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BV – 족 계수를 갖는 푸리에 급수의 $L^1(T^1)$ – 수렴성에 관하여

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On $L^1(T^1)$ – Convergence of Fourier Series with BV – Class Coefficients

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Abstract

In general the Banach space $L^1(T^1)$ doesn't admit convergence in norm. Thus the convergence in norm of the partial sums can not be characterized in terms of Fourier coefficients without additional assumptions about the sequence $\{\hat{f}(\xi)\}$. The problem of $L^1(T^1)$-convergence consists of finding the properties of Fourier coefficients such that the necessary and sufficient condition for (1.2) and (1.3). This paper showed that let $\{a_k\}\in BV$ and $\xi\Delta a_\xi = o(1)$, $\xi\to\infty$. Then (1.1) is a Fourier series if and only if $\{a_k\}\in\Gamma$.

Key Words : Convergence of Fourier series, Fourier coefficients

1. Preliminaries

Let $L^1(T^1)$ denote the $L^1(T^1)$ space of all complex-valued Lebesgue integrable functions on $T^1=\{t\in R|-\pi\leqq t\leqq\pi\}$ Also, its usual norm is

$$\|f\|_{L^1(T^1)}=\frac{1}{2\pi}\int_{T^1}|f(t)|dt$$

where $f(t)\in L^1(T^1)$, $t\in T^1$.

The Fourier series $S[f]$ of a function $f(t)$

$$S[f]\sim\sum_{|n|<\infty}\hat{f}(n)e^{int}(n\in Z).$$

Also,

$$\hat{f}(n)=\frac{1}{2\pi}\int_{T^1}f(t)e^{-int}dt, |n|<\infty$$

are Fourier coefficients of f.

The n-th partial sum of $S[f]$ will be denoted by

$$Sn(f)=Sn(f;t)=Sn(f(t))=\sum_{|k|\leq n}\hat{f}(k)e^{ikt}.\ n=0,\ 1,\ 2,...$$

The Dirichlet kernel is defined by

$$D_n(t)=\sum_{|k|\leq n}e^{ikt}=\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{1}{2}\right)t}.$$

Note that $S_n(f)=(D_n*f)(t)$.

The Cesaro summable arithmetic mean (C,1) will be written as

$$a_n(t)=\frac{(S_0+S_1+\cdots+S_n)(t)}{(n+1)}=\frac{1}{n+1}\sum_{k=0}^{1}S_k(t).$$

The problem of $L^1(T^1)$-convergence of the Fourier cosine series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nt \tag{1.1}$$

can be approached by finding the properties of Fourier coefficients such that the necessary and sufficient condition for

$$\|S_n(f)-f\|_{L^1(T^1)}=o(1), n\to\infty, \tag{1.2}$$

is given in the form

$$\hat{f}(n)\log|n|=o(1), |n|\to\infty. \tag{1.3}$$

The partial sums of a real cosine and sine series

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will be denoted by

$$S_n(t)=\frac{a_0}{2}+\sum_{k=1}^{n}a_k\cos kt \qquad (1.4)$$

and

$$S_n(t)=\sum_{k=1}^{n}b_k\sin kt,$$

respectively, where $t\in(0,\pi)$. The Banach space in the real case will be denoted by $L^1(0,\pi)$, and norm by $\|\cdot\|_{L^1(T^1)}$.

2. Some Results of $L^1(T^1)$ -Convergence Classes

$L^1(T^1)$-convergence of the Fourier cosine series has been studied. There are two kinds of results in the classical and neoclassical development. In the first kind, the conditions on the coefficients are strong enough to guarantee the integrability and pointwise convergence, hence the Fourier character of the considered series. In the second kind, the conditions on the coefficients are not strong enough to guarantee the integrability. Thus, the integrability that is the Fourier character of the considered series has to be assumed. In both cases, these conditions define the so called $L^1(T^1)$-convergence classes. Namely the classes of Fourier coefficients such that the convergence in $L^1(T^1)$-*norm* is guaranteed.

Proposition 2.1. (Young[1]). Let $\{a_n\}_{n=0}^{\infty}$ be a null sequence and $\{a_n\}_{n=0}^{\infty}(\Delta^2 a_n\geqq 0$, for *every* n). (1.1) is the Fourier series of some function and the relation (1.2) being satisfied if and only if

$$a_n\log n=o(1),\ n\to\infty. \qquad (2.1)$$

W. H. Young found that (2.1) is a necessary and sufficient condition for cosine series with convex ($\Delta^2 a_n\geqq 0$) coefficients.

Proposition 2.2. (Kolmogorov[2]). If $a_n=o(1)$, $n\to\infty$ and the series

$$\sum_{n=0}^{\infty}(n+1)|\Delta^2 a_n|<\infty,$$

where

$\Delta^2 a_n=\Delta a_n-\Delta a_{n+1}$, $\Delta a_n=a_n-a_{n+1}$, $n=1, 2,...,$

then the cosine series (1.1) is the Fourier series of some function and the relation (1.2) being satisfied if and only if (2.1).

A. N. Kolmogorov extended that result to the cosine series with quasi-convex

$\left(\sum_{n=0}^{\infty}(n+1)|\Delta^2 a_n|<\infty\right)$ coefficients. The null sequences that satisfy Kolmogorov's condition $\sum_{n=0}^{\infty}(n+1)|\Delta^2 a_n|<\infty$ are called quasi-convex.

Proposition 2.3. (Sidon[3]). Let $\{a_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences such that $|a_n|\leqq 1$, for every n. If $\sum_{n=1}^{\infty}|p_n|<\infty$. and

$$a_n=\sum_{k=n}^{\infty}\frac{p_k}{k}<\sum_{i=n}^{k}a_i \quad \text{n = 1, 2,...,}$$

then the cosine series (1.1) is the Fourier series of some function f and the relation (1.2) being satisfied if and only if (2.1).

The class S is defined by reformulating Sidon's condition: a null sequence $\{a_n\}_{n=0}^{\infty}$ belongs to the class S if there exists a monotonicly decreasing sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty}A_n$ converges, and $|\Delta a_n|\leqq A_n$ as $a_n\to 0$, for every n.

Class S is also the extension of the class of the quasi-convex sequences.

Proposition 2.4. (Telyakovskii[4]). Let $\{a_n\}_{n=0}^{\infty}\in S$. Then the cosine series (1.1) is the Fourier series of some function f and the relation (1.2) being satisfied if and only if (2.1).

The importance of S. A. Telyakovskii's contributions are twofold. First, he expressed Sidon's conditions in a succinct equivalent form. Secondly, he showed that the class S is also a class of $L^1(T^1)$-convergence which in turn led to numerous, more general results.

Proposition 2.5. (Fomin[5]). Let $a_n=o(1)$, $n\to\infty$. If for some $1<p\leqq 2$ series

$$\sum_{n=1}^{\infty}\left(\frac{|\Delta a_n|^p+|\Delta a_{n+1}|^p+\cdots}{n}\right)^{1/p}<\infty,$$

then the cosine series (1.1) is the Fourier series of some function f and the relation (1.2) being satisfied if and only if (2.1).

G. A. Fomin showed that for cosine series with monotone coefficients (2.1) is a sufficient condition and

$$a_{2n}^2/a_n \log n = o(1),\ n \to \infty,$$

is necessary one. It is easy to see that $\Delta a_n \geqq 0$ and (2.2) imply (2.1). Hence for cosine series with monotone coefficients such that (2.2) holds, the condition (2.1) is necessary and sufficient for $L^1(T^1)$-convergence. This theorem is a generalization of theorem of Telyakovskii[4].

Proposition 2.6 (Fomin and Telyakovskii[6]). Let $\{a_n\}_{n=0}^{\infty}$ be a quasi-monotone sequence such that

$$a_n/n^a \downarrow,\ n \to \infty, \text{ for some } a \geqq 0$$

If the cosine series (1.1) is the Fourier series of some function f, then the relation (1.2) being satisfied if and only if (2.1) .

G.A. Fomin and S.A. Telyakovskii[6] obtained a similar result for Fourier series with quasi-monotone coefficients. Namely, a sequence $\{a_n\}$, $a_n \to 0$, $n \to \infty$, is called quasi-monotone if for some $\alpha > 0$ the sequence a_n/n^a is monotonically decreasing (for $\alpha = 0$, it is monotone).

3. The Main Theorem

The problem of $L^1(T^1)$-convergence consists of finding the properties of Fourier coefficients such that the necessary and sufficient condition for (1.2) and (1.3).

The classical A. N. Kolmogorov[2] results was extended by S. A. Telyakovskii[4] to a class S of a Fourier coefficients that contains the class of quasi-convex coefficients introduced by Kolmogorov. The class of all convex sequences is a subclass of all null sequences of bounded variation(BV) class. From S. Sidon[3]'s conditions it follows that the sequence $\{a_n\}_{n=0}^{\infty} \in BV$. If there exists a monotonicly decreasing sequence $\{A_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} A_k$ converges, and

$$|\Delta a_k| \leqq A_k \text{ as } a_k \to 0, \text{ for every } k,$$

then a null sequence $\{a_k\}_{k=0}^{\infty}$ belongs to the class S i.e. $\{a_k\} \in S \Rightarrow BV$. From this, the series $\sum_{k=1}^{\infty} |\Delta a_k|$ converges.

To prove theorem 3.3, we use the following lemma 3.1 and 3.2.

Lemma 3.1. Let $\{a_n\}_{n=0}^{\infty} \in BV$. If there exists $\delta(\varepsilon) > 0$, for every $\varepsilon > 0$, independent of n, such that

$$\int_0^{\delta} |\Delta a_1 D_1(x) + \Delta a_2 D_2(x) + \cdots| dx < \varepsilon \quad \text{for every } n,$$

then the cosine series (1.1) is the Fourier series of some function f and the relation (1.2) being satisfied if and only if (2.1).

Lemma 3.2. *Let*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series with quasi--monotone coefficients. Then

$$\|S_n - \sigma_n\|_{L^1} = o(1),\ n \to \infty, \tag{3.1}$$

if and only if

$$(a_n + b_n) \log n = o(1),\ n \to \infty.$$

Proof. we need to show that $\{a_n\}$is a quasi-monotone sequence such that (2.1).

Then $\frac{1}{n}\sum_{k=1}^{n} k|\Delta a_k| \log k = o(1)$, $n \to \infty$. It suffices to show the sufficient condition.

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{n-1} \Delta(k a_k) \sum_{j=1}^{k} \frac{1}{j} + a_n \sum_{j=1}^{n} \frac{1}{j}$$

we obtain

$$\frac{1}{n}\sum_{k=1}^{n-1} k \Delta a_k \sum_{j=1}^{k} \frac{1}{j} = \frac{1}{n}\sum_{k=1}^{n} a_k - a_n \sum_{j=1}^{n} \frac{1}{j} + \frac{1}{n}\sum_{k=1}^{n-1} a_{k+1} \sum_{j=1}^{k} \frac{1}{j}$$

The quasi-monotonicity of the $\{a_n\}$ yields

$$|\Delta a_k| \leqq \Delta a_k + 2\alpha a_k/k \text{ for some } \alpha > 0.$$

Hence,

$$\frac{1}{n}\sum_{k=1}^{n-1} k|\Delta a_k| \sum_{j=1}^{k} \frac{1}{j}$$
$$\leq -a_n \sum_{j=1}^{n-1} \frac{1}{j} + \frac{1}{n}\sum_{k=1}^{n} a_k + \frac{2a}{n}\sum_{k=1}^{n-1} a_k \sum_{j=1}^{k} \frac{1}{j} + \frac{1}{n}\sum_{k=1}^{n-1} a_{k+1} \sum_{j=1}^{k} \frac{1}{j}\ .$$

Each term on the right-hand side is $o(1)$ as $n \to \infty$. Now that

$$\|S_n - \sigma_n\| = \frac{1}{n+1}\left\|\sum_{k=1}^{n} k a_k \cos kx\right\|$$
$$\le \frac{1}{n+1}\left\|\sum_{k=1}^{n-1} k\Delta a_k\left[D_k(x) - \frac{1}{2}\right]\right\|$$
$$+\frac{1}{n+1}\left\|\sum_{k=1}^{n-1} k\Delta a_{k+1}\left[D_k(x) - \frac{1}{2}\right]\right\| + a_n\left\|D_n(x) + \frac{1}{2}\right\|$$

where $D_n(x)$ is the Dirichlet kernel.

Or, since $\left\|D_n(x) - \frac{1}{2}\right\| = 0$ (log n), for some $B > 0$

$$B\|S_n - \sigma_n\| \le \frac{1}{n+1}\sum_{k=1}^{n-1} k|\Delta a_k|\log k$$
$$+\frac{1}{n+1}\sum_{k=1}^{n-1} a_{k+1}\log k + a_n \log n.$$

From this sufficient condition it follows that (3.1). For the necessary condition notice that

$$\|S_n - \sigma_n\| + \|\sigma_n - f\| \ge \|S_n - f\| \ge C\sum_{k=1}^{n}\frac{a_{n+k}}{k}$$

where C is a positive constant. Since $f \in L^1$, we have that

$$\|\sigma_n - f\| = o(1),\ n \to \infty. \tag{3.2}$$

Assume that (3.1). Then

$$\sum_{k=1}^{n}\frac{a_{n+k}}{k} = o(1),\ n \to \infty.$$

From the fact that the sequence $\{a_n\}$ is quasi-monotone, we have

$$\sum_{k=1}^{n}\frac{a_{n+k}}{k} \ge n^a \sum_{k=1}^{n}\frac{1}{k}\frac{a_{n+k}}{(n+k)^a} \ge \frac{n^a a_{2n}\log n}{(2n)^a} = \left(\frac{1}{2}\right)^a a_{2n}\log n.$$

Thus, (2.1) holds.

The class of null sequences that satisfy the condition that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, independent of n, such that

$$\int_0^\delta |\Delta a_1\, D_1(x) + \Delta a_2\, D_2(x) + \cdots| dx < \varepsilon \ \textit{for every } n,$$
$$= \int_0^\delta \left|\sum_{k=n}^{\infty}\Delta a_k\, D_k(x)\right| dx < \varepsilon,$$

is denoted by Γ.

The proof of proposition 2.1 is based on the use of modified cosine sums defined by

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n}\Delta a_k + \sum_{k=1}^{n}\left[\left(\sum_{i=k}^{n}\Delta a_i\right)\cos kx\right]$$
$$= S_n(x) - a_{n+1}D_n(x)$$

We consider the class Γ in the following theorem 3.3.

Theorem 3.3. Let $\{a_k\} \in BV$ and let

$n\Delta a_n = o(1),\ n \to \infty$

Then (1.1) is a Fourier series if and only if $\{a_k\} \in \Gamma$

Proof. Let $\{a_k\} \in BV$. Then $\{a_k\} \in \Gamma$ is always a sufficient condition for $f \in L^1(0,\pi)$. Thus it remains to show that under the condition of this theorem $f \in L^1(0,\pi)$, it follows that . Notice that for

$$\|g_n - \sigma_n\|_{L^1(T^1)} = o(1),\ n \to \infty,$$

from (3.2) it follows

$$\|g_n - f\|_{L^1(T^1)} = o(1),\ n \to \infty. \tag{3.3}$$

But $f \in L^1$ implies that (3.2), and (3.3), if and only if $\{a_k\} \in \Gamma$. However $\{a_k\} \in BV$. Hence from

$$\|g_n - \sigma_n\|_{L^1(T^1)} = O\left(\frac{1}{n}\sum_{k=1}^{n} k|\Delta a_k|\right),\ n \to \infty.$$

it follows that $\|g_n - \sigma_n\|_{L^1(T^1)} = o(1),\ \ n \to \infty.$

This theorem prove that (3.3) is equivalent to

$\{a_n\}_{n=0}^{\infty} \in \Gamma \cap\ BV$

Hence, with the BV class, condition Γ is a sufficient condition for the integrability of the cosine series. proposition 2.1 contains all previous results as special cases.

Corollary 3.4. If $\{\hat{f}(n)\}$ is even and satisfies

$$\frac{1}{n}\sum_{k=1}^{n} k|\Delta\hat{f}(k)| = o(1),\ n \to \infty,$$

and

$$n\Delta\hat{f}(n) = o(1),\ n \to \infty,$$

then (1.2) being satisfied if and only if (2.1).

Corollary 3.5. Let $f \in L^1(T)$. If for some $1 < p \le 2$

$$\lim_{\lambda \to 1} \overline{\lim_{n \to \infty}} \sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta \hat{f}(k)|^p = 0,$$

then (1.2) is equivalent to (1.3).

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