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Statistical Inference of Some Semi-Markov Reliability Models[1]

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Abstract. The objective of this paper is to discuss the stochastic analysis and the statistical inference of a three-states semi-Markov reliability model. Using the maximum likelihood procedure, the parameters included in this model are estimated. Based on the assumption that the lifetime and repair time of the system are generalized Weibull random variables, the reliability function of this system is obtained. Then, the distribution of the first passage time of this system is derived. Many important special cases are discussed. Finally, the obtained results are compared with those available in the literature.
Key words: Maximum likelihood estimators, Semi-Markov model, System reliability, Active unit, Repair facility

1 INTRODUCION

Stochastic modeling is an important tool for reliability theory, probability models, social security policy analysis and many other different applications see for example Medhi (1982), Korolyuk and Swishchuk (1994) and Janssen and Manca (2002). There two types of random disturbances present. The first kind, termed measurement noise, arises because of imprecise measurement instruments, inaccurate recording systems and so on. The second kind can be termed system noise, in which the system itself is subjected to random disturbances. Stochastic models like generalized semi-Markov processes have a long history of application, but they do not provide primitives for modelling of concurrency aspects Kulkarni (1995). They also lack mechanisms for compositional specification. Thus models of larger systems tend to be very complex.

A Markov chain analysis can be used to describe patterns of deposition and conditional probability of occurrence of different rock types through transition probability matrices see for example Anderson and Goodman (1957). In these models, the stochastic process used to represent the disposition sequence typically is assumed to be homogeneous along the profile. Morkov chain models have also been used for subsurface modelling. The occurrence of lithologies is viewed as a stochastic

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process. The stochastic analysis of a semi-Markov reliability model is rarely investigated during the last two decades. For a more extensive overview of the reliability theory of repairable systems, see the well-known books Medhi (1982) and ,Korolyuk and Swishchuk (1994).

In this section, we will display some important definitions and properties of a semi-Markov process and its kernel. The evolution of many systems naturally ends as the first failure occurs, because external intervention is not practicable. These systems are non-repairable systems. For other systems, generally of high complexity, renewal possibilities exist, and their effectiveness therefore depends not only on their intrinsic reliability but also on the characteristics of maintenance and repair actions.

To take the repair actions into account, the concept of reliability must be reinterpreted as an overall capacity of the system to accomplish a specified task.

To define the discrete renewal process, consider the continuous time interval $(0, t)$. The number of renewals $N(t)$ occurring in this interval is a discrete stochastic process, called a renewal process. Once the characteristics of this process are known the reliability model, as predictions of the evolution of the system, can be made. Preventive maintenance is scheduled downtime, usually periodically, in which defined set of tasks, such as inspection and repair, replacement, cleaning, lubrication, adjustment and alignment, are preformed.

To discuss the stochastic analysis of our reliability model, we present some important definitions. A semi Markov process $\{X(t) : t \geq 0\}$ is a stochastic process in which changes of state occur according to a Markov chain and the time interval between two successive transitions is a random variable whose distribution depends on the state from which the transition takes place as well as the state to which the next transition takes place Korolyuk and Swishchuk (1994). Generally a semi-Markov process with discrete state space can be defined as a Markov renewal process Grabski(1999). Assuming that the state space S is finite, we can define the renewal kernel as follows:

Definition 1.1 *The stochastic matrix $Q(t) = [Q_{ij}(t); i, j \in S]$, $t \geq 0$ is said to be a renewal kernel if and only if the following conditions are satisfied:*

1. *The functions $Q_{ij}(t)$ are nondecreasing functions in t.*
2. *$\sum_{j\in S} Q_{ij} = G_i(t)$ are distribution functions in t.*
3. *$[Q_{ij}(+\infty) = P_{ij}, i, j \in S] = P$ is a stochastic matrix.*

Definition 1.2 *A two-dimensional Markov process $\{\xi_n, \vartheta_n, n \in N\}$ with values in $S \times [0, \infty)$ is called a Markov renewal process if and only if*

1. *$Q_{ij} = P\{\xi_{n+1} = j, \vartheta_{n+1} \leq t | \xi_n = i, \vartheta_n = t_n, \ldots, \xi_0 = i_0, \vartheta_0 = t_0\} = P\{\xi_{n+1} = j, \vartheta_{n+1} \leq t | \xi_n = i\}$.*
2. *$P\{\xi_0 = i, \vartheta_0 = 0\} = p_{i0}$.*

From this definition it is follows directly that the Markov renewal process is a homogeneous two declensional Markov chain such that the probabilities of transition depend only the discrete component (do not depend on the second components). In the Markov renewal process, the non-negative random variables $\vartheta_n, n \geq 1$, can defined as the interval between Markov renewal times:

$$\tau_n = \sum_{k=1}^{n} \vartheta_k, \ \ n \geq 1, \tau_0 = 0 \tag{1.1}$$

Now, let

$$\nu(t) := \sum_{n=1}^{\infty} I_{[0,t]}(\tau)$$

The process $\nu(t)$ is called a counting process. It determines the number of renewal times on the segment $[0, t]$.

In what follows, we will display some important concepts and definitions which will be used throughout this paper and details about these definitions can be found in Grabski (1999).

Definition 1.3 *A stochastic process* $\{X(t) : t \geq 0\}$ *where* $X(t) = \xi_{\nu(t)}$ *is called a semi-Markov process that generated by the Markov renewal process with initial distribution* $P_i^0 = p(\xi_0 = i)$ *and the kernel* $Q(t), t \geq 0$.

Since the counting process $\nu(t)$ keeps constant values on the half-interval $[t_n, t_{n+1})$ and is continuous from the right, then the semi-Markov process keeps also constant values on the half intervals $[\tau_n, \tau_{n+1})$: $X_n(t) = \xi_n$ for $t \in [\tau_n, \tau_{n+1})$. Moreover the sequence $\{X(\tau_n) : n \in N\}$ is a Markov chain with transition probability matrix $P = \{p_{ij} = Q_{ij}(\infty), i, j \in S\}$ that is called an embedded Markov chain. The concept of a Markov renewal process is a natural generalization of the concept of the ordinary renewal process given by a sequence of independent identically non-negative random variables $\theta_n, n \geq 1$. The random variables θ_n can be interpreted as lifetimes.

Now, using the definition (1.3), the following lemma can be formulated.

Lemma 1.4 *If* $\{X(t) : t \geq 0\}$ *is a semi-Markov process with renewal kernel*

$$Q(t) = Q_{ij}(t), i, j \in S, \ t \in [0, \infty)$$

then

$$P\{\xi_0 = i_0, \vartheta_0 = 0, \xi_1 = i_1, \vartheta_1 \leq u_1, \ldots, \xi_n = i_n, \vartheta_n \leq u_n\} = p_{i_0} \prod_{k=1}^{n} Q_{i_{k-1}i_k}(u_k) \tag{1.2}$$

This lemma will be used to construct the likelihood function of some semi-Markov reliability models.

2 LIKLIHOOD FUNCTION

Assuming that the semi-Markov renewal kernel of the reliability model depends upon a vector of unknown parameters $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_k)$. that is

$$Q(t|\underline{\theta}) = \{Q_{ij}(t|\underline{\theta}) : i, j \in S\}, \tag{2.1}$$

Our aim in this paper is to find both Maximum likelihood estimators of those unknown parameters, based on the realization of the semi-Markov process. Let us assume that there is a sequence of random observations $(i_0, t_0), (i_1, t_1), \dots, (i_n, t_n)$ of the random vector $(\xi_0, \vartheta_0), (\xi_1, \vartheta_1), \dots, (\xi_n, \vartheta_n)$. Suppose z denotes the observation $(i_0, t_0), (i_1, t_1), \dots, (i_n, t_n)$. We assume that there exist functions denoted by $q_{ij}(t|\theta), i, j \in S$ such that

$$Q_{ij}(t|\underline{\theta}) = \int_0^t q_{ij}(u|\underline{\theta}) du \tag{2.2}$$

Using lemma (1.4), the likelihood function for the given random observations of the semi-Markov process becomes

$$L(z; \underline{\theta}) = p_{i_0} \prod_{s=1}^{n} q_{i_{s-1} i_s}(t_s|\underline{\theta}) \tag{2.3}$$

In the Bayesian procedure, it is assumed that $\underline{\theta}$ is a vector of random variables. Then these random variables have a joint probability density function, say $g(\underline{\theta})$, called a joint prior probability distribution function of $\underline{\theta}$. If the loss incurred when the vector $\underline{\theta}$ of the unknown parameters estimated by $\hat{\theta}$ is quadratic, then the value of the Bayes estimator for θ_i becomes the posterior expectation, given by:

$$\hat{\theta}_i = E(\theta_i|z) = \int \theta_i g(\theta_i|z) d\theta_i, \ i = 1, 2, \dots k. \tag{2.4}$$

Now, we proceed to apply maximum likelihood procedure to obtain estimators of the unknown parameters included in a three-state semi Markov reliability model. It is assumed that the lifetime of the system has a generalized Weibull distribution with three parameters. Under the assumptions, that the life and repair times of the standby system with repair are generalized Weibull, the reliability function of the system is derived. The distribution of the first passage time of the system is obtained.

3 SEMI-MARKOV STANDBY MODEL

The semi-Markov process is a convenient tool to describe many reliability models. The model of this is slight modification of well known reliability model introduced by Barlow and Proschan (1965), Pogosian (1973). In order to describe a reliability model of a standby system with a repair facility, the following assumptions are adopted:

1. The system consists of one active unit, an identical spare, a switch and a repair facility.

2. When the operating unit fails, the spare is put in motion by the switch immediately.

3. The failed units can be repaired by the repair facility and the repair is fully restore the units. This means that the repaired element can be considered as new one.

4. The system fails when the active unit fails and repair has not been finished yet or when the active unit fails and the switch fails .

5. The lifetimes of the active units can be represented by independent and identical non-negative random variables ζ with probability density function $f(t), t \geq 0$.

6. The lengths of repair periods of the units can be represented by independent and identical non-negative random variable γ with the distribution function $H(t) = P\{\gamma \leq t\}$.

7. The event E denotes the switch-over as the active unit fails. Then the probability that the switch performs when required is represented by $P(E) = a$.

8. The whole system can also be repaired, and the failed system is replaced by a new identical one.

9. The replacing time is represented by a non-negative random variable k with distribution function $C(t) = P\{k \leq t\}$.

10. Finally, we assume that all the random variables described above are independent.

The underlying reliability model can be described by a semi-Markov process with three states. Under the model assumptions, the states of the prescribed system can be considered as follows:

1. The system failure represents the first state of the semi-Markov describing the model and denoted by (0).

2. The failed unit is repaired and the standby unit is operating represents the second state of the semi-Markov describing the model and denoted by (1).

3. Both active and standby units are "Up" represents the third state of the semi-Markov describing the model and denoted by (2)

Let $\tau_0^*, \tau_1^*, \tau_2^*, \ldots$ denote the instants when of the state of the system changes where $\tau_0^* = 0$ and let $\{Y(t) : t \geq 0\}$ be a stochastic process with state space $S = \{0, 1, 2\}$. This process keeps constant values on the half intervals $[\tau_n^*, \tau_{n+1}^*)$ and is continuous from the right. Therefore, it is not a semi-Markov process.

Let us define a new stochastic process as follows:

Assuming that $\tau_0 = 0$ and $\tau_n, n = 1, 2, \ldots$ represent the instants, when the components of the system failed or the whole system renewal. The stochastic process $\{X(t) : t \geq 0\}$ defined by

$$X(0) = 0, X(t) = Y(\tau_n) \, \text{for} \, t \in [\tau_n, \tau_{n+1}) \tag{3.1}$$

is a semi-Markov process and its kernel is given by the following matrix

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & 0 & Q_{02} \\ Q_{10} & Q_{11} & 0 \\ Q_{20} & Q_{21} & 0 \end{bmatrix} \tag{3.2}$$

The semi-Markov process $\{X(t), t \geq 0\}$ is completely specified by its semi-Markov kernel. Let us deduce the elements of the semi-Markov kernel as follows:

$$\begin{aligned}
Q_{02}(t) &= P\{X(\tau_{n+1}) = 2, \vartheta_{n+1} \leq t | \, X(\tau_n) = 0\} \\
&= P\{k \leq t\} = K(t), \\
\\
Q_{10}(t) &= P\{X(\tau_{n+1}) = 0, \vartheta_{n+1} \leq t | \, X(\tau_n) = 1\} \\
&= P\{\zeta \leq t, \gamma > \zeta\} + P\{\bar{A}, \zeta \leq t, \gamma < \zeta\} \\
&= \int_0^t [1 - H(t)] dF(t) + (1 - a) \int_0^t H(x) dF(x) \\
&= F(t) - a \int_0^t H(x) dF(x) \\
\\
Q_{11}(t) &= P\{X(\tau_{n+1}) = 1, \vartheta_{n+1} \leq t | X(\tau_n) = 1\} \\
&= P\{E, \xi_1 \leq t, \xi_2 > \xi_1\} = a \textstyle\int_0^t G(u) dF(u) \\
&= P\{E, \zeta \leq t, \gamma < \zeta\} = a \int_0^t H(x) dF(x) \\
\\
Q_{21}(t) &= P\{X(\tau_{n+1}) = 1, \vartheta_{n+1} \leq t | X(\tau_n) = 2\} \\
&= P\{E, \zeta \leq t\} = a \, F(t) \\
\\
Q_{20}(t) &= P\{X(\tau_{n+1}) = 0, \vartheta_{n+1} \leq t | Y(\tau_n) = 2\} \\
&= P\{\bar{A}, \, \zeta \leq t\} = (1 - a) \, F(t)
\end{aligned} \tag{3.3}$$

Using the relations between the elements of the semi-Markov kernel and their corresponding densities $q_{ij}, i, j \in S$ we get:

$$\begin{aligned}
q_{02}(t) &= k(t), t \geq 0, \\
q_{10}(t) &= f(t) - a \, H(t) f(t), t \geq 0, \\
q_{11}(t) &= a \, H(t) f(t), t \geq 0, \\
q_{20}(t) &= (1 - a) f(t), t \geq 0, \\
q_{21}(t) &= a \, f(t), \, t \geq 0.
\end{aligned} \tag{3.4}$$

Many of statisticians are interested to search for new families of distributions or generalized some of the presented distributions such that they have some properties which enable to describe the lifetimes of some reliability systems and devices or to describe sets of real data. It is well known that, some of the statistical distributions have a constant failure rate such as the exponential distribution Gupta and Kundu (2001), and other distributions have increasing failure rates such as linear failure rate distribution, and some others with decreasing failure rates such as Weibull distribution with shape parameter does not exceed one and other distributions with all of these types of failure rates on different periods of time such as those distributions having failure rate of the bath-tub curve shape see for example Lai, et al. (2001) and Lawless (2003).

Now, we assume that the lifetime of the active units have identically generalized Weibull distribution with the parameters α, β and γ. That is,

$$f(t) = \alpha\,\beta\,\gamma\; t^{\beta-1}\, e^{-\alpha\, t^{\beta}}\, [1 - e^{-\alpha t^{\beta}}]^{\gamma-1},\; \alpha, \beta, \gamma > 0,\; t \geq 0 \tag{3.5}$$

Substituting from (3.5) into the densities (3.4) of the semi-Markov kernel, we get

$$\left.\begin{aligned}
q_{10}(t\,|\,\underline{\theta}) &= \alpha\,\beta\,\gamma t^{\beta-1}\,(1 - a\,H(t))\, e^{-\alpha\, t^{\beta}}\, [1 - e^{-\alpha t^{\beta}}]^{\gamma-1},\; a,\; \alpha, \beta, \gamma > 0,\; t \geq 0 \\
q_{11}(t\,|\,\underline{\theta}) &= a\,\alpha\,\beta\,\gamma t^{\beta-1}\; H(t)\, e^{-\alpha\, t^{\beta}}\, [1 - e^{-\alpha t^{\beta}}]^{\gamma-1},\; a,\; \alpha, \beta, \gamma > 0,\; t \geq 0 \\
q_{20}(t\,|\,\underline{\theta}) &= (1-a)\,\alpha\,\beta\,\gamma t^{\beta-1}\, e^{-\alpha\, t^{\beta}}\, [1 - e^{-\alpha t^{\beta}}]^{\gamma-1},\; a,\; \alpha, \beta, \gamma > 0,\; t \geq 0 \\
q_{21}(t\,|\,\underline{\theta}) &= a\,\alpha\,\beta\,\gamma t^{\beta-1}\; e^{-\alpha\, t^{\beta}}\, [1 - e^{-\alpha t^{\beta}}]^{\gamma-1},\; a,\; \alpha, \beta, \gamma > 0,\; t \geq 0
\end{aligned}\right\} \tag{3.6}$$

Next, we go to discuss the maximum likelihood estimation of the parameters a, α, β and γ include in our reliability model.

4 MAXIMUM LIKLIHOOD ESTIMATES

In this section the maximum likelihood estimators of the unknown vector $\underline{\theta} = (a, \alpha, \beta, \gamma)$ included in the generalized exponential reliability model are presented. Suppose that z denotes the observations $\{(i_0, t_0), (i_1, t_1), \ldots, (i_n, t_n)\}$ of two dimensional random vector of variables, $\{(\xi_0, \vartheta_0), (\xi_1, \vartheta_1), \ldots, (\xi_n, \vartheta_n)\}$ where $i_0, i_1, \ldots, t_n$ and $t_0, t_1, \ldots, t_n \in [0, \infty)$. Further, we assume that this observation is classified as follows:

Let

$$A_{ij} = \{k : i_{k-1} = i, i_k = j, k = 1, 2, \ldots, n\}$$

be the set of numbers of direct observed transition from the state i to the state j and n_{ij} is the cardinal number of the set A_{ij} which represents the number of direct transitions from the state i to state j . In the present case we find that

$$n_{02} + n_{10} + n_{11} + n_{20} + n_{21} = n \tag{4.1}$$

Based on the above observation, the sample likelihood function $L(z;\underline{\theta})$ can be obtained as follows:

Using (2.3) and (3.4) the sample likelihood function $L(z;\underline{\theta})$ takes the form

$$L(z;\underline{\theta}) = \prod_{i\in A_{02}} q_{02}(t_i) \prod_{i\in A_{10}} q_{10}(t_i) \prod_{i\in A_{11}} q_{11}(t_i) \prod_{i\in A_{20}} q_{20}(t_i) \prod_{i\in A_{21}} q_{21}(t_i) \tag{4.2}$$

Substituting the semi-Markov densities from (3.6) into (4.2) we get

$$L(z;\underline{\theta}) = C\, a^{n_{11}+n_{21}}\,(1-a)^{n_{20}}\,W(a)\,(\alpha\,\beta\,\gamma)^m \prod_{i\in L} t_i^{\beta-1}\, e^{-\alpha\, t_i^\beta}[1-e^{-\alpha\, t_i^\beta}]^{\gamma-1}, \tag{4.3}$$

where

$$\left.\begin{array}{ll} W(a) = \prod_{i\in A_{10}} [1-aH(t_i)], & C = \prod_{i\in A_{02}} c(t_i), \\ L = A_{10}\cup A_{11}\cup A_{20}\cup A_{21}, & m = n_{10}+n_{11}+n_{20}+n_{21} \end{array}\right\}$$

Finally, the log of the sample likelihood function L can be written in the following form

$$\left.\begin{array}{rcl} \mathcal{L} &=& (n_{11}+n_{21})\ln a + n_{20}\ln(1-a) + \ln W(a) + m\ln(\alpha\,\beta\,\gamma)+ \\ && (\beta-1)\sum_{i\in L}\ln t_i - \alpha\sum_{i\in L} t_i^\beta + (\gamma-1)\sum_{i\in L}\ln[1-e^{-\alpha\, t_i^\beta}] + \ln C \end{array}\right\} \tag{4.4}$$

The maximum likelihood estimators $\hat{a}, \hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ are the values of a, α, β and γ, respectively that maximize the sample likelihood $\mathcal{L}$. Equivalently a, α, β and γ maximize the log sample likelihood since it is a monotone function of $L(z,\underline{\theta})$.

The maximum likelihood equations are given by :

$$\frac{\partial\mathcal{L}}{\partial a} = 0,\ \frac{\partial\mathcal{L}}{\partial \alpha} = 0,\ \frac{\partial\mathcal{L}}{\partial \beta} = 0,\ \frac{\partial\mathcal{L}}{\partial \gamma} = 0. \tag{4.5}$$

Using (4.4) and (4.5) the maximum likelihood equations are

$$\begin{array}{rcl} \frac{\partial\mathcal{L}}{\partial a} &=& \frac{n_{11}+n_{21}}{a} - \frac{n_{20}}{1-a} + \frac{1}{W(a)}\frac{\partial W(a)}{\partial a} = 0, \\ \frac{\partial\mathcal{L}}{\partial \alpha} &=& \frac{m}{\alpha} - \sum_{i\in L} t_i^\beta - (\gamma-1)\sum_{i\in L}\frac{t_i^\beta\, e^{-\alpha\, t_i^\beta}}{1-e^{-\alpha\, t_i^\beta}} = 0, \\ \frac{\partial\mathcal{L}}{\partial \beta} &=& \frac{m}{\beta} + \sum_{i\in L}\ln t_i - \alpha\sum_{i\in L} t_i^\beta \ln\, t_i + \alpha(\gamma-1)\sum_{i\in L}\frac{t_i^\beta\,\ln t_i\, e^{-\alpha\, t_i^\beta}}{1-e^{-\alpha\, t_i^\beta}} = 0, \\ \frac{\partial\mathcal{L}}{\partial \gamma} &=& \frac{m}{\gamma} + \sum_{i\in L}\ln[1-e^{-\alpha\, t_i^\beta}] = 0, \end{array} \tag{4.6}$$

The maximum likelihood estimators (MLEs) $\hat{a}$, $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ for the unknown parameters a, α, β and γ are the solution of the non-linear system (4.6). As it seems, the general solution of this system is very difficult to find in a closed form. The general solution is intractable and numerical procedures are required El-Gohary and Sarhan (2003, 2004).

From (4.6), we obtain the MLE of γ as a function of α and β, namely $\hat{\gamma}(\alpha,\beta)$ where

$$\hat{\gamma}(\alpha,\beta) = \frac{-m}{\sum_{i\in L}\ln(1-e^{-\alpha t_i^{\beta}})} \tag{4.7}$$

Substituting $\hat{\gamma}(\alpha,\beta)$ from (4.7) into (4.6) we obtain

$$\begin{aligned} g_1(\alpha,\beta) &= \frac{m}{\alpha} - \sum_{i\in L} t_i^{\beta} + \left[\frac{m+\sum_{i\in L}(1-e^{-\alpha\, t_i^{\beta}})}{(1-e^{-\alpha\, t_i^{\beta}})}\right] \sum_{i\in L} \frac{t_i^{\beta}\, e^{-\alpha\, t_i^{\beta}}}{1-e^{-\alpha\, t_i^{\beta}}} = 0, \\ g_2(\alpha,\beta) &= \frac{m}{\beta} + \sum_{i\in L}\ln t_i - \alpha \sum_{i\in L} t_i^{\beta}\ln t_i + \alpha \left[\frac{m+\sum_{i\in L}(1-e^{-\alpha\, t_i^{\beta}})}{(1-e^{-\alpha\, t_i^{\beta}})}\right] \\ &\quad \sum_{i\in L}\frac{t_i^{\beta}\,\ln t_i\, e^{-\alpha\, t_i^{\beta}}}{1-e^{-\alpha\, t_i^{\beta}}} = 0 \end{aligned} \tag{4.8}$$

Therefore, MLEs of both α and β namely $\hat{\alpha}$ and $\hat{\beta}$ can be obtained by solving (4.8) with respect to α and β respectively.

Next, we discuss some important special cases of both the time lengths of the repair periods of the units and the lifetimes of the active units.

5 SPECIAL CASES

This section is devoted to study some important special cases. Such these cases occur when, both of the time lengths of the repair periods of the units and the lifetimes of the active units are exponentially and generalized exponential random variables.

In order to obtain the first special case, the following assumptions are needed:

1. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1 - aH(t_i) = 1 - a$ for every $i \in A_{10}$.

2. The lifetimes of the active units can be represented by identically generalized exponential random variables with one parameter. That is, $\alpha = \beta = 1$.

   In this case, the MLEs are given by:

$$\hat{a} = \frac{n_{22}+n_{12}}{m}, \ \ \hat{\gamma} = \frac{-m}{\sum_{i\in L}\ln(1-e^{-t_k})}. \tag{4.9}$$

The second special case can be obtained by considering the following assumptions:

1. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1 - aH(t_i) = 1 - a$ for every $i \in A_{10}$.

2. The lifetimes of the active units can be represented by identically exponential random variables with parameter α. That is, $\beta = \gamma = 1$

   In this case, the MLEs are given by:

$$\hat{a} = \frac{n_{22} + n_{12}}{m}, \;\; \hat{\alpha} = \frac{\tau}{m}, \;\; \tau = \sum_{i \in L} t_i. \tag{4.10}$$

   The third special case can be obtained by considering the following assumptions:

3. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1 - aH(t_i) = 1 - a$ for every $i \in A_{10}$.

4. The lifetimes of the active units can be represented by identically generalized exponential random variables with two parameters α and γ. That is, $\beta = 1$

   In this case, the MLEs are given by:

$$\hat{a} = \frac{n_{22} + n_{12}}{m}, \;\; \hat{\alpha} = \frac{\tau}{m}, \;\; \tau = \sum_{i \in L} t_i. \tag{4.11}$$

From (4.6), we obtain the MLE of γ as a function of α, namely $\hat{\gamma}(\alpha)$ where

$$\hat{\gamma}(\alpha) = \frac{-m}{\sum_{i \in L} \ln(1 - e^{-\alpha\, t_i})} \tag{4.12}$$

Putting $\hat{\gamma}(\alpha)$ in (4.4) we obtain

$$\begin{aligned} g(a, \alpha) = {} & (n_{22} + n_{12}) \ln a + n_{13} \ln(1 - a) + m \ln \alpha - \tau\alpha \\ & -m \;\ln \left(\sum_{i \in L} \text{-log}(1 - e^{-\alpha t_i}) \right) - \sum_{i \in L} \ln(1 - e^{-\alpha t_i}) \end{aligned} \tag{4.13}$$

Therefore, MLEs of both a and α, namely $\hat{a}$ and $\hat{\alpha}$ can be obtained by maximizing (4.13) with respect to a and α respectively. It is observed that both $\hat{a}$ and $\hat{\alpha}$ can be obtained from the fixed point solution of $h_1(a)$ and $h_2(\alpha)$ respectively, where

$$h_1(a) = \left(n_{22} + n_{12} + n_{23} - \sum_{i=1}^{n_{23}} \frac{1}{1 - aH(t_i)} \right) \left[m + \sum_{i=1}^{n_{23}} \frac{1}{1 - aH(t_i)} \right]^{-1}, \tag{4.14}$$

$$h_2(\alpha) = \left[\frac{\sum_{i \in L} t_i e^{-\alpha t_i} / (1 - e^{-\alpha t_i})}{\sum_{i \in L} \ln(1 - e^{-\alpha t_i})} + \frac{1}{m} \sum_{i \in L} \frac{t_i}{1 - e^{-\alpha t_i}} \right]^{-1}, \tag{4.15}$$

where, the function $H(t_i)$ can be considered as a known function of the observation data z. An iterative procedure can be used to solve the Eqs (4.14) and (4.15) . The MLEs $\hat{a}, \hat{\alpha}$ can be obtained from (4.14) and (4.15).

Next, we discuss in details the reliability of our semi-Markov model that consists of one active unit, an identical spare, a switch, and a repair facility.

6 SYSTEM RELIABILITY

In what follows, we will obtain the system reliability of the semi-Markov reliability model. Now, we will define the first passage time. In order to define the first passage time, we should find an accurate answer for the question "how many transitions will the process take to reach state j for the first time if the system is in state i at time zero". The first passage time of the continuous-time semi-Markov process can be measured in time or in terms of the number of transitions. We will obtain the distribution $\Theta_{iA}(t)$ of the first passage time from the state i to a state in a subset $A \subset S$ given that state i was entered at time zero and zeroth transition.

Assuming that $A \subset S = \{0, 1, 2\}$ and $\bar{A} = S - A$, we introduce the following notations

$$\Delta_A = \inf\{n \in N : X(\tau_n) \in A\}, \tag{5.1}$$

and

$$f_{iA}(n) = P\{\Delta_A = n | X(0) = i\}, T_A = \tau_{\Delta A} \tag{5.2}$$

Thus, the function $\Theta_{iA}(t)$ is given by

$$\Theta_{iA}(t) = P\{T_A \leq t | X(0) = i\},\ i \in \bar{A} \tag{5.3}$$

represents the distribution of the first passage time of the semi-Markov process $\{X(t) : t \geq 0\}$, from the state $i \in \bar{A}$ to state in the subset A.

Now, we will define, a mean and the second moment of the first passage time distribution as follows

$$\bar{\Theta}_{iA} = \int_0^\infty t d\Theta_{iA}(t),\ and,\quad \bar{\Theta}^2_{iA} = \int_0^\infty t^2 d\Theta_{iA}(t), \tag{5.4}$$

If A denotes the subset of the failed states of the model and $i \in \bar{A}$ is an initial operating state such that $P\{X(0) = i\} = 1$, then the random variable T_A represents the lifetime or the time to failure of our system. That is, the reliability of the system is

$$R(t) = 1 - \Theta_{iA}(t),\ t \geq 0 \tag{5.5}$$

Using Grabski (1999), some of the reliability characteristics of the system can be defined as follows:

$$\bar{q}_{ik} = \int_0^\infty t q_{ik}(t)dt,\ \bar{q}^2_{ik} = \int_0^\infty t^2 q_{ik}(t)dt \tag{5.6}$$

To derive the reliability of the system, we will establish the following theorem.
Theorem 5.1 *Consider the following systems:*

1.

$$\Theta_{iA}(t) = \sum_{j \in A} Q_{ij}(t) + \sum_{k \in \bar{A}} \int_0^t \Theta_{kA}(t-u) q_{ik}(u)du,\ i \in \bar{A} \tag{5.7}$$

2.

$$\bar{\theta}_{iA} = \bar{g}_i + \sum_{k\in\bar{A}} p_{ik}\bar{\theta}_{ik}, \quad i \in \bar{A} \tag{5.8}$$

3.

$$\bar{\theta}_{iA} = \bar{g}_i^2 + 2\sum_{k\in\bar{A}} \bar{q}_{ik}\bar{\theta}_{kA} + \sum_{k\in\bar{A}} p_{ik}\bar{\theta}^2_{ik}, \quad i \in \bar{A}, \tag{5.9}$$

which consist of a system of integral equations (5.7) and two linear algebraic systems of equations (5.8) and (5.9). These systems have the only solution $\Theta_{iA}(t), \bar{\theta}_{iA}$ and $\bar{\theta}^2_{iA}$ respectively, if the following conditions are satisfied

1.

$$f_{iA} = 1 \ \forall \ i \in \bar{A} \tag{5.10}$$

2.

$$\forall i, j \in S \ \exists \ d > 0 \ s.t. \ \bar{q}^2_{ij} < d \tag{5.11}$$

3.

$$\sum_{k=1}^{\infty} k^2 f_{iA} < \infty \ \forall, \quad i \in \bar{A} \tag{5.12}$$

The system of integral equations (5.7) is equivalent to its Laplace-Stieltjes system

$$\tilde{\vartheta}_{iA}(s) = \sum_{j\in A} \tilde{q}_{ij}(s) + \sum_{k\in\bar{A}} \tilde{q}_{ik}(s)\tilde{\vartheta}_{kA}(s), \ i \in \bar{A} \tag{5.13}$$

where

$$\tilde{\vartheta}_{iA}(s) = \int_0^{\infty} e^{-st}\frac{d\Theta_{iA}(t)}{dt}dt, \ \tilde{q}_{ij}(s) = \int_0^{\infty} e^{-st}q_{ij}(t)dt \tag{5.14}$$

In the present model $A = \{0\}$ and $\bar{A} = \{1, 2\}$. From the solution of the system (5.14), we have

$$\tilde{\vartheta}_{20}(s) = \frac{\tilde{q}_{10}(s)}{1 - \tilde{q}_{11}(s)}, \quad \tilde{\vartheta}_{20}(s) = \tilde{q}_{20}(s) + \frac{\tilde{q}_{21}\tilde{q}_{10}}{1 - \tilde{q}_{11}(s)} \tag{5.15}$$

Using the Laplace transformation, the reliability function (5.5) of the present model is given by

$$\tilde{R}(s) = \frac{1 - \tilde{\vartheta}_{20}(s)}{s} \tag{5.16}$$

From the system of equations (5.8), we can get

$$\bar{\theta}_{20} = \bar{g}_2 + \frac{p_{21}\bar{g}_1}{1 - p_{11}} \tag{5.17}$$

For the present model we have:

$$\bar{g}_1 = \bar{g}_2 = E(\zeta) = \alpha\,\beta\,\gamma \int_0^{\infty} t^{\beta-1}\, e^{-\alpha\, t^{\beta}}(1 - e^{-\alpha\, t^{\beta}})^{\gamma-1}\, dt$$

For $\beta = 1$, we find that

$$\bar{g}_1 = \frac{1}{\alpha}\{\Psi(\gamma+1) - \Psi(1)\} \tag{5.18}$$

where $\Psi(.)$ denotes the digamma function.

$$P_{21} = \int_0^\infty q_{21}(t)dt = a$$

Substituting (5.17) into (5.16) we can obtain the mean of the lifetime of the present system as follows:

$$E(T_A|X(0) = 2) = \bar{\theta}_{20} = \frac{1}{\alpha}\{\Psi(\gamma+1) - \Psi(1)\}\{1 + \frac{a}{1-p_{11}}\} \tag{5.19}$$

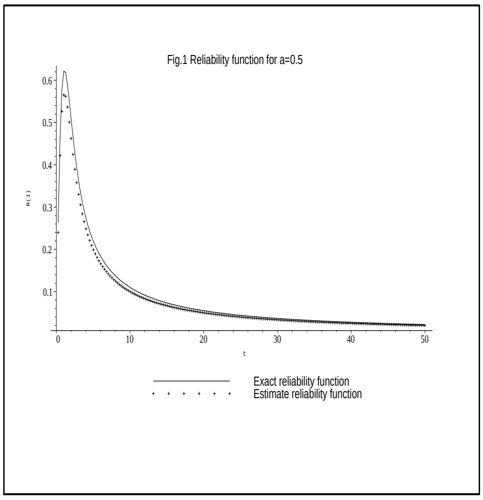
where

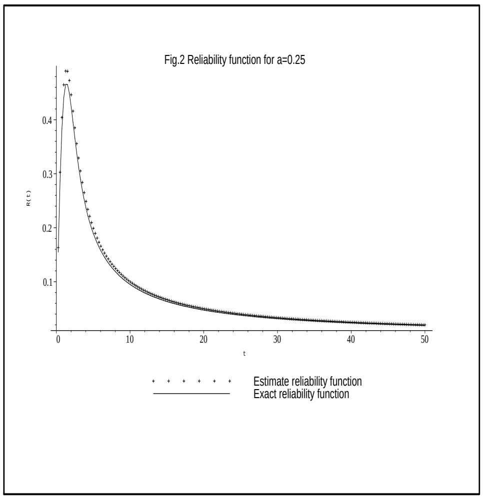
$$p_{11} = a\gamma\alpha \int_0^\infty H(t)(1 - e^{-\alpha\, t})\, e^{-\alpha\, t}\, dt \tag{5.20}$$

Now, an important special case can be obtained when $\gamma = 1$. In this case, the lifetimes of the active units can be represented by identically exponential random variables with parameter α. That is, $\gamma = 1$ and the mean of the lifetime of the system is given by

$$E(T_A|X(0) = 2) = \bar{\theta}_{20} = \frac{1}{\alpha} + \frac{a}{\alpha(1-p_{11})},\ \ p_{11} = a\alpha \int_0^\infty H(t)e^{-\alpha\, t} \tag{5.21}$$

This result, as a special case from the presented results, agrees with the result obtained by Grabski (1999) . This shows the effectiveness of the present method.





Figures 1 and 2 display the graph of the system reliability function against time for the set values of the switch probability parameter $a = 0.5$ and $a = 0.25$ respectively and the set of sub-states $A = \{0\}$ and $\bar{A} = \{1, 2\}$. The solid curves represent the reliability function corresponding to the estimated values of switch probability parameter a while the dot curves represent the reliability function corresponding to the exact values of switch probability parameter a.

7 CONCLUSION

Finally, we conclude that the stochastic analysis of the semi-Markov reliability model is discussed and the likelihood procedure is used to obtain estimators of the parameters included in a three-state standby with repair semi-Markov model. The distribution of the first passage time is discussed. The reliability function of this model is derived. Many special cases are discussed.

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