathematical thinking is usually conducted through the use of formula and working on "application problem" or "word problems". However, there are many other ways that can promote mathematical thinking, and investigation on mathematical structure by using counting is one important source for promoting mathematical thinking for primary school children, as every children can count and hence a well designed question that can be solved by counting can enable children of different abilities to work together and obtain different results.

Keywords: counting, mathematical thinking
ZDM Classification: C32, F32
MSC2000 Classification: 97C50

INTRODUCTION

Hargreaves, Threlfall, Frobisher & Shorrocks-Taylor (1999) identified how children deal with linear sequences and quadratic sequences. The identification of sequences helped children to look for the structure of the pattern. For example, the sequence "4, 7, 10, 13, …" is one more than the 3 times the table of "3, 6, 9, 12…" and children can refer the sequence to $4 = (3\times1) + 1$, $7 = (3\times2) + 1$, $10 = (3\times3) + 1$ and so on.

Vergnaud (1988) adopted a broad view of multiplication and division, regarding them as a part of larger multiplicative conceptual field and argued that such conceptual fields takes considerable time to develop. English (1998) confirmed that children were not successful in solving the task sets when they could not identify the relational (or structural) properties of a problem or detect common relational properties.

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Ginsburg (1977) researched on how children discovered the mathematical rule of community and that children inclined to believe what they discovered rather than the rule they learned. Van Hiele (1986) has mentioned children learned mathematics through the stages of visual recognition, analysis of part and whole, abstraction, informal deduction and formal deduction.

Alexander, White & Daugherty (1997) proposed that there is a strong relationship of analogical reasoning and mathematical learning. English (1998) proposed that analogies and images are important vehicles for mathematical reasoning.

The investigation used two questions for the primary four and primary five children in Hong Kong (ages 9 to 10) and investigates how they solve such problem through counting. One of the questions is the counting of number of rectangles of a grid of $1 \times k$ (where k will be extended to test the ability of individual child). Base on the results of counting grids of different size 1×2, 1×3, 1×4 etc., the children are guided to find the closed form of the solution for grid $1 \times k$, which is $k \times (k + 1) \div 2$. This result will also be used as base or reference for children to work on the problem of grid $2 \times k$, through working on grid 2×1, 2×2, 2×3, etc to obtain the closed form $3k \times (k+1) \div 2$.

Another question is the counting of route from one corner of a $1\times k$ grid. The solving of the question is through counting the route for cases 1×2, 1×3, 1×4 etc to $1\times k$. The result of the question will then be used to solve the case of $2 \times k$ grids.

It is found that, with the design of counting backward, the recognition of the pattern and the formation of solution of large value of k, children are able to obtain the mathematical structure or the closed form of the solution.

THE RESEARCH QUESTION

The question is how to help children to discover mathematical structure through their counting process in solving a mathematics problem.

There are many combinatory problems which appear in the secondary curriculum that could be solved by primary students by using counting. However, counting alone may be able to provide answer of a certain problem but it would be desirable to obtain the closed form of the solution that is the structure of the solution.

The research is based on the following assumption:

1. Counting is a basic ability of every children;
2. Children are able to look for pattern with their counting ability;
3. Primary four and five students in Hong Kong can use algebraic symbol to represent an object or expression;
4. Children can map a simple structure (within 3 operations) to another identical

structure.

The above assumption in ability enables children to obtain the closed form of the solution of combinatory problem, with the suitable choice of question and also design of question.

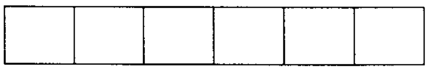
The teaching process involved

1. Counting simple cases,
2. Listing table for the result,
3. Observe the pattern of the results,
4. Testing large value of the pattern through the using of forward and backward deduction,
5. Generate a closed form of the structure of solution.

The following are the two questions that were used in Hong Kong school for this research.

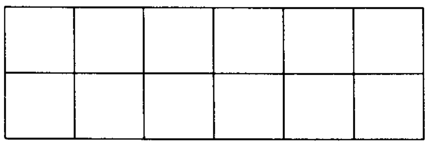
Question 1:

How many rectangles can be counted from the figure of 6-grids?
How about the case of k-grids?



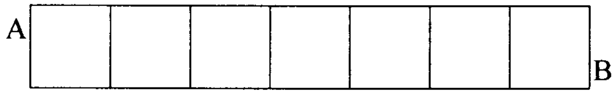
Question 1 extended:

How many rectangles can be counted from the figure (2×6 grids)?
How about the case of $2\times k$ grids)?



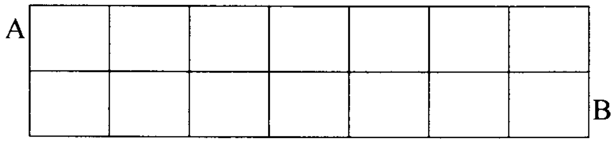
Question 2:

If one start to walk along the line from corner A to corner B, only able to go right and down (no left or up allowed), how many possible ways are there to reach corner B from corner A?



Question 2 extended:

If one start to walk along the line from corner A to corner B, only able to go right and down (no left or up allowed), how many possible ways are there to reach corner B from corner A?



SOLUTION STRUCTURE OF THE TWO QUESTIONS

For question 1, the closed form of the solution with k-grid has $[k(k+1)\div 2]$ number of rectangles.

For question 1 extended, the solution structure is $[3k(k+1)\div 2]$.

For question 2, the number of routes from A to B is $(k+1)$.

For question 2 extended, the solution structure ($2\times k$ grid) is $[(k+1) + k\times(k+1)\div 2]$.

Work on Question 1

Question 1:

How many rectangles can be counted from the figure of 6-grids?

How about the case of k-grids?

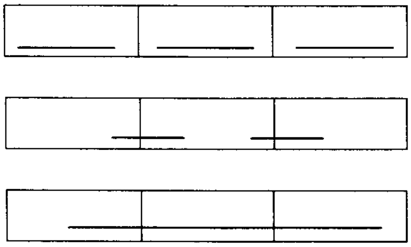
Children need to start working on some very small number such as 1×3, 1×4, 1×5 and extend to higher case of 1×10. When children are asked to find the number of rectangles in a 1×3 grid, they use different counting method. The following describe their work for simple cases.

Counting method

From the students work, the following three counting methods are observed

Counting method 1

Using line to distinguish the counting cases.



3 rectangles

2 rectangles

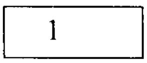
1 rectangles

Hence there are 1 + 2 + 3 = 6 rectangles in total.

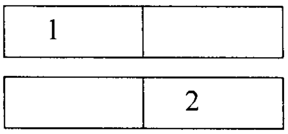
Counting method 2

Some of the children use number to denote the rectangle. For example, for the case of 1 grid, it is denoted by 1, and for the case of 2 grids, it is denoted by number 1 and 2 etc.

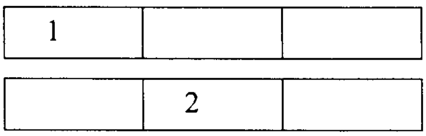
In the following case, the only rectangle formed is 1.

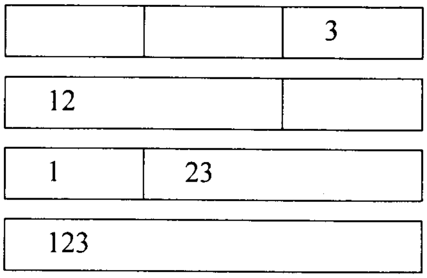


In the second case, there are 3 rectangles formed. They are 1, 2 and 12.



For the case of three rectangles, there are 6 rectangles formed. They are 1, 2, 3, 12, 23, and 123.





And for the case of 4 rectangles, there are 10 rectangles formed. They are 1, 2, 3, 4, 12, 23, 34, 123, 234 and 1234. By using such symbols, children can find out the number of rectangles formed easily.

Counting Method 3

Some children count the highest number of grid. For example, in the case of 4-grids, children counted in the following ways:

"4-rectangle", (1 of them)
"3-rectangle", (2 of them)
"2-rectangles" (3 of them)
"1-rectangle" (4 of them)

They count and adding the sum "1+2+3+4" = 10, which is the answer to the question.

This is more successful than those counting 4+3+2+1 as not really every one can count to the final number 4 in this case.

For the case of 5 small grids, there are 5 one-grid, 4 two-grid, 3 three-grid, 2 four-grids and 1 five-grid.

So there are in total 5+4+3+2+1 = 15 answers.

For the case of 6 small grids, there are 6 one-grid, 5 two-grid, 4 three-grid, 3 four-grids, 2 five-grid and 1 six-grid.

So there are in total 6+5+4+3+2+1 = 21 answers.

Gradually, children then guess the answer for the case of 7 grids, 8 grids, 9 grids and 10 grids. And obtain answer and list in a table.

Observation of pattern by listing table

Children are required to fill in the answer in table.

Number of grids	Answer Number of rectangles
1	1
2	3
3	6
4	10
5	15
6	21
7	28

Using forward and backward deduction.

Children are given an answer for a particular case and then guess the answer for the next case or the previous case. Three cases in forward deduction and three cases for backward deduction are used in the questions.

Forward deduction

Children are required to fill in the number of cases for forward deduction.

1	If there are 78 answers for the case of 12-grid, then the answer for the case of 13 grids will be (91 rectangles).
2	If 14-grids give the number of 105 rectangles, then 15 grids can give 105 + 15=120 rectangles.
3	If 15-grids give the number of 120 rectangles, then 16 grids can give 120 + 16 = 136 rectangles.

Backward deduction

Children are required to fill in the number of cases for backward deduction.

1	If 20-grids give the number of 210 rectangles, then 19 grids can give 210-20 = 190 rectangles.
2	If 21-grids give the number of 231 rectangles, then 20-grids can give 231-21 = 210 rectangles.
3	If 22-grids give the number of 253 rectangles, then 21 grids can give 253-22 = 231 rectangles.

A deduction for the general term and structure

By the knowledge they get from forward and backward deduction, children try to obtain a general deduction of the general term.

1	Given that there are 78 rectangles for the case of "k-grids". For the case of one more grid "k+1 grid", there are "78 + k+1" rectangles. For the case of one less grid "k-1 grids", there are "78-k" rectangles.
2	Given that there are 55 rectangles for the case of "k-grids". For the case of one more grid "k+1 grid", there are "55 + k+1" rectangles. For the case of one less grid "k-1 grids", there are "55-k" rectangles.
3	Given that there are 210 rectangles for the case of "k-grids". For the case of one more grid "k+1 grid", there are "210 + k+1" rectangles. For the case of one less grid "k-1 grids", there are "210-k" rectangles.

The above process allows children to connect the relation of the answer and the number of grids. Then children fill in the following table to discover the relationship of the answer and form an expression.

Number of grids	Answer Number of rectangles	Relationship
1×1	1	1
1×2	3	1+2
1×3	6	1+2+3
1×4	10	1+2+3+4
1×5	15	1+2+3+4+5
1×6	21	1+2+3+4+5+6
1×7	28	1+2+3+4+5+6+7
1×10	55	.....
1×11	66	.....
1×12	78	.....
1×13	91	.....

Most of the children can obtain the answer of counting as $1+2+3+.....+k$.

From this point, children are asked to connect the expression $(1+2+3+.....+k)$ with a closed form.

Obtain the relation of numerical vales and expression. Students are asked to observe the relationship of the answer and the number of grids.	
Expression	Closed form
For the case of 20-grids, 210 rectangles can be counted, which is 1+2+...+20=210.	210 $= 21 \times 20 \div 2 = 210$
For the case of 24-grids, 300 rectangles can be counted, which is 1+2+...+24=300.	300 $= 25 \times 24 \div 2 = 300$

Result of the achievement by students

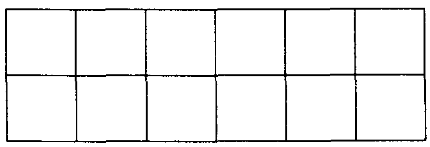
Achievement	Percentage of students Primary 4	Percentage of students Primary 5
Counting correct answers	65%	80%
Listing table	55%	70%
Forward and backward calculation	20%	44%
Obtain closed form $k\times(k+1)\div 2$.	5%	30%

Question 1 extended

Question 1 extended:

How many rectangles can be counted from the figure (2×6 grids)?

How about the case of 2× *k* grids)?



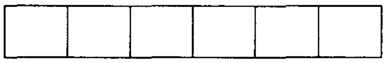
Question 1 is extended to the form of the following diagram (the case 2×3, general case is 2× *k*).

Childrens' Strategy in solving the extended problem

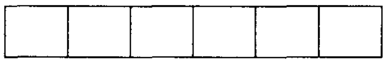
Some children break down the problem by into three cases

Case 1) first row
Case 2) second row
Case 3) vertical row

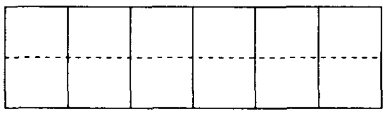
Case 1) for a row of 6-grid, the answer is 21 rectangles.



Case 2) for a row of 6-grid, the answer is 21 rectangles.



Case 3) The two rows can be combined and count as one row, so again the answer is 21 rectangles.



Hence, the number of rectangles is then 21×3 = 63.

Listing of table

Children observed that the number of rectangles of 2×k grids is always 3-times the number of 1×k grid.

Number of grids	Answer Number of rectangles	Number of grids	Answer Number of rectangles
1×1	1	2×1	3
1×2	3	2×2	9
1×3	6	2×3	18
1×4	10	2×4	30
1×5	15	2×5	45
1×6	21	2×6	63
1×7	28	2×7	84

Deduction of the case of 3×*k* grid

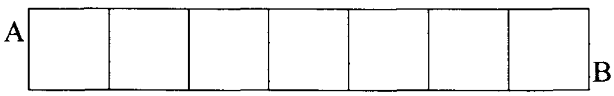
Not many children can deduce the answer for the case of 3×k grids, but some children have borrowed the concept of multiplication. For 3×k grids, the answer is the product of number in a row of 1×k (horizontal row), and 1×3 (vertical row). For those who can deduce to the case of 3×k, they proved themselves to be capable for more extension of the problem, say, finding the answer for 4×k, 5×k, etc. The case of 3×k and 4×k prove to be difficult for children at primary four and five.

Achievement $2\times k$ grid	Percentage of students Primary 4	Percentage of students Primary 5
Counting correct answers	2×6 grids 50%	2×6 grids 60%
Relationship between $2\times k$ and $1\times k$	30%	44%
Using the idea of Multiplication	0%	13%%
Extension to the case of $3\times k$, $4\times k$	0%	6%

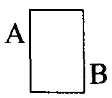
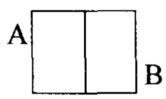
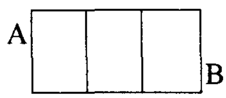
Work on question 2

Question 2:

If one start to walk along the line from corner A to corner B, only able to go right and down (no left or up allowed), how many possible ways are there to reach corner B from corner A?



We wish children to find the answer by counting and also summaries the answer to general solution. Children are given work sheets to try out the number of different grids. The following three exercises served as starter for children.

1×1 grid	1×2 grid	1×3 grid
		
There are 2 ways to go from A to B.	There are 3 ways to go from A to B, namely 2+1=3.	There are 4 ways to go from A to B, namely 3+1 = 4.

Children are asked to observe the relationship of the three answers. Then they have to work on question of 1×7 grids. From there they guess the answer to be 8 and they work out a general formula for "$1\times$any number" and the answer to this is "any number plus 1"

Listing of Table

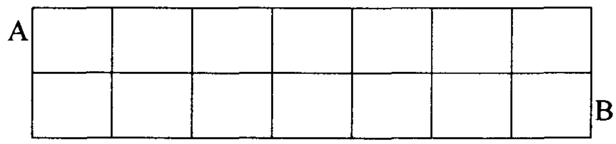
It is not difficult for children to list a table and then observed the closed form of the solution.

Number of grids	Number of ways
1	2
2	3
3	4
4	5
5	6
6	7
k	k+1

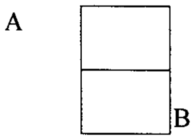
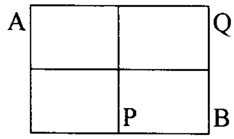
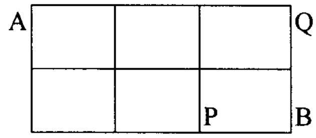
The relationship of k grid gives k+1 ways is not difficult to obtain. However, it is more difficult to obtain the closed form of the following extended question.

Question 2 extended:

If one start to walk along the line from corner A to corner B, only able to go right and down (no left or up allowed), how many possible ways are there to reach corner B from corner A?



It is not easy for children to get the answer, as the counting processes are more tedious. Some simple cases are introduced. As usual, children start working for simple cases and observe the pattern.

1×1 grid	1×2 grid	1×3 grid
		
There are 3 ways to go from A to B.	There are 3 ways to go from A to P and 3 ways from A to Q. Hence there are 3+3 = 6 ways to go from A to B	There are 6 ways to go from A to P and 4 ways from A to Q. Hence there are 6 + 4 = 10 ways to go from A to B

Working on question with identical structure

In order to provide another chance for children to solve the problem, another set of questions with identical structure was designed. Children were asked to count the number of steps that the 「 • 」 at the upper left corner can move to the lower right corner to the position 「 Δ 」. For example, the following diagram gives the number of each move. So it is 4 moves in total.

•			
			Δ

•	1	1	1
1	2	3	4

The above solution structurally corresponded to the question of finding the number of route from A to B.

A				
	1	2	3	4

The following is the case for 2×3 grids and the number of routes is 4+6 = 10.

A				
	1	2	3	4
	1	3	6	10

With the understanding of the correspondence, children are asked to find the number of ways to move 「 • 」 to 「 Δ 」 for the following diagram.

•			
			Δ

•	1	1	1
1	2	3	4
1	3	6	10

The using of this representation helps children to write down the solution of the problem easily, but still the counting process could not consolidate to a closed form. However, with the above counting and the result in a table, it is easier for children to guess the process.

A

1	2	3	4	5
1	3	6	10	15

With the listing of table. The case of 2×7, which has answer 36 can be obtained.

•	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8
1	3	6	10	15	21	28	36

Listing of the table

Number of grids	Number of ways	relationship
2×1	2 +1 =3	
2×2	3 + 3 = 6	
2×3	4 + 6 = 10	
2×4	5 + 10 = 15	
2×5	6 + 15 = 21	
2×6	7 + 21 = 28	
2×7	8 + 28 = 36	8 + 7×8÷2 = 36
…	…	…
2×k		$(k+1) + k\times (k+1)\div 2$

The above table tries to link the numerical values with the corresponding process of calculation.

Achievement 2×k grid	Percentage of students Primary 4	Percentage of students Primary 5
Counting correct answers	2×6 grids 60%	2×6 grids 65%
Correspondence of the two question	21%	34%
Finding the structure	0%	8%

CONCLUSION

Children can use counting to obtain solution of a simple problem and through the observation of the structure pattern. However, the breaking down of the questions and form guided-questions are helpful for children to solve the problem. Especially students of different levels of achievement may tackle with different level of results.

The using of table to list the answer and observing the pattern is important in forming a link of the structure and the numerical values. Such a correspondence in can help children to explain the formation of the solution.

The using of different question format (in question 2) helps children to relate the counting and the original problem. Sometimes to teaching effectively, a kind of alternatively question with identical structure is helpful for students. The successful solving of one question may lead to the successful understanding of a related question.

The process of identifying the pattern of numerical values is best obtained with original expression, such as $8 + 7\times8\div2 = 36$. The simplified value of 36 erases all possible links of the solution.

The suitable choice of questions arose the interest of children in mathematical thinking, though the percentage of successful solving of the extended problem in primary 4 is low.

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