Journal of the Korea Society of Mathematical Education Series D:
Research in Mathematical Education
Vol. 12, No 2, June 2008, 85–108

韓國數學教育學會誌 시리즈 D:
<數學教育研究>
제 12 권 제 1 호 2008 년 6 월, 85–108

Authentic Investigative Activities for Teaching Ratio and Proportion in Elementary and Middle School Mathematics Teacher Education[1]

Ben-Chaim, David*
Faculty of Science and Science Education, University of Haifa-Oranim, Kiryat Tivon 36006, Israel;
E-mail: david.ben-chaim@weizmann.ac.il

Ilany, Bat-Sheva
Beit-Berl College, Doar Beit-Berl 44905, Israel

Keret, Yaffa
Weizmann Institute of Science, Israeli Science Teaching Center, Solar Tower, Rehovot 76100, Israel;
E-mail: yaffa.keret@weizmann.ac.il

(Received March 24, 2008. Accepted March 30, 2008)

In this study, we created, implemented, and evaluated the impact of proportional reasoning authentic investigative activities on the mathematical content and pedagogical knowledge and attitudes of pre-service elementary and middle school mathematics teachers. For this purpose, a special teaching model was developed, implemented, and tested as part of the pre-service mathematics teacher training programs conducted in Israeli teacher colleges. The model was developed following pilot studies investigating the change in mathematical and pedagogical knowledge of pre- and in-service mathematics teachers, due to experience in authentic proportional reasoning activities. The conclusion of the study is that application of the model, through which the pre-service teachers gain experience and are exposed to authentic proportional reasoning activities with incorporation of theory (reading and analyzing relevant research reports) and practice, leads to a significant positive change in the pre-service teachers' mathematical content and pedagogical knowledge. In addition, improvement occurred in their attitudes and beliefs towards learning and teaching mathematics in general, and ratio and proportion in particular.

Keywords: authentic investigative activities, ratio and proportion, proportional reasoning, pre-service mathematics teachers.

[1] This paper will be presented at the 40th Korean National Meeting of Mathematics Education at Seoul National University, Seoul, Korea; May 16–17, 2008.

* Corresponding author

ZDM Classification: B53, C73, D43, F83
MSC2000 Classification: 97B50, 97C740, 97D40

BACKGOUND AND PURPOSE OF THE STUDY

Proportional reasoning (PR) is at the heart of mathematics in the upper grades of the elementary schools and in the middle schools. Ratio and proportion are important mathematical topics that have received much attention during ancient times and continue to do so today (Ben-Chaim, Keret & Ilany, 2007). In principle, PR deals with mathematical relations, which are multiplicative in nature, in contrast to additive mathematical relations, that are typical for many young children in elementary schools (Harel & Confrey, 1994).

According to the Curriculum and Evaluation Standards (NCTM, 1989),

> "The ability to reason proportionally develops in students throughout grades 5–8. It is of such great importance that it merits whatever time and efforts that must be expended to assure its careful development." (p. 82).

Therefore, the topics of ratio and proportion should have central part in mathematics curriculum for children in school as well as for pre-service mathematics teacher education.

Nevertheless, research has consistently shown that (Post, Behr & Lesh, 1988, p. 78)

> "...relatively few junior high students of average ability use proportional reasoning in a consistent fashion,"

or successfully cope with it (Ben-Chaim *et al.*, 1998). The topic even "remains problematic for many college students" (Lawton, 1993), and "There is evidence that a large segment of our society never acquires fluency in proportional thinking" (Hofer, 1988, p. 285). Furthermore, recent research findings all over the world have indicated many gaps in the content knowledge of pre- and in-service teachers in mathematical subjects taught in elementary and middle schools, including the topics of ratio and proportion. Frequently, existing knowledge is technical, schematic, unconnected, and incoherent. As a result, difficulties arise which are evidence of the pre-service teachers' lack of understanding of mathematical concepts, including ratio and proportion, and feeling that they are incapable of both, coping with the material, and teaching it (Fischbein, Jehiam, & Cohen, 1994; Sowder *et al.*, 1998; Keret, 1999; Ben-Chaim, Ilany & Keret, 2002).

The conclusion reached from these studies is that in order to improve the situation, we need to implement the research findings in teaching the topic of PR and, in parallel, to deal this topic differently in pre-and in-service teacher education. According to our philosophy, pre-and in-service training should include three central components:

cognitive, affective and the component of teacher behavior which is evident by his/her readiness to cope with constructing teaching units and planning the teaching of the subject.

In our attempts to investigate and contribute to the field of PR, we decided to adopt the recommendations proposed by Sowder *et al.* (1998) regarding how to educate pre-service mathematics teachers to teach multiplicative structures including the PR topic. It should be indicated that our approach is in agreement with Jaworski's (2005) idea that

> "didactics of mathematics teacher education is about creating tasks, or situations, through which teachers will have opportunity to learn concepts in mathematics teaching" (p. 359).

There are three main purposes to this report:

(1) To present, discuss and analyze some of our authentic investigative PR activities;
(2) To introduce a PR teaching model for training pre- and in-service elementary and middle school mathematics teachers; and
(3) To provide evidence of prospective teachers' learning, for enhancing their PR pedagogical content knowledge, and for improving their attitudes toward mathematics in general and all the components and aspects of the PR topic in particular, as a result of taking and passing the PR course.

THE AUTHENTIC INVESTIGATIVE PR ACTIVITIES

The driving motive for the design of our PR authentic tasks is the following quotation from the NCTM Principles and Standards for School Mathematics (2000):

> "In effective teaching, worthwhile mathematical tasks are used to introduce important mathematical ideas and to engage and challenge students intellectually. Well-chosen tasks can pique students' curiosity and draw them into mathematics.... Regardless of the context, worthwhile tasks should be intriguing with a level of challenge that invites speculation and hard work." (pp. 18–19).

In a pilot study by Ben-Chaim, Ilany & Keret (2001), nineteen PR activities were developed and conducted in order to assess the influence of exposing pre-service mathematics elementary teachers to authentic investigative PR activities. The chosen activities usually present problems from the children's world and everyday life, problems that raise non-routine open questions, which in turn give rise to thinking and reflection. This enables a deep investigation of the learner's responses and disclosure of many of his/her mathematical features, such as originality and self criticism. Those activities are called authentic activities that require the learner to activate judgment and use of knowledge to solve problems with relation to everyday life. The tasks within the activities, consist of several levels, are spiral in their structure as is described in Fried and Amit (2005) and usually have more than one correct answer. We believe that authentic investigative activities cultivate higher order thinking and the ability to solve problems, abilities that might be useful for the individual as well as for society. In evaluating the learner's mathematical capability, it is very important to take into account the applied solution and thinking procedures. Many of the activities are in the appropriate form for presentation to children in the upper classes of elementary school and in the classes of middle school.

All of the tasks were developed content wise, based on review of the PR literature (such as: Sowder *et al.*, 1998; Harel & Confrey, 1994; Tourniaire & Pulos, 1985) and on the results of our qualitative and quantitative pre and post testing of student teachers at the beginning and at the end of each PR course during the pilot stage of our study. The results showed that at the beginning of the PR course, the majority of the student teachers were unfamiliar with the PR subject, indicating that this topic is not specifically treated within the school system at any level. A typical task is introduced as a worksheet to the student teachers to work on, preferably in heterogeneous groups of 3–4 participants, and later on to be discussed with the whole class. For most of the tasks, the teacher educators are equipped with our detailed remarks, optional solutions, and recommendations for assignment of reading material, including suitable research reports on PR, as well as how to conduct and expand the activity, based on our numerous pilot results of presenting the PR tasks to many groups of student teachers, and in some cases to elementary school students as well.

The activities include mathematical tasks, which require quantitative and qualitative numerical comparisons between ratios and finding a missing value. The tasks involve small and large integer numbers, fractions, decimals, and percents. For each activity there is a basic part, to build the basic knowledge needed regarding ratio and proportion. Then there is an extension part to further develop understanding. The activities establish the understanding of many concepts related to ratio and proportion topics and are focused on the three main categories of PR problems: Rate and Density, Ratio, and Scaling. The distinction between rate, ratio and the scaling types are according to Freudenthal's categorization of the proportional reasoning problems (Freudenthal, 1978; 1983, pp. 178–209). We created two additional types of PR activities, one is introductory PR activities and the second is indirect PR activities. In the following, we will shortly relate to the different types of the activities and present for example several of them.

An example for an introductory activity is the "teaching incident task" in which the student teachers are asked to help a sixth grade teacher to decide whether or not her students are ready to study the topic of ratio. The sixth grade students are asked to write statements about the following story: Yael and Uri decided to surprise their mother for her birthday by bringing her a flower bouquet that consisted of three tulips and nine roses. The student teachers are asked to find out what characterizes a certain set of statements and which mathematical connections they introduce. For example, one set of students' statements includes "there are more roses than tulips", and "the bouquet of flowers includes 6 tulips less than the number of roses", while the second set includes "the number of roses is 3 times larger than the number of tulips", and "25% of the flowers are tulips and 75% of the flowers are roses." The main goals of this task and the other introductive tasks are for the student teachers to be aware of the distinction between additive and multiplicative thinking, to be exposed to and deal intuitively with the concepts of ratio and proportion, and to be confronted with real class situations that require the teacher's reaction.

The rate activities emphasize the creation of new units by comparing magnitudes of different quantities with an interesting connection, as in "miles per gallon", or "people per square kilometer", or "kilograms per cubic meter", or "unit price". All of those are related to authentic real-life situations in our everyday life, and are very rich in their mathematical connections. For example, one of the rate tasks presents several bargains through which one has to decide "which is a better buy". Our experience with introducing such a task to student teachers as well as to students in middle school classes shows that they apply a variety of strategies, some of them correct and others erroneous (Ben-Chaim *et al.*, 1998). The important point for the student teachers is the opportunity to discuss and test each strategy, look at its mathematical validity and evaluate its efficiency, in addition to tracking misconceptions and finding techniques of solving them.

Two other rate activities deal with the density concept. Students are given a bag with an unknown number of white beans and a cup with a certain number of brown beans. They create mixtures of the beans, and then by sampling and counting procedures, they estimate the number of the white beans in the bag. In the second activity they experience methods of how to estimate the number of individuals in a huge demonstration of people or the number of birds or animals in a certain area. For example see the activity in the Appendix 1.

Two of the ratio activities are presented in Appendix 2 and Appendix 3. One deals with a survey of preference between two different flavors of coca-cola, and the second with a situation of sharing pizza. In both cases students compare ratios, apply ratio properties and find a part of a given whole, or the whole through a given part. An additional ratio task deals with the division of profit between partners. The ratio activities emphasize connections to the topics of fractions, decimals, and percentages. In addition, while conducting a discourse with the students that relates to their actual strategies and attempts to prepare concrete examples for teaching the topic of ratio, the discussion led to clarifying the meanings of the arithmetic operations, especially the different meanings of the division operation regarding division by inclusion, or division by parts.

The scaling activities are considered to be the most difficult ones, since they are related to many different mathematical topics such as measurements, spatial visualization, two and three dimensional objects, linear, quadratic and cubic scaling, conversion of units, perimeter and area and volume formulas, and in many cases complicated computations. Hence we propose the largest number of authentic scaling tasks — seven tasks that deal with most of the above topics in different contexts. For example, one of the tasks "Wimpy in the Amusement Park" simulates what happens in the mirror room by creating images of an original Wimpy figure with up and down scaling and in some cases the mirror affecting only one dimension. The students have to measure and decide what is the linear scale factor used and what is the scale factor for the area that we relate to as quadratic scaling. Later on within the expanded part of the Wimpy task, the students are asked to create their own mirrors to produce similar shapes of Wimpy according to given scales of enlargement or reduction by a certain percentage, and find out what are the results on Wimpy of several successive operations of scaling. In another authentic task, based on the article by Tracy & Hague (1997), the student teachers are asked to read the article, analyze its findings and prepare an activity for middle school pupils that relates to different toys and models emphasizing their scale factor (linear, quadratic, and cubic) compared to reality. Another example is the "The Task of the Bet-Shean Temple" (BST-task) given in Appendix 4.

An example for the indirect PR activities is presented in Appendix 5. In this activity the learners are asked to figure out "What Is the Weight of a Meteoroid?" Through this and additional indirect PR activities, the basic concepts of indirect ratio and proportion are clarified.

The findings of the pilot study showed that after exposing and experiencing with authentic ratio and proportion investigative activities, the pre-service teachers were more successful in solving ratio and proportion problems, exhibited different strategies for solving the problems, and were more capable of providing a good quality of written and oral explanations (during the interviews) to their work. In addition, they improved their attitudes toward mathematics in general and all the components and aspects of ratio and proportion in particular (Ben-Chaim, Ilany & Keret, 2002). A replication of this study with several classes of pre-service teachers in 3 different Israeli colleges showed similar findings. From these research findings, the conclusion was that it is necessary to teach the ratio and proportion topics in pre- and in-service mathematics elementary and junior teacher education, by applying PR authentic investigative activities, such as those developed and conducted through the pilot study. As a result, a special model for teaching the ratio and proportion topics in mathematics teacher education was developed. This model is introduced next.

A MODEL FOR TEACHING PR ACTIVITIES

The model was developed following pilot studies investigating the change in mathematical and pedagogical knowledge of pre- and in-service mathematics teachers due to experience in authentic proportional reasoning activities. The proposed model incorporates the main areas in the training of pre-service teachers, particularly in the areas of content knowledge and pedagogical content knowledge, pedagogical reasoning, training, and beliefs. The model is comprised of 4 components with interaction between them (see Figure 0).

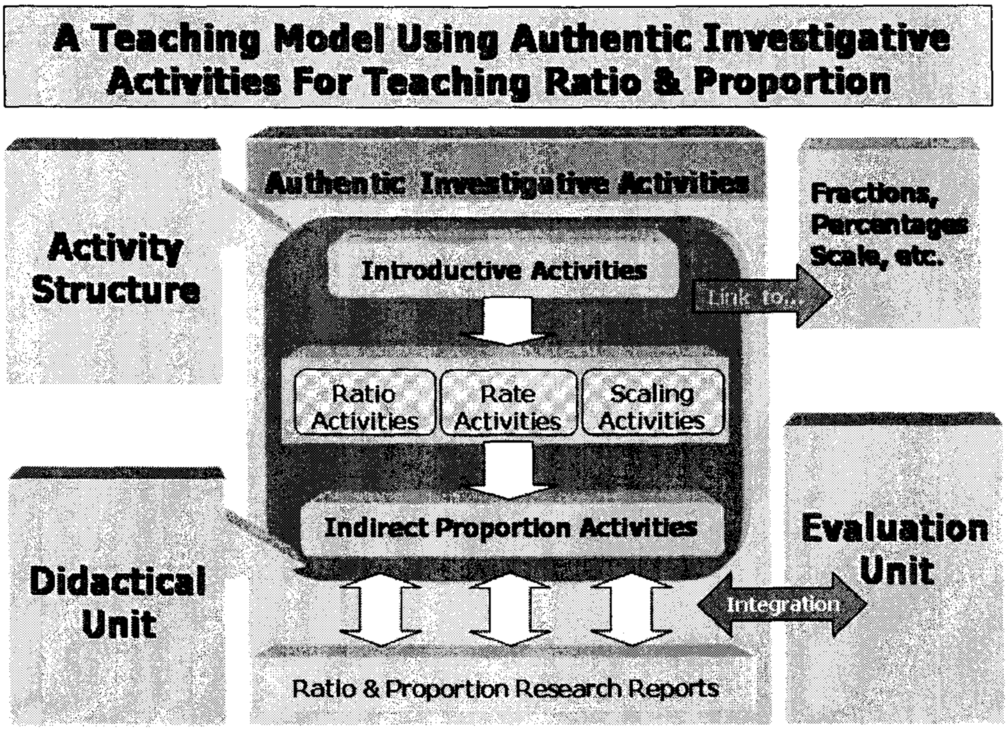


Figure 0. A Teaching Model Using Authentic Investigative Activities

The first component — the core of the model includes the authentic investigative activities with 5 types of activities: Introductive activities, investigative activities dealing with ratio, dealing with rate, dealing with scaling (the first four are introduced as direct PR tasks), and dealing with indirect proportion (for examples of the activities see Appendix 1– Appendix 5).

In parallel to the engagement in the investigative activities, the pre-service teachers are referred to articles, dealing with the topics of ratio and proportion. The articles are related to the mathematical as well as to the didactical-pedagogical aspects of the topics. The analysis of the research articles, enable the students to be aware of their mathematical knowledge, while the presentation of the research findings might lead to a wide perspective and deeper discussion regarding suitable teaching strategies for teaching the subject. In this case, the intention was to create integration between theory and practice (Greeno, 1994), or as Lienhardt, Young & Merriman's (1995) notation of

"... knowledge learned in the academy vs. in practice".

They claim that professional knowledge can vary by the location of the learning (in the academy or in practice), the type of knowledge (declarative or procedural), the generality of knowledge (abstract or specific), and the nature of principles (conceptual or pragmatic).

The second component includes the structure of the activities. As indicated before, they are structured as authentic investigative problems related to content and context familiar to the prospective teachers and to children in the elementary and junior level. As can be seen in the given examples (Figures 1–5), the activities include types of tasks reported in the professional literature as appropriate for assessment for PR (Cramer, Post & Currier, 1993):

(a) Missing value problems, where three pieces of information are given and the task is to find the fourth or missing piece of information;
(b) Numerical comparison problems, where two complete ratios/rates are given and a numerical answer is not required, but the ratios or rates are to be compared;
(c) Qualitative prediction and comparison problems which require comparisons not dependent on specific numerical values.

The third component includes the structure of the didactical unit. It includes a unit around a concept; for example: A didactical unit that its role to impart the concept of ratio. The structure of the didactical unit includes:

(a) Working by groups;
(b) Discussion of results with the whole class;
(c) Mathematical and didactical summary; and
(d) Homework.

The fourth component includes the evaluation unit with several types of assessment instruments:

(a) An instrument to evaluate student's mathematical knowledge, and mathematical didactical knowledge;

(b) Attitude questionnaire;

(c) Portfolio - An instrument for alternative assessment related to learning processes of pre-service prospective teachers during their studies; and

(d) An instrument for assessing research reports during and after the course studies. For a more detailed description of the model see Keret, Ben-Chaim & Ilany (2003) and Ilany, Keret & Ben-Chaim (2004).

AN IMPLEMENTATION OF THE PR TEACHING MODEL

In the following, we report a research study on the implementation of the PR teaching model within the program of pre-service elementary teacher education in one of the Israeli academic colleges.

METHODOLOGY

Our report will be related to the 2004/2005 academic year cycle of teaching the PR semester course, during the second semester of the academic year, 14 weekly teaching sessions, each of 90 minutes.

Sample: 15 Pre-service teachers from an Israeli academic teacher college, as part of their training to teach mathematics in elementary and middle school. The participants were in their 4th year of studies toward earning a B.Ed degree including a teaching certificate with a specialization in math.

Research instruments: In addition to the model, the activities and the theoretical material (articles and research reports), the research instruments included:

1) A proportional reasoning questionnaire which included 5 rate and density problems, 5 ratio problems, and 6 scaling problems. In each problem, the participants were asked to provide support work by giving reasons for their answers. For a more detailed description of this instrument, see Ben-Chaim et al. (1998).
2) Attitude questionnaire, which includes 22 items and 3 open questions. Twenty of the items of the attitude questionnaire belong to four categories as presented in Table 1. For more details, see Ben-Chaim, Ilany & Keret (2002).
3) Observations that were aimed at a formative evaluation of the instructional model and at the follow up of the pre-service teachers' procedures and change in behavior.

4) Personal interviews that were aimed at in-depth examination of the participants' opinions on the impact of the course. For this purpose, a representative sample of 5 students out of the 15 students in the course was interviewed. Each one of the representative sample was interviewed individually.

The instructor of the course was one of the authors (Dr. Ilany). She implemented the model including all of its components using the teaching method of inquiry.

RESULTS AND DISCUSSION

To illustrate the complexity of the PR activities, one authentic scaling task is presented in depth, pointing to the dynamic nature of task design, and the added value stimulated by the PR component entailed in the task in terms of mathematical and pedagogical notions. The activity is called "The Task of the Bet-Shean Temple" (BST-task) given in the Appendix 5. This task was conducted with several groups of pre-service student teachers at the college level. The following illustration is belonging to the sample of this study.

The student teachers, worked on the BST-task in small heterogeneous groups (3-4 students each). From the outset, it was evident, that the students encountered some difficulties in working on the BST-task. They had first to visualize the sketch of the temple and identify the main hall and other parts of the temple. Then they had to decide if they needed to measure the interior or the exterior walls of the main hall. In any case, the measurements of the sides of the main hall are not integers, and in addition the shape of the main hall is a quadrilateral that is not a rectangle. Hence, they had to discuss and decide how to compute its area approximately. It is built-in the structure of the BST-task that the students would clarify to themselves the connection between the scale of the sketch and the ratio between the linear measures of the sketch and the same linear measures in reality. It took them some time to accept the idea that the scale is the same as the above ratio. Several students still had reservations about that "since the scale should be 1 to a number, rather than a ratio between any two numbers", especially when they had non-integer measures.

During the first part of the activity, it was suggested to the student teachers to prepare many similar pairs of rectangles, and to find the scale factor with relation to their sides and also to compute the area of each one and find the ratio between the areas of similar pairs of rectangles. The student teachers were also referred to the rectangular classroom desks, asking them to draw sketches of the desks on different scales and compute the linear and quadratic (area) ratios between the sketched desks and the real ones. Nevertheless, when the student teachers had to work on question d) of the BST-task (see Appendix 5), the vast majority of them picked the given linear scale. This phenomenon is known as the "illusion of linearity" or the "linear trap" (De Bock, Verschaffel & Janssens, 1998) and is related to application of the linear model, whether or not it is appropriate. This "linear illusion" is widely prevalent and used by students of all ages in their solutions of word problems involving length, area and volume of similar objects. Our experience with implementing the PR scaling tasks indicates that there is a need for long engagement of the students with many different tasks, as well as working with concrete objects, in order for the students not to fall into the "linear trap". For many students, the analysis of the quantitative data, of the pre and post testing on the scaling items, indicated this problematic situation.

It should be noted that the students, were very enthusiastic and interested in working on this task. During the discussion of the activity, many ideas were raised regarding on how to present it to the children in the elementary classes and how to connect it to geography and to being acquainted with maps. It was evident that the students could cope with the problems raised, but in many cases they argued with each other within the working groups until they reached an agreement or were convinced by the explanations of their peers. They formed tables to organize the results of their measurements and computations, and only several of them were able to understand immediately the effect of changing the scale on the size of the sketch or how to execute the backward procedure needed for solving question h) of the BST-task (see Appendix 5). For this task, it is also recommended in our guidance to the teachers to use computer software, such as Sketchpad, to produce the sketches with different scales and present the measures in order to create a dynamic changing situation and enhance the work on the important ideas rather than on the technical parts. Many of the student teachers reported later, that they applied the advised strategies of teaching and the above ideas successfully.

The second level questions of the BST-task were too demanding for the student teachers, since the numbers were relatively larger, even though they used calculators, and in addition many of them were unfamiliar with the formulas of finding volume of either the rectangular pool or the cylindrical well. However, in both cases, enough time was devoted to explaining and dealing with the questions posed. Towards the end of engagement with this activity, one of the student teachers summarized: "The task is very good summary of what we should teach the children: scales, ratio, conversion of units, measurements, perimeters, areas, three dimensional objects, and volumes. The task makes connections among all those topics."

The student teachers, in the PR course, experienced several activities of each PR type. For example, they experienced at least 3–4 PR scaling tasks, including the Wimpy task and the BST-task. In addition, they were required to read, analyze and discuss research reports. During the PR course, it was obvious that the student teachers were mainly concerned not only with their lack of PR content knowledge, but also with their future teaching on their own. Hence, the practical research reports on trials of PR activities with children seemed to us to be the most appropriate for them. For example, after having experience with two research reports that are related to the PR scaling tasks: "Toys 'r' math" (Tracy & Hague, 1997) and "Fractions attack! Children thinking and talking mathematically" (Alcaro, Alston & Katims, 2000) two of the student teachers were interviewed to find out what they learned from reading the reports. Ofra: "Regarding my teaching, it provided me with more devices to implement while opening the topic of PR in my class; it gave me backup; that is what they did and those are the results; it will be interesting what will happen in my class compared to the report in the article." Orit: "There are more experiments, and I like to be involved in experiments of methods and strategies in addition to numbers; here they showed me the experiment, it was more meaningful and interesting for me." In the next section we report the pre and post results of testing the student teachers who participated in the PR course at the college level.

The layout of the study is one group pre-test – post-test. The attitude (qualitative) and the proportional reasoning (qualitative and quantitative) questionnaires were administered before and after implementation of the PR teaching model, including the presentation of the different types of the PR activities.

At the outset of reporting the results of pre and post testing, we should admit that we are aware of the small number of participants in our study ($n = 15$), which is rather small to be suitable for quantitative analysis of the pre- and post- data. Nevertheless, it has been very encouraging for us to discover for the past five years, similar results for larger samples. Hence, it is justified to consider the quantitative results of this study, which are supported by the previous evaluations of the impact of similar courses on student teachers' PR content knowledge.

The attitudes toward the topic of ratio and proportion were evaluated in order to detect changes that took place between pre and post exposure to the PR authentic investigative activities. The assessment was conducted by a Likert scale of 1–5. Table 1 presents the pre-post results, which are quite similar to those presented and discussed in a previous study by Ben-Chaim *et al.* (2002). The data in Table 1 indicates that the pre-service teachers of this study improved their attitudes toward teaching mathematics in general, even though they started with quite a positive level. The standard deviation measure of post testing indicates that all the participants enjoyed teaching mathematics. In addition, they were much more confident in their ability to deal with ratio and proportion at the end of the course.

Nevertheless, there is a difference which is noticeable in the responses to the three open questions at the end of the attitude questionnaire, asking them to mention a situation of ratio and a situation of proportion, in addition to concepts and words that are related to these topics. In most cases, in the pre-test the pre-service teachers wrote: "*I don't* *remember*", or "*I don't know*", especially for proportion. On the post-test, every one of the 15 participants provided several correct examples of ratio and proportion situations with indication of properly related concepts and words, even using mathematical notations such as $a/b = c/d$, $a,b,c,d \neq 0$.

To those who were exposed to the PR activities, teaching ratio and proportion seemed to be more complicated than before the exposure. The interviews strengthened this finding, for example: "I thought that the topic of ratio and proportion was easy and that I knew how to teach it, but today after I have learned, I realize that it is very complicated and that I still lack knowledge"; or another quotation: "It exposed me to a topic about which I did not know how much I don't know and how unready I am to teach it".

Another change that occurred following the exposure to the PR activities is connected to the need to include ratio and proportion as part of pre- and in-service teacher training. Before the exposure, not all the examinees thought it was very important to include a course related to ratio and proportion; however, after the exposure, almost all the participants thought that it was very important to include it. For example, one of the interviewees said: "It is very important to teach this topic of ratio and proportion in college, since it is the A B C of mathematics. If you are not familiar with the topic, it is impossible to teach the other topics". These findings indicate a significant improvement in the participants' attitudes toward the overall components and aspects of ratio and proportion.

Table 1. Summary of Attitudes toward Ratio and Proportion, Scale (1–5)

Number of Items	Items' Relation	Mean Before Instruction $n = 15$	Mean After Instruction $n = 15$
4	Items relating to attitudes toward teaching mathematics in general.	4.45	4.81
7	Items relating to confidence in ability to deal with ratio and proportion.	2.81	3.89
3	Items relating to attitudes toward difficulties in teaching ratio and proportion.	3.22	3.25
6	Items relating to attitudes toward the importance of teaching ratio and proportion.	4.62	4.59

The quantitative analysis of the participants' performance on the PR questionnaire indicates a very significant progress from pre- to post-test: 45% to 90%. The range on the pre-test was 24%–74% and on the post-test 80%–100%. On each one of the subtests: Rate (5 items), Ratio (5 items), and Scaling (5 items) the improvement was remarkable, 69% to 97%, 30% to 88%, and 36% to 86% accordingly. It should be noted that the assessment tasks are different from those appearing in standard tests. They stem from familiar situations such as: buying soft drinks, riding bicycles, population density, bussing to school, and a visit to the photo shop. Moreover, the problems and the situations are different from those presented within the investigative tasks during the teaching sessions. Even though the participants in this study started with lower achievement than in the previous study (Ben-Chaim *et al.*, 2002), their improvement was much better. Another indication is a better quality of the explanations provided by the participants in this study regarding their methods of solving the problems, especially following instruction.

The PR questionnaire included six exercises in fractions in order to examine if the students had difficulties in comparing fractions and/or working with "naked" numbers/fractions similar to those that appeared within the verbal problems. In this part, there were no differences found between pre and post testing results. The students had no special difficulties in solving the fraction exercises.

The in-class observations and the in-depth interviews with the representative sample of the participants strongly supported the findings, suggesting that an implementation of our model of teaching the PR topic leads not only to acquiring mathematical content knowledge and pedagogical-didactical knowledge, but also to a profound change in the pre-service teachers' opinions regarding teaching and learning mathematics. For example, the interviewees indicated that they liked the strategy of "not just providing the answer, because it is possible to get there through a wrong procedure, hence one needs to add verbal explanations", or "I liked the strategy of inquiry, challenges, cooperation with colleagues, the tasks were very interesting and can be presented to school children, the theoretical materials (articles and research reports) were very helpful", or "The activities, the theoretical material, and the guidance during the teaching sessions helped me to move from technical knowledge to a knowledge of understanding". It is interesting to note that most of the participants felt that the teaching style of the PR course caused them to construct their understanding of the PR concepts through an enjoyable and efficient process. Nevertheless, they also felt that the course should be longer, since they were not confident enough in their knowledge.

The most important finding of our study is that many of the student teachers who participated in the PR college courses attempted to present our authentic tasks to their pupils in their first years of teaching. Usually, they ask for our help in preparing the class activities and are willing to invite us to observe them and provide them with feedback on their presentations.

CONCLUSION

The findings of this study indicate that the type of authentic investigative PR activities developed and used in this study were found to be suitable, effective and contributory to pre-service elementary mathematics teachers for acquiring and understanding the topics of ratio and proportion. Through implementation of our PR teaching model, using hands-on experience with a variety of authentic PR tasks, the student teachers gained insight regarding the relationship between a task and the kind of thinking that task requires of students. This is in accord with Stein & Lane's (1996) appreciation of the influence tasks have on the learning of mathematics.

In general, the particular method of designing such authentic mathematical tasks, being plausible and within reach, encourages the pre-service teachers to implement similar tasks in their own classrooms on a regular basis, when they start their own teaching. Ball (2000) claims that "Acquiring the ability to think with precision about mathematical tasks and their use in class can equip teachers with more developed skills in the ways they select, modify, and enact mathematical tasks with their students" (p. xii). These skills are important ones for teachers to develop, so much so that Ball considers them "a core domain of teachers' work" (p. xii).

Following learning the ratio and proportion topics through the authentic tasks, the pre-service teachers not only significantly improved their content and pedagogical knowledge, but also positively changed their attitudes toward mathematics in general and all the components and aspects of ratio and proportion specifically.

Given the evidence for weakness of pre-service elementary mathematics teachers (and in many cases of in-service teachers too) in content and pedagogical knowledge of the proportional reasoning topic, it is highly recommended that at least one semester dealing with this important topic by conducting authentic PR tasks similar to those proposed in the present study be included in their educational training programs. In fact, the authors of this article have recently completed writing a book for college mathematics teacher educators which includes the theoretical and practical material needed — authentic tasks with explanations and extensions — for teaching the PR topic at college level to pre-service mathematics teachers and in in-service mathematics teacher training (Ben-Chaim, Keret & Ilany, 2006).

REFERENCES

Alcaro, P. C.; Alston, A. S. & Katims, N. (2000). Fractions Attack! Children Thinking and Talking

Teach. Child. Math. **6 (9)**, 562–567. MATHDI **2002d.**03121

Ball, D. (2000). Foreword. In: Mary K. Stein, Margaret M. Smith, Marjorie A. Henningsen & Edward A. Silver (Eds.), *Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development* (pp. ix–xiv). New York: Teacher College Press. ERIC ED442659

Ben-Chaim, D.; Fay, J. T., Fitzgerald; M. W., Benedetto, C. & Miller, J. (1998). Proportional reasoning among 7th grade students with different curricular experiences. *Educ. Stud. Math.* **36(3)**, 247–273. MATHDI **1999c.**01418

Ben-Chaim, D.; Ilany, B. & Keret, Y. (2001). Ratio and Proportion — Mathematical and pedagogical knowledge of pre- and in-service elementary teachers, before and after learning these topics by a new approach. Research Report (In Hebrew). Tel-Aviv, Israel: Mofet Publication.

_____ (2002). Mathematical and pedagogical knowledge of pre- and in- service elementary teachers, before and after experience in proportional reasoning activities. In: Anne D., Cockburn & Elena Nardi (Eds.), *Proceedings of the 26th Annual Meeting of the International Group for the Psychology of Mathematics Education (held at Norwich, England; July 21–26, 2002), Volume 2* (pp. 81–88). Available at http://www.eric.ed.gov/ERICDocs/data/ericdocs2sql/content_storage_01/0000019b/80/29/d4/dc.pdf

Ben-Chaim, D.; Keret, Y. & Ilany, B. (2006). *Ratio and Proportion – Research and Teaching in Mathematics Teacher Training* (In Hebrew). Tel-Aviv, Israel: Mofet and Ach Publication.

_____ (2007). Designing and implementing authentic investigative proportional reasoning tasks: the impact on pre-service mathematics teachers' content and pedagogical knowledge and attitudes. *J. Math. Teach. Educ.* **10(4–6)**, 333–340. ERIC EJ785267

Cramer, K.; Post, T. & Currier, S. (1993). Learning and teaching ratio and proportions: Research implications. In: D. T. Owens (Ed.), *Research Ideas for the Classroom, Middle Grades Mathematics* (pp. 159–178). New York: MacMillan Publishing Company. ERIC ED404143

De Bock, D.; Verschaffel, L. & Janssens, D. (1998). The Predominance of the Linear Model in Secondary School Students' Solutions of Word Problems Involving Length and Area of Similar Plane Figures. *Educ. Stud. Math.* **35(1)**, 65–83. ERIC EJ561805

Fischbein, E.; Jehiam, R. & Cohen, D. (1994). The irrational number and the corresponding epistemology and obstacles. In: Joao Pedro da Ponte & Joao Filipe Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education (PME-18) held at Lisbon, Portugal: University of Lisbon; July 29–August 3, 1994, Vol. 2*, (pp. 352–359) . ERIC ED383537 Available at http://www.eric.ed.gov/ERICDocs/data/ericdocs2sql/content_storage_01/0000019b/80/13/fa/77.pdf

Freudenthal, H. (1978). Weeding and Sowing: A Preface to a Science of Mathematical Education. Dordrecht, Netherlands: Riedel. MATHDI **1979b.**00145

_____ (1983). Didactical Phenomenology of Mathematical Structures. Dordrecht, Netherlands: Riedel. MATHDI **1984c.**02648

Fried, M. N. & Amit, M. (2005). A Spiral Task as a Model for In-service Teacher Education. *J. Math. Teach. Educ.* **8(5)**, 419–436. MATHDI **2006a.**00086

Greeno, J. G. (1994). Some further observations of the environment/model metaphor. *J. Res. Math. Educ.* **25(1)**, 94–99. MATHDI **pre**02307304

Harel, G. & Confrey, J. (1994). The Development of Multiplicative Reasoning in the Learning of Mathematics. Albany, NY: SUNY Press. MATHDI **1994f.**02496

Hoffer, A. (1988). Ratios and proportional thinking. In: T. Post (Ed.), *Teaching mathematics in grades K-8: Research based methods* (pp. 285–313). Boston: Allyn and Bacon. ERIC ED286753

Ilany, B.; Keret, Y. & Ben-Chaim, D. (2004). Implementation of a Model Using Authentic Investigative Activities for Teaching Ratio & Proportion in Pre-Service Elementary Education. In: Marit Johnsen Høines & Anne Berit Fuglestad (Eds.), *Proceedings of the 28th international conference of the International Group for the Psychology of Mathematics Education (PME-28) held at Bergen, Norway: Bergen University College; July 29–18, 2004, Vol. 3*, (pp. 81–88). MATHDI **2007c.**00052 ERIC ED489557 Available at
http://www.univie.ac.at/EMIS/proceedings/PME28/RR/RR032_Ilany.pdf
http://www.eric.ed.gov/ERICDocs/data/ericdocs2sql/content_storage_01/0000019b/80/1b/ba/ef.pdf

Jaworski, B. (2005). Tools and Tasks for Learning and Meta-Learning. *J. Math. Teach. Educ.* **8**, 359–361.

Keret, Y. (1999). Change processes in adult proportional reasoning: Student teachers and primary mathematics teachers, after exposure to ratio and proportion study unit. In: Orit Zaslavsky (Ed.), *Proceedings of the Conference of the International Group for the Psychology of Mathematics* Education *(PME-23) held at Haifa, Israel July 24– 30, 1999, Vol. 3*, (pp. 145–152). MATHDI **2000a.**00380 ERIC ED436403 Available at
http://www.eric.ed.gov/ERICDocs/data/ericdocs2sql/content_storage_01/0000019b/80/15/f9/0e.pdf

Keret, Y.; Ben-Chaim, D. & Ilany, B. (2003). A model using authentic investigative activities for teaching ratio and proportion in pre-service elementary education. *Proceedings of SEMT '03 - International Symposium Elementary Math's Teaching (pp. 94–99)*, Prague, Czech Republic.

Lawton, C. A. (1993). Contextual factors affecting errors in proportional reasoning. *Journal for Research in Mathematics Education* **24(5)**, 460–466. ERIC EJ480148

Leinhardt, G.; Young, K. M. & Merriman, J. (1995). Integrating professional knowledge: The theory of practice and the practice of theory. *Learning and Instruction* **5**, 401–408.

National Council of Teachers of Mathematics (NCTM) (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: NCTM. MATHDI **1996f.**03595

_____ (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM. MATHDI **1999f.**03937

Post, T. R.; Behr, M .J. & Lesh, R. (1988). Proportionality and the development of prealgebra understanding. In: A. Coxford and A. Schute (Eds.), *The Ideas of Algebra, K-12, 1988 Yearbook of the National Council of Teachers of Mathematics, Reston, VA.: The Council* (pp. 78–90).

Sowder, J.; Armstrong, B.; Lamon, S.; Simon, M.; Soweder, L. & Tompson, A. (1998). Educating Teachers to teach Multiplicative Structures in the Middle Grades. *J. Math. Teach. Educ.* **1(2)**, 127–155. MATHDI **1999c.**01765

Stein, M. K. & Lane, S. (1996). Instructional Tasks and the Development of Student Capacity to Think and Reason: An Analysis of the Relationship Between Teaching and Learning in a Reform Mathematics Project. *Educational Research and Evaluation* **2**, 50–80.

Tourniaire, F. & Pulos, S. (1985). Proportional reasoning: a review of the literature. *Educ. Stud. Math.* **16**(2), 181–204. MATHDI **1985k.**03817

Tracy, D. & Hague, M. (1997). Toys 'R' Math. *Mathematics Teaching in the Middle School* **2(3)**, 140–145. MATHDI **1997e.**03076

APPENDIX 1.

AUTHENTIC INVESTIGATIVE PROPORTIONAL REASONING ACTIVITY – RATE: ESTIMATING THE SIZE OF A CROWD

Stage 1 of the activity

Figure 1a. Estimating the Size of a Crowd 1

Usually, journalists like to estimate the number of people who participate in demonstrations, parades and festivals. For example, in one of the TV news editions, a political demonstration was reported. The TV reporter stood within the crowd and told the audience that the "town square is full with demonstrators; at least 200,000 people are here and in the adjacent streets". In parallel, the radio narrator reported: "the police announced that there are about 100,000 people in the demonstration and that the order is kept strictly".

Discuss the following points:

Why, in spite the fact that both reporters report from the same place, there are significant differences regarding the estimation of the number of people in the crowd?

In your opinion, how the reporters estimate the number of people in a crowd?

Suggest a method, by which the reporters can get a better estimation for the number of people in a crowd — get help from the second stage of the problem.

Stage 2 of the activity

Sometimes the size of a crowd is estimated from aerial photographs. Imagine that the illustration below is an aerial photograph of a crowd at a rally. Each dot represents one person.

Estimate how many people attended the rally. Explain the method you used to arrive at your answer.

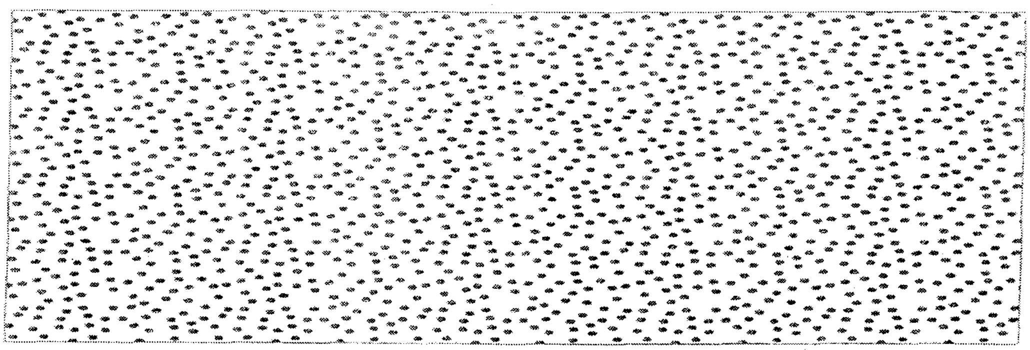


Figure 1b. Estimating the Size of a Crowd 2

APPENDIX 2

AUTHENTIC INVESTIGATIVE PROPORTIONAL REASONING ACTIVITY – RATIO: PREFERENCE OF COLA

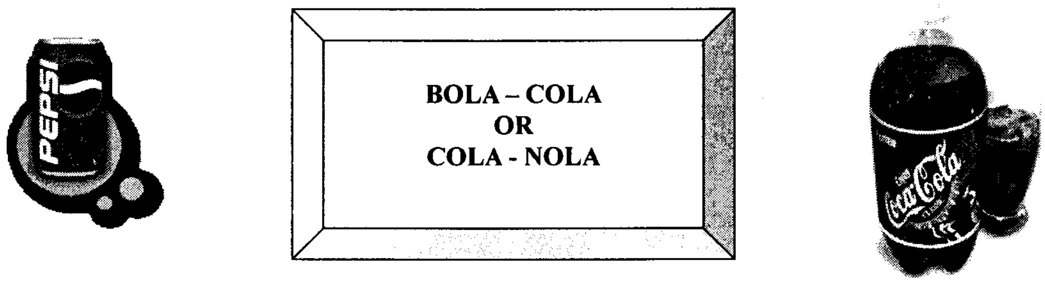


Figure 2. Preference of Cola

The following results are related to a test of preference between BOLA-COLA and COLA-NOLA:

The ratio of those who preferred BOLA-COLA than COLA-NOLA is 3 to 2.

The numbers of those who preferred BOLA-COLA than COLA-NOLA are in ratio of 17,139 to 11,426.

5,713 more participants preferred BOLA-COLA than COLA-NOLA.

Decide if the above three statements are necessarily extracted from the same survey? Explain!

Which statement describes most accurately the results of comparison between BOLA-COLA and COLA-NOLA? Explain!

If you need to advertise the results, which statement seems to be the most effective for advertisement? Why?

Suggest other possible ways for comparison between the popularity results of the two kinds of cola.

APPENDIX 3

AUTHENTIC INVESTIGATIVE PROPORTIONAL REASONING ACTIVITY – RATIO: SHARING PIZZA

Every month Danny's friends meet at a restaurant for a pizza party. Danny as usual is late, but his friends like him a lot and they are waiting for him to come. They reserved for him a seat in each of the two tables: Table 1 and Table 2.

Table 1

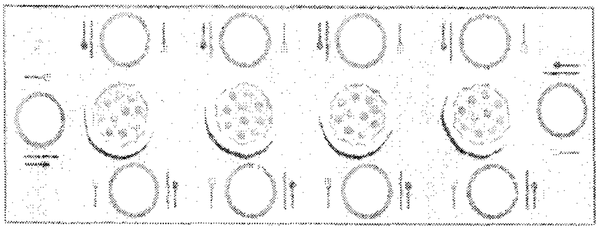


Table 2

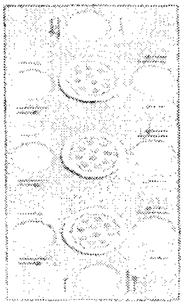


Figure 3. Sharing Pizza

Finally he arrived and then he had to decide where to seat: should he join his friends at Table 1 in which there are 4 large pizzas and 9 persons or Table 2 in which there are 3 large pizzas and 7 persons?

Problems for Discussion

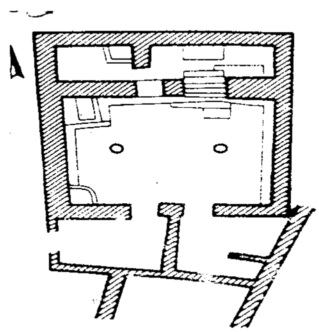
What is your suggestion? Which table should Danny choose? Explain your reasoning.

The ratio of large tables (Table 1 with 10 seats) to small tables (Table 2 with 8 seats) in the restaurant is 9 to 5. There are exactly enough seats for 390 persons. How many tables of each kind are in the restaurant?

APPENDIX 4

AUTHENTIC INVESTIGATIVE PROPORTIONAL REASONING ACTIVITY – SCALING: THE TASK OF BET-SHEAN TEMPLE

Given is a sketch of Bet-Shean Temple (BST) restoration. The sketch is drawn by a scale of 1:200



(It is taken from the "new encyclopedia of Archeological excavations in the land of Israel").

Figure 4a. A Sketch of Bet-Shean Temple

Find:

a) What are the dimensions of the main hall of BST in reality?
b) What is the ratio between the perimeters of the main hall in the sketch to the perimeter of the main hall in reality? What is the connection between these ratios to the given scale?
c) What is the ratio between the perimeters of the main hall in reality to the perimeter of your classroom?
d) What is the ratio between the areas of the main hall in the sketch to its area in reality?
e) If the sketch of BST was given by a scale of 1:400, how the dimensions of the main hall in reality (the length, the width, the perimeter, and the area) would be affected compared to the previous sketch? Explain your answer.

If the sketch of BST is drawn by a scale of 1:100, what is the ratio between the areas of the main hall in the new sketch to the area of the main hall in the original sketch? Explain your answer.

The sketch is enlarged in an enlargement machine such that the area of the main hall after the enlargement is 243cm square. Find what is the scale of the new sketch? Explain your answer.

The Task of BST—Second Level

A model of the BST was constructed by a scale of 1:50 in one of the museums. In the backyard of the BST there is a rectangular pool used to store water. Its dimensions in the model are: 4cm by 6cm by 8cm.

(1) How many meter cube of water could be stored in this pool?
(2) What is the ratio between the volumes of the pool in the model to the volume of

the pool in reality? How is this ratio connected to the given scale? Explain your answer.

In the backyard of the BST model there is an additional rounded water well that its diameter is 5cm and its depth is 6cm.

(1) How many meter cube of water could be stored in this well?
(2) What is the ratio between the volumes of the well in the model to the volume of the well in reality? How is this ratio connected to the given scale? Explain your answer.

Figure 4b. A Rounded Water Well

APPENDIX 5

AUTHENTIC INVESTIGATIVE PROPORTIONAL REASONING ACTIVITY – INDIRECT RATIO: WHAT IS THE WEIGHT OF THE METEOROID?

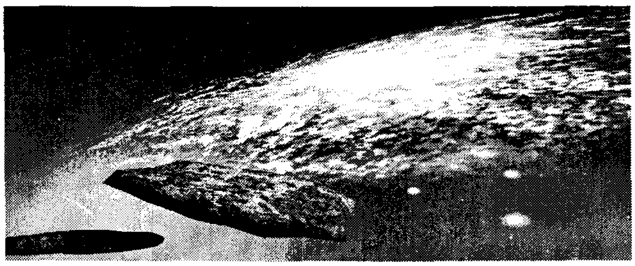


Figure 5. A Meteoroid

University scientists acquired a small meteoroid that was brought from the moon. When they came to the laboratory to weigh the meteoroid, they found out that the special device for weighing small objects was sent for new calibration. Their curiosity led them to look for another way, in order to find out the weight of the meteoroid (W).

In the laboratory, there was an unbalanced scale (with uneven arms $a \neq b$). After a few minutes of consultation, the scientists found a way to use the unbalanced scale to weigh the small meteoroid (W).

They acted in the following way:

Figure a:

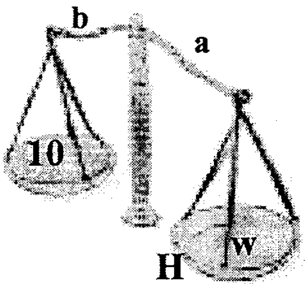


Figure b:

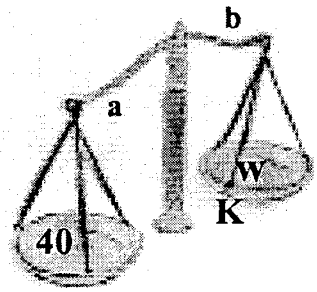


Figure 5-1. Scales

They placed the meteoroid (W) on one of the balances marked by H (see Fig. a) and found a weight of 10 gr. Then, they placed the meteoroid (W) in the second balance marked by K (see Fig. b) and found a weight of 40 gr. They made some calculations and were very happy to find the weight of the meteoroid (W).

A. Try to explain how the scientists found the weight of the meteoroid (W)?
B. Write down the reasoning that guided you in finding the solution.
C. Suggest a way to explain to elementary school students, the meaning of the concept "equilibrium" ("balance") of a scale, or the meaning of a "balanced scale".