

INEQUALITIES FOR CHORD POWER INTEGRALS

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ABSTRACT. For convex bodies, chord power integrals were introduced and studied in several papers (see [3], [6], [14], [15], etc.). The aim of this article is to study them further, that is, we establish the Brunn-Minkowski-type inequalities and get the upper bound for chord power integrals of convex bodies. Finally, we get the famous *Zhang projection inequality* as a corollary. Here, it is deserved to mention that we make use of a completely distinct method, that is using the theory of inclusion measure, to establish the inequality.

1. Preliminaries

The setting for this paper is n -dimensional Euclidean space R^n . We will denote by *convex figure* a compact convex subset of R^n , and by *convex body* a convex figure with nonempty interior. Let S^{n-1} denote the unit sphere centered at the origin o in R^n , and write α_{n-1} for the $(n-1)$ -dimensional volume of S^{n-1} . Let B_n be the closed unit ball in R^n , write ω_n for the n -dimensional volume of B_n . Note that

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}, \quad \alpha_{n-1} = n\omega_n.$$

By a *direction*, we mean a unit vector, that is, an element of S^{n-1} . If u is a direction, we denote by u^\perp the $(n-1)$ -dimensional subspace orthogonal to u and by l_u the line through the origin parallel to u .

Denote by $AG_{i,n}$ the affine Grassmann manifold of i -dimensional planes in R^n . It is a homogeneous space under the action of the motion group $G(n)$ (see [7], p.199). Let $d\xi_k$ be the normalized invariant measure of $AG_{i,n}$ whose restriction to the Grassmann manifold $G_{i,n}$ is the invariant probability measure. Let ξ_1 be a random line intersecting K . Then $vol_1(K \cap \xi_1)$ is the chord length of the intersection $K \cap \xi_1$. The *chord power integrals* of K are defined by

$$I_\lambda(K) = \frac{2\alpha_{n-1}}{n} \int_{\xi_1 \in AG_{1,n}} vol_1(K \cap \xi_1)^\lambda d\xi_1, \quad 0 \leq \lambda < \infty.$$

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Here the normalization says that

$$\int_{B_n \cap \xi_1 \neq \emptyset} d\xi_1 = \omega_{n-1}.$$

Chord power integrals are generalizations of the surface area $S(K)$ and the volume $V(K)$ of convex body K . There are several interesting integral formulas for chord power integrals (see [7]):

$$I_0(K) = \frac{\omega_{n-1}}{n} S(K),$$

$$I_1(K) = \frac{\alpha_{n-1}}{n} V(K),$$

$$I_{n+1}(K) = (n+1)V(K)^2.$$

Also, chord power integrals have strong relations with *inclusion measure* of convex body. Let $G(n)$ be the group of rigid motions in \mathbb{R}^n . Each element, $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$, of $G(n)$ can be represented by

$$g: x \rightarrow ex + b,$$

where $b \in \mathbb{R}^n$ and e is an orthogonal matrix of determinant 1. Let μ be the Haar measure on $G(n)$ normalized as follows: Let $\varphi: \mathbb{R}^n \times SO(n) \rightarrow G(n)$ be defined by $\varphi(t, e)x = ex + t$, $x \in \mathbb{R}^n$, where $SO(n)$ is the rotation group of \mathbb{R}^n . If ν is the unique invariant probability measure on $SO(n)$, η is the Lebesgue measure on \mathbb{R}^n , then μ is chosen as the pull back measure of $\eta \otimes \nu$ under φ^{-1} .

The *inclusion measure* of a convex figure L contained in a convex body K is defined by

$$m_K(L) = m(L \subseteq K) = \int_{\{g \in G(n): gL \subseteq K\}} d\mu(g).$$

It gives the measure of the set of copies congruent to convex figure L contained in a fixed convex body K . When L is a line segment with length l , chord power integrals and inclusion measures have the following relation (see [15]):

$$(1.1) \quad I_\lambda(K) = \frac{1}{2} \lambda(\lambda - 1) \int_0^\infty m_K(l) l^{\lambda-2} dl, \lambda > 1.$$

Now, we give another integral formula for the chord power integral and it will be used in the following.

The position of an oriented line ξ_1 in R^n can be fixed by specifying its direction u and the point p , where ξ_1 intersects the orthogonal hyperplane through the origin. The "integral-geometric density" $d\xi_1$, for oriented lines is then given by

$$d\xi_1 = dpdu,$$

where dp is the $(n-1)$ -dimensional volume element in the orthogonal hyperplane u^\perp , and du is the element of surface area of the unit sphere S^{n-1} . Let

$\sigma(p, u) = \text{vol}_1(K \cap \xi_1)$. Then we get another integral formula for chord power integral

$$(1.2) \quad I_\lambda(K) = \frac{1}{n} \int_{S^{n-1}} \int_{K|u^\perp} \sigma(p, u)^\lambda dp du, 0 \leq \lambda < \infty.$$

To establish the Brunn-Minkowski-type inequalities for chord power integrals, we need more facts about convex body.

Associated with convex body K is its *support function* h_K defined on \mathbb{R}^n by

$$h_K(x) = \max\{\langle x, y \rangle : y \in K\},$$

where $\langle x, y \rangle$ is the usual inner product of x and y in \mathbb{R}^n . The function h_K is positively homogeneous of degree 1. We will usually be concerned with the restriction of the support function to the unit sphere S^{n-1} .

The *Minkowski addition* of two convex bodies K and L is defined as

$$K + L = \{x + y : x \in K, y \in L\}.$$

The *scalar multiplication* λK of K , where $\lambda \geq 0$, is defined as

$$\lambda K = \{\lambda x : x \in K\}.$$

For convex figure $\lambda K + \mu L$, the support function is

$$h_{\lambda K + \mu L} = \lambda h_K + \mu h_L.$$

The *projection body* of K is the centered convex body ΠK defined by

$$h_{\Pi K}(u) = V_{n-1}(K|u^\perp)$$

for each $u \in S^{n-1}$, where $K|u^\perp$ is the orthogonal projection of K on u^\perp . We denote the *polar body* of K by K^* , and call $\Pi^* K$, the polar body of ΠK , the *polar projection body* of K .

The *difference body* of convex body K , denoted by DK , is the centrally symmetric convex body (centered at the origin) defined by,

$$DK = K + (-K) = \{x - y : x \in K, y \in K\}.$$

It is well known that DK can be equivalently described as follows,

$$DK = \{x : (x + K) \cap K \neq \emptyset\}.$$

If K is a convex body that contains the origin in its interior, then we associate with K its *radial function* ρ_K on S^{n-1} by:

$$\rho_K(u) = \max\{\lambda > 0 : \lambda u \in K\}.$$

It is not difficult to verify that the *radial function* ρ_{DK} of the difference body DK is

$$\rho_{DK}(u) = \max_{y \in u^\perp} \text{vol}_1(K \cap (l_u + y)), u \in S^{n-1}.$$

The function

$$g_K(x) = V(K \cap (K + x)),$$

for $x \in R^n$, is called the *covariogram* of K . Note that $g_K(o) = V(K)$ and that if $u \in S^{n-1}$, then $g_K(ru) = 0$ for $r \geq \rho_{DK}(u)$.

For convex bodies K, L in R^n , the Brunn-Minkowski inequality is

$$V^{\frac{1}{n}}(K + L) \geq V^{\frac{1}{n}}(K) + V^{\frac{1}{n}}(L),$$

with equality if and only if K and L are homothetic.

2. Main results

Theorem 1. *Let K, L be convex bodies in R^n . Then for $0 < \lambda \leq 1$,*

$$I_\lambda^{\frac{1}{\lambda}}(K + L) \geq I_\lambda^{\frac{1}{\lambda}}(K) + I_\lambda^{\frac{1}{\lambda}}(L).$$

Proof. Let $\xi_1 \in AG_{1,n}$. It is easy to prove that

$$(K + L) \cap \xi_1 \supseteq (K \cap \xi_1) + (L \cap \xi_1).$$

Hence, according to Brunn-Minkowski inequality and Minkowski inequality, we have

$$\begin{aligned} I_\lambda^{\frac{1}{\lambda}}(K + L) &= \left(\frac{2\alpha_{n-1}}{n} \int_{\xi_1 \in AG_{1,n}} vol_1((K + L) \cap \xi_1)^\lambda d\xi_1\right)^{\frac{1}{\lambda}} \\ &\geq \left(\frac{2\alpha_{n-1}}{n} \int_{\xi_1 \in AG_{1,n}} vol_1[(K \cap \xi_1) + (L \cap \xi_1)]^\lambda d\xi_1\right)^{\frac{1}{\lambda}} \\ &\geq \left(\frac{2\alpha_{n-1}}{n} \int_{\xi_1 \in AG_{1,n}} [vol_1(K \cap \xi_1) + vol_1(L \cap \xi_1)]^\lambda d\xi_1\right)^{\frac{1}{\lambda}} \\ &\geq \left(\frac{2\alpha_{n-1}}{n} \int_{\xi_1 \in AG_{1,n}} vol_1^\lambda(K \cap \xi_1) d\xi_1\right)^{\frac{1}{\lambda}} \\ &\quad + \left(\frac{2\alpha_{n-1}}{n} \int_{\xi_1 \in AG_{1,n}} vol_1^\lambda(L \cap \xi_1) d\xi_1\right)^{\frac{1}{\lambda}} \\ &\geq I_\lambda^{\frac{1}{\lambda}}(K) + I_\lambda^{\frac{1}{\lambda}}(L). \end{aligned}$$

This completes the proof. □

For the case of $\lambda > 1$, we can make use of inclusion measures to establish the similar inequality.

Theorem 2. *Let K, L be convex bodies in R^n . Then for $\lambda > 1$,*

$$I_\lambda(K + L) > I_\lambda(K) + I_\lambda(L).$$

To prove the theorem, we should establish the similar results for inclusion measures and then using the relation (1.1) to finish our proof. For this aim, we prove several lemmas in advance.

Firstly, we define the set

$$C(K, L, \lambda) = \{x : x + \lambda L \subseteq K\}.$$

The following Lemma 1 is explicitly contained in the paper [13].

Lemma 1 ([13]). *If K is a convex body and L is a convex figure in \mathbb{R}^n , then the inclusion measure of L contained in K is*

$$m_K(L) = \int_{SO(n)} V(C(K, eL, 1)) d\nu(e),$$

where ν is the unique invariant probability measure on $SO(n)$.

Lemma 2. *If K is a convex body and L is a convex figure in \mathbb{R}^n , then*

$$C(\alpha K, L, \lambda) = \alpha C(K, \frac{1}{\alpha}L, \lambda) = \alpha C(K, L, \frac{\lambda}{\alpha}), \alpha > 0, \lambda \geq 0.$$

In addition, if $0 < \alpha < 1$, then $C(\alpha K, L, \lambda) \subseteq \alpha C(K, L, \lambda)$; if $\alpha > 1$, then $C(\alpha K, L, \lambda) \supseteq \alpha C(K, L, \lambda)$.

Proof. From the definition of the set $C(K, L, \lambda)$, we can get

$$\begin{aligned} C(\alpha K, L, \lambda) &= \{x \in \mathbb{R}^n : x + \lambda L \subseteq \alpha K\} \\ &= \{x \in \mathbb{R}^n : \frac{1}{\alpha}(x + \lambda L) \subseteq K\} \\ &= \{x \in \mathbb{R}^n : \frac{1}{\alpha}x \in K - \frac{\lambda}{\alpha}L\} \\ &= \{\alpha x \in \mathbb{R}^n : x \in K - \frac{\lambda}{\alpha}L\} \\ &= \alpha C(K, \frac{1}{\alpha}L, \lambda) \\ &= \alpha C(K, L, \frac{\lambda}{\alpha}). \end{aligned}$$

Furthermore, if $0 < \alpha < 1$, then

$$C(K, L, \frac{\lambda}{\alpha}) \subseteq C(K, L, \lambda).$$

So

$$\alpha C(K, L, \frac{\lambda}{\alpha}) \subseteq \alpha C(K, L, \lambda).$$

Hence,

$$C(\alpha K, L, \lambda) \subseteq \alpha C(K, L, \lambda).$$

Similarly, if $\alpha > 1$, then

$$C(K, L, \lambda) \subseteq C(K, L, \frac{\lambda}{\alpha}).$$

So

$$\alpha C(K, L, \lambda) \subseteq \alpha C(K, L, \frac{\lambda}{\alpha}).$$

Hence,

$$\alpha C(K, L, \lambda) \subseteq C(\alpha K, L, \lambda).$$

This completes the proof. □

In the following, we will denote $C(K) = C(K, L, \lambda)$, then we can get the following results.

Lemma 3. *Let $K_i, i = 1, 2, \dots, s, s \in \mathbb{N}$, be convex bodies and L be a convex figure in \mathbb{R}^n . Then for $s > 1$, there holds*

$$\sum_{i=1}^s C(K_i) \subseteq C\left(\sum_{i=1}^s K_i\right).$$

Proof. For any $x \in \sum_{i=1}^s C(K_i)$, x can be expressed as

$$x = x_1 + x_2 + \dots + x_s, \quad x_i \in C(K_i), \quad i = 1, 2, \dots, s.$$

From the definition of $C(K_i)$, we can get

$$\begin{aligned} \langle x_1, u \rangle &\leq h_{K_1}(u) - \lambda h_L(u), \\ \langle x_2, u \rangle &\leq h_{K_2}(u) - \lambda h_L(u), \\ &\dots\dots\dots, \\ \langle x_s, u \rangle &\leq h_{K_s}(u) - \lambda h_L(u) \end{aligned}$$

for $u \in S^{n-1}$. So

$$\left\langle \frac{x_1 + x_2 + \dots + x_s}{s}, u \right\rangle \leq \frac{h_{K_1}(u) + h_{K_2}(u) + \dots + h_{K_s}(u)}{s} - \lambda h_L(u).$$

That is,

$$\begin{aligned} \left\langle \frac{x}{s}, u \right\rangle &\leq \frac{h_{K_1+K_2+\dots+K_s}(u)}{s} - \lambda h_L(u), \\ \frac{1}{s} \langle x, u \rangle &\leq h_{\frac{K_1+K_2+\dots+K_s}{s}}(u) - \lambda h_L(u). \end{aligned}$$

Hence,

$$\frac{\sum_{i=1}^s C(K_i)}{s} \subseteq C\left(\frac{\sum_{i=1}^s K_i}{s}\right).$$

Since $0 < \frac{1}{s} < 1$, from Lemma 2 we can get

$$\frac{\sum_{i=1}^s C(K_i)}{s} \subseteq C\left(\frac{\sum_{i=1}^s K_i}{s}\right) \subseteq \frac{1}{s} C\left(\sum_{i=1}^s K_i\right).$$

So

$$\sum_{i=1}^s C(K_i) \subseteq C\left(\sum_{i=1}^s K_i\right).$$

□

Lemma 4. *Let $K_i, i = 1, 2, \dots, s, s > 1, s \in \mathbb{N}$, be convex bodies and L be a convex figure in \mathbb{R}^n . Then*

$$m_{K_1+K_2+\dots+K_s}(L) > m_{K_1}(L) + m_{K_2}(L) + \dots + m_{K_s}(L).$$

Proof. Firstly, we consider the case when $s = 2$. From Lemma 2, we can get

$$m_{K_1+K_2}(L) = \int_{SO(n)} V(C(K_1 + K_2, eL, \lambda))d\nu(e).$$

From Lemma 3 and Brunn-Minkowski inequality, there holds

$$V^{\frac{1}{n}}(C(K_1 + K_2)) \geq V^{\frac{1}{n}}(C(K_1) + C(K_2)) \geq V^{\frac{1}{n}}(C(K_1)) + V^{\frac{1}{n}}(C(K_2)).$$

So,

$$V(C(K_1 + K_2)) > V(C(K_1)) + V(C(K_2)).$$

Therefore,

$$\begin{aligned} m_{K_1+K_2}(L) &= \int_{SO(n)} V(C(K_1 + K_2, eL, \lambda))d\nu(e) \\ &> \int_{SO(n)} V(C(K_1, eL, \lambda))d\nu(e) + \int_{SO(n)} V(C(K_2, eL, \lambda))d\nu(e) \\ &= m_{K_1}(L) + m_{K_2}(L). \end{aligned}$$

By the finite induction, it can be easily get

$$m_{K_1+K_2+\dots+K_s}(L) > m_{K_1}(L) + m_{K_2}(L) + \dots + m_{K_s}(L).$$

This completes the proof. □

Proof of the Theorem 2. When $\lambda > 1$, from the equation (1.1) and the above Lemma 4 we have

$$\begin{aligned} I_\lambda(K + L) &= \frac{1}{2}\lambda(\lambda - 1) \int_0^\infty m_{K+L}(l)l^{\lambda-2}dl \\ &> \frac{1}{2}\lambda(\lambda - 1) \int_0^\infty m_K(l)l^{\lambda-2}dl + \frac{1}{2}\lambda(\lambda - 1) \int_0^\infty m_L(l)l^{\lambda-2}dl \\ &= I_\lambda(K) + I_\lambda(L). \end{aligned}$$

This completes the proof. □

Question. Let K, L be convex bodies in R^n . Does the following inequality hold for all $\lambda > 0$?

$$I_\lambda^{\frac{1}{\lambda}}(K + L) \geq I_\lambda^{\frac{1}{\lambda}}(K) + I_\lambda^{\frac{1}{\lambda}}(L).$$

Theorem 3. Let K be a convex body in R^n . Then for $\lambda \geq 1$,

$$I_{\lambda+1}(K) \leq (\lambda + 1)B(\lambda + 1, n)V(K) \int_{S^{n-1}} \left(\frac{nV(K)}{h_{\Pi K}(u)}\right)^\lambda du,$$

with equality holds if and only if K is a simplex.

Proof. Let $x = ru$, where $r \geq 0$ and $u \in S^{n-1}$, and define

$$g_K(r, u) = g_K(ru).$$

According to the concavity of $g_K(ru)^{\frac{1}{n}}$ on its support DK (see the Lemma 3.2 in [5]) and

$$\frac{\partial}{\partial r} \left(\frac{g_K(r, u)}{V(K)} \right)^{\frac{1}{n}} \Big|_{r=0} = -\frac{h_{\Pi K}(u)}{nV(K)},$$

we have

$$g_K(ru) \leq V(K) \left(1 - \frac{h_{\Pi K}(u)}{nV(K)} r \right)^n$$

for $0 \leq r \leq \rho_{DK}(u)$. Therefore,

$$\begin{aligned} \int_0^{\rho_{DK}(u)} g_K(ru) r^{\lambda-1} dr &\leq V(K) \int_0^{\rho_{DK}(u)} \left(1 - \frac{h_{\Pi K}(u)}{nV(K)} r \right)^n r^{\lambda-1} dr \\ &\leq V(K) \left(\frac{nV(K)}{h_{\Pi K}(u)} \right)^\lambda \int_0^1 (1-t)^n t^{\lambda-1} dt \\ &= B(\lambda, n+1) V(K) \left(\frac{nV(K)}{h_{\Pi K}(u)} \right)^\lambda. \end{aligned}$$

On the other hand, from the proof of Theorem 3 in [3], we have

$$\int_0^{\rho_{DK}(u)} g_K(ru) r^{\lambda-1} dr = \frac{1}{\lambda(\lambda+1)} \int_{K|u^\perp} \sigma(p, u)^{\lambda+1} dp.$$

From the formula (1.2), we can get

$$\begin{aligned} I_{\lambda+1}(K) &= \frac{1}{n} \int_{S^{n-1}} \int_{K|u^\perp} \sigma(p, u)^{\lambda+1} dp du \\ &\leq \frac{\lambda(\lambda+1)}{n} \int_{S^{n-1}} B(\lambda, n+1) V(K) \left(\frac{nV(K)}{h_{\Pi K}(u)} \right)^\lambda du \\ &= \frac{\lambda(\lambda+1)}{n} B(\lambda, n+1) V(K) \int_{S^{n-1}} \left(\frac{nV(K)}{h_{\Pi K}(u)} \right)^\lambda du \\ &= (\lambda+1) B(\lambda+1, n) V(K) \int_{S^{n-1}} \left(\frac{nV(K)}{h_{\Pi K}(u)} \right)^\lambda du, \end{aligned}$$

since $\lambda B(\lambda, n+1) = nB(\lambda+1, n)$.

Equality holds if and only if $g_K(ru)^{\frac{1}{n}}$ is linear in r for each $u \in S^{n-1}$, and hence, if and only if K is a simplex. This completes the proof. \square

Remark. Theorem 9 in paper [13] gives the upper bound for the integration of inclusion measure of convex bodies. We point out that when the contained convex figure L is a line segment with length l , we also can get the same result as the above Theorem 3 from Theorem 9 directly. So, Theorem 3 here is a corollary of Theorem 9 in [13].

From Theorem 3, we can get the following reverse form of the Petty projection inequality, that is, the famous Zhang projection inequality.

Corollary 1. *Let K be a convex body in R^n . Then*

$$n^{-n} \binom{2n}{n} \leq V(K)^{n-1} V(\Pi^* K),$$

with equality holds if and only if K is a simplex.

Proof. When $\lambda = n$, the equality in Theorem 3 is

$$I_{n+1}(K) \leq (n+1)B(n+1, n)V(K) \int_{S^{n-1}} \left(\frac{nV(K)}{h_{\Pi^* K}(u)} \right)^n du.$$

Since

$$I_{n+1}(K) = (n+1)V(K)^2, \quad \frac{1}{h_{\Pi^* K}(u)} = \rho_{\Pi^* K}(u),$$

we have

$$\begin{aligned} (n+1)V(K)^2 &\leq (n+1) \frac{n!(n-1)!}{(2n)!} n^n V(K)^n \int_{S^{n-1}} \rho_{\Pi^* K}(u)^n du \\ &= (n+1) \frac{n!(n-1)!}{(2n)!} n^{n+1} V(K)^n V(\Pi^* K). \end{aligned}$$

That is,

$$n^{-n} \binom{2n}{n} \leq V(K)^{n-1} V(\Pi^* K).$$

The condition of the equality can be get from the above Theorem 3 directly.

This completes the proof. \square

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