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A FREDHOLM MAPPING OF INDEX ZERO

José M. Soriano Arbizu

Abstract. Sufficient conditions are given to assert that between any two Banach spaces over $\mathbb{K}$ Fredholm mappings share exactly N values in a specific open ball. The proof of the result is constructive and is based upon continuation methods.

1. Preliminaries

Let X and Y be two Banach spaces. If $F : X \to Y$ is a continuous mapping, then one way of solving the equation

$$F(x) = 0 \tag{1}$$

is to embed (1) in a continuum of problems

$$H(x,t) = 0 \quad (0 \leq t \leq 1), \tag{2}$$

which is resolved when $t = 0$. When $t = 1$, the problem (2) becomes (1). In the case when it is possible to continue the solution for all t in $[0,1]$ then (1) is solved. This method is called continuation with respect to a parameter [1]-[23].

In this paper, sufficient conditions are given in order to prove that two differentiable mappings share exactly N values in a specific open ball. Other conditions, sufficient to guarantee the existence of zero points in finite and infinite dimensional settings, have been given by the author in several other papers [10]-[23]. In this paper we use continuation methods. The proof supplies the existence of implicitly defined continuous mappings whose ranges reach zero points [5]-[7]. The key is the use of the Continuous Dependence theorem on a parameter in Banach spaces [25] , properties of Fredholm C^1-mappings [25, 26], the Weierstrass theorem relative to extremum points [26], and a consequence of the properties of the algebra of Banach whose elements are the linear continuous mappings from a Banach space into itself.

We briefly recall some theorems and concepts to be used.

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Theorem 1 ([25, pp. 17-19] Continuous Dependence Theorem). *Let the following conditions be satisfied:*

(i) *P is a metric space, called the parameter space.*

(ii) *For each $p \in P$, the mapping T_p satisfies the following hypotheses:*

(1) *$T_p : M \subseteq X \to M$, i.e., M is mapped into itself by T_p;*

(2) *M is a closed non-empty set in a complete metric space (X, d);*

(3) *T_p is k-contractive for fixed $k \in [0, 1)$.*

(iii) *For a fixed $p_0 \in P$, and for all $x \in M, \lim_{p \to p_0} T_p(x) = T_{p_0}(x)$.*

Thus, for each $p \in P$, the equation $x_p = T_p x_p$ has exactly one solution, where $x_p \in M$ and $\lim_{p \to p_0} x_p = x_{p_0}$.

Definition ([25, 26]). We will assume X and Y are Banach spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Mapping $F : D(F) \subseteq X \to Y$, is said to be *compact* whenever it is continuous and the image $F(B)$ is relatively compact (i.e., its closure $\overline{F(B)}$ is compact in Y) for every bounded subset $B \subset D(F)$.

Mapping F is said to be *proper* whenever the pre-image $F^{-1}(K)$ of every compact subset $K \subset Y$ is also a compact subset of $D(F)$.

If $D(F)$ is open, then mapping F is said to be a *Fredholm* mapping if and only if both F is a C^1-mapping and $F'(x) : X \to Y$ is a Fredholm linear mapping for all $x \in D(F)$. That $L : X \to Y$ is a *linear Fredholm* mapping means that L is linear and continuous and both the numbers $\dim(\ker(L))$ and $\operatorname{codim}(R(L))$ are finite, and therefore $\ker(L) = X_1$ is a Banach space and has topological complement X_2, since $\dim(X_1)$ is finite. The integer number $\operatorname{ind}(L) = \dim(\ker(L)) - \operatorname{codim}(R(L))$ is called the *index* of L, where dim signifies dimension, codim codimension, ker kernel and $R(L)$ stands for the range of mapping L.

Let $\mathcal{F}(X, Y)$ denote the set of all linear Fredholm mappings $A : X \to Y$. Let $\mathcal{L}(X, Y)$ denote the set of all linear continuous mappings $L : X \to Y$. Let $\operatorname{Isom}(X, Y)$ denote the set of all the isomorphisms $L : X \to Y$.

Let $B(x_0, \rho)$ denote the open ball of centre x_0 and radius ρ, and $S(x_0, \rho)$ the sphere of centre x_0 and radius ρ. If $u : X \to Y$ is a linear continuous bijective operator, the inverse linear continuous operator to u will be denoted by u^{-1}.

Theorem 2 ([27, pp. 23-24]). (*a*) *The set* $\operatorname{Isom}\mathcal{L}(X, Y)$ *is open in* $\mathcal{L}(X, Y)$.

(*b*) *The mapping* $\beta : \operatorname{Isom}(X, Y) \to \mathcal{L}(Y, X), \beta(u) := u^{-1}$ *is continuous.*

Theorem 3 ([26, p. 296]). *Let $g : D(g) \subset X \to Y$ be a compact mapping, where $a \in D(g)$. If the derivative $g'(a)$ exists, then $g'(a) \in \mathcal{L}(X, Y)$ is also a compact mapping.*

Theorem 4 ([26, p. 366]). *Let $S \in \mathcal{F}(X, Y)$. The perturbed mapping $S + C$ verifies $S + C \in \mathcal{F}(X, Y)$ and* $\operatorname{ind}(S + C) = \operatorname{ind}(S)$ *if $C \in \mathcal{L}(X, Y)$ and C is a compact mapping.*

Definition ([26, p.318]). Let $F : X \to Y$ be a C^1-mapping.

The point $u \in X$ is called a *regular point* of F if and only if $F'(u) \in \mathcal{L}(X,Y)$ maps onto Y, and $\ker(F'(x))$ *splits* X into a topological direct sum.

The point $v \in Y$ is called a *regular value* of A if and only if the pre-image $F^{-1}(v)$ is empty or consists solely of regular points.

2. A Fredholm mapping

If we can say $u := f - g$, then u has a zero if and only if f and g share a value, that is, there is $x \in X$ with $f(x) = g(x)$. We thereby establish our result in terms of f, g.

Theorem 5. *Let $f, g : X \to Y$ be two C^1-mappings, where X and Y are two Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.*

(i) *f is a proper and Fredholm mapping of index zero and g is a compact mapping.*

(ii) *Mapping f has N zeros, $x_i, i = 1, \ldots, N$ in $B(x_0, \rho)$.*

(iii) *Zero is a regular value of the mapping $f(\cdot) - tg(\cdot) : X \to Y$ for each parameter $t \in [0,1]$.*

(iv) *If $(x,t) \in S(x_0, \rho) \times [0,1]$ then $f(x) \neq tg(x)$.*

Hence the following statement holds true:

(*a*) *f and g share exactly N values in the open ball $B(x_0, \rho)$.*

Proof. (*a*) Henceforth $X \times \mathbb{R}$ is provided by the topology, given by the product norm. $\mathcal{L}(X,Y), \mathcal{L}(Y,X)$ are provided by the topologies given by their respective operator norm.

(a1) Let us construct the following homotopy $H : X \times [0,1] \to Y$, $H(x,t) := f(x) - tg(x)$, which is a C^1-homotopy between the mappings f and $f - g$.

Henceforth partial derivatives will generally be denoted by writing initial spaces as subindices of mappings.

We will see here for any $(x,t) \in X \times [0,1]$ that $H_X(x,t) = f'(x) - tg'(x)$ verifies $H_X(x,t) \in \mathcal{F}(X,Y)$, and $\mathrm{ind} H_X(x,t) = 0$.

Since g is a compact mapping and the derivative $g'(x)$ exists for any fixed $x \in X$, Theorem 3 implies that $g'(x) \in \mathcal{L}(X,Y)$ is a compact mapping and therefore for any $(x,t) \in X \times [0,1], tg'(x) \in \mathcal{L}(X,Y)$, is also a compact mapping.

Since f is a Fredholm mapping of index zero, then $f'(x) \in \mathcal{F}(X,Y), \forall x \in X$ and $\mathrm{ind}(f'(x)) = 0, \forall x \in X$.

These results together with Theorem 4 imply that $H_X(x,t) \in \mathcal{F}(X,Y)$, and $\mathrm{ind} H_X(x,t) = 0, \forall (x,t) \in X \times [0,1]$.

(a2) We will now prove that, if $H(x,t) = 0, (x,t) \in B(x_0, \rho) \times [0,1]$, then $H_X(x,t) \in \mathrm{isom}(X,Y)$.

Let $(x,t) \in X \times [0,1], H(x,t) = 0$. Since zero is a regular value of $f(x) - tg(x)$, therefore $H_X(x,t)$ maps onto Y, therefore $\mathrm{codim}(\mathrm{R}(H_X(x,t))) = \dim(Y/Y) = 0$ and hence $\mathrm{ind}(H_X(x,t)) = \dim(\ker(H_X(x,t)))$.

Furthermore, since $\text{ind}(H_X(x,t)) = 0$, therefore $\dim(\ker(H_X(x,t))) = 0$, and hence $H_X(x,t)$ is also injective. Thus, $H_X(x,t)$ is a bijective linear continuous mapping, and since Y is a Banach space, the linear inverse mapping $H_X(x,t)^{-1} \in \mathcal{L}(Y,X)$ is also continuous. Hence, $H_X(x,t) \in \text{isom}(X,Y)$.

(a3) We will prove the existence of a compact set V'' which contains all $x \in X$ such that $f(x) - tg(x) = 0$, when $(x,t) \in B(x_0,\rho) \times [0,1]$. Let us define the set $V := g(D)$, where

$$D := \{x \in B(x_0,\rho) : \exists t \in [0,1],\ t = t(x),\ \text{such that } f(x) = tg(x)\}.$$

Since $f(x_i) = 0 = f(x_i) - 0g(x_i)$, $i = 1,\ldots,N$, therefore $x_i \in D$, $i = 1,\ldots,N$, hence D is not empty. Owing to $V \subset g(B(x_0,\rho))$, we know that V is a bounded set, and with g as a compact mapping, then V is a relatively compact set.

We now construct the set $V' := \{ty : t \in [0,1], y \in \overline{V}\}$. V' is a compact set in Y due to the fact that it can be written in the following way $V' = v(\overline{V} \times [0,1])$, where v is the continuous mapping $v : \overline{V} \times [0,1] \subset Y \times [0,1] \to Y$, $v(y,t) = ty$, and $\overline{V} \times [0,1]$ is a compact set in the topological product space $Y \times \mathbb{R}$.

Since f is a proper mapping and V' is a compact set on Y, the pre-image of V' under f, $V'' := f^{-1}(V')$ is a compact set on X, which contains all x which verify the following $f(x) - tg(x) = 0, (x,t) \in B(x_0,\rho) \times [0,1]$.

(a4) We will prove that there is a real number $C > 0$ such that if

$$(x,t) \in (H^{-1}\{0\}) \cap (B(x_0,\rho) \times [0,1]),$$

where $H^{-1}\{0\}$ is the pre-image of zero under H, then $\|H_X(x,t)^{-1}\| \leq C$, where $H_X(x,t)^{-1}$ is the inverse mapping of $H_X(x,t)$.

Since H is a C^1-mapping, the mapping $H_X : X \times \mathbb{R} \to \mathcal{L}(X,Y), (x,t) \mapsto H_X(x,t)$ is continuous. From (a2) if (x,t) belongs to $(H^{-1}\{0\}) \cap (B(x_0,\rho) \times [0,1])$, then $H_X(x,t) \in \text{Isom}(X,Y)$. From Theorem 2, the mapping inverse formation $\beta : \text{Isom}(X,Y) \subset \mathcal{L}(X,Y) \to \mathcal{L}(Y,X), \beta(u) = u^{-1}$, is a continuous mapping. Consequently, by composition of continuous mappings, the mapping

$$\|\cdot\| \circ \beta \circ H_X : H^{-1}\{0\} \cap (V'' \times [0,1]) \subset X \times \mathbb{R} \to \mathbb{R},\ (x,t) \mapsto \|H_X(x,t)^{-1}\|,$$

is continuous.

Since $H : X \times [0,1]$ is a continuous mapping, $H^{-1}\{0\} \subset X \times [0,1]$ is a closed set, and as $V'' \times [0,1] \subset X \times \mathbb{R}$ is a compact set, therefore $H^{-1}\{0\} \cap (V'' \times [0,1]) \subset X \times \mathbb{R}$ is a compact set. Weierstrass Theorem implies that there is maximum of $\|H_X(x,t)^{-1}\|$ when $(x,t) \in H^{-1}\{0\} \cap (V'' \times [0,1])$, and hence, there is a real number $C > 0$, such that $\|H_X(x,t)^{-1}\| \leq C$, $\forall (x,t) \in (H^{-1}\{0\}) \cap (V'' \times [0,1])$.

(a5) Let suppose that $(x_a,t_a) \in B(x_0,\rho) \times [0,1]$ and that $H(x_a,t_a) = 0$. Therefore:

From (a3), $(x_a,t_a) \in V'' \times [0,1]$.

From (a2), $H_X(x_a,t_a) \in \text{Isom}(X,Y)$.

We will now prove the existence of $r_0 > 0, r > 0$ and the existence a continuous mapping $x(\cdot) : [t_a - r_0, t_a + r_0] \cap [0,1] \to X$, which verifies

$$\|x(t)\| < r, H(x_a + x(t), t) = 0, \forall t \in [t_a - r_0, t_a + r_0] \cap [0,1].$$

To this end, we define $G(x,t) := H(x_a + x, t), \forall x \in X$, and we solve the equation

$$G(x,t) = 0 \tag{3}$$

for x. Obviously, we have $G(0, t_a) = H(x_a, t_a) = 0$, and furthermore, $G_X(0, t_a)$ verifies $G_X(0, t_a) = H_X(x_a, t_a)$.

We transform Equation (3) into the following equivalent equation:

$$H_X(x_a, t_a)^{-1}[H_X(x_a, t_a)(x) - G(x,t)] = x. \tag{4}$$

Equation (4) leads us to define the two following mappings

$$h(x,t) := H_X(x_a, t_a)(x) - G(x,t),$$

and

$$T_t(x) := H_X(x_a, t_a)^{-1}((h(x,t)),$$

where h is a C^1-mapping, and

$$h(0, t_a) = 0. \tag{5}$$

Equation (4) is equivalent to the following "key equation"

$$T_t(x) = x. \tag{6}$$

Let us observe that t in the definition of T_t is an index and not a partial derivative as is usually written. Equation (3) is equivalent to the Fixed Point Equation (6), which will be studied below.

Let $x, x' \in B(x_0, \rho)$; $t, t_a \in [0,1]$ such that $|t - t_a|$, $\|x\|$, $\|x'\| < r$, $|t - t_a| < r_0$, where r, r_0 will be fixed at a later stage.

Since $h_X(x,t) = H_X(x_a, t_a) - G_X(x,t)$, hence

$$h_X(0, t_a) = 0. \tag{7}$$

From Equation (7) and since $h_X : X \times [0,1] \to \mathcal{L}(X,Y)$, $(x,t) \mapsto h_X(x,t)$ is continuous, the Taylor theorem implies that

$$\begin{aligned} \|h(x,t) - h(x',t)\| &\le \sup\{\|h_X((x' + \theta(x - x'), t)\| : \theta \in [0,1]\}\|x - x'\| \\ &= o(1)\|x - x'\|, \quad o(1) \to 0 \quad \text{as} \quad r \to 0. \end{aligned} \tag{8}$$

Due to Equations (5) and (8), and since h is a continuous mapping, therefore

$$\|h(x,t)\| \le \|h(x,t) - h(0,t)\| + \|h(0,t)\| = o(1)\|x\| + o'(1),$$

$$o(1) \to 0 \quad \text{as} \quad r \to 0, \quad o'(1) \to 0 \quad \text{as} \quad r'_0 \to 0.$$

Hence

$$\|T_t(x)\| \le \|H_X(x_a, t_a)^{-1}\|\|h(x,t)\| \le \|H_X(x_a, t_a)^{-1}\|(o(1)\|x\| + o'(1)),$$

$$o(1) \to 0 \quad \text{as} \quad r \to 0, \quad o'(1) \to 0 \quad \text{as} \quad r'_0 \to 0. \tag{9}$$

Now r is fixed so that $o(1) \leq \frac{1}{2C}$, and then the closed and non-empty set $M := \{x \in X : \|x\| \leq r\}$ is constructed. We are now able to fix r_0', so that $o'(1) < \frac{r}{2C}$, and the set $M' := \{t \in [0,1] : |t - t_a| \leq \min\{r, r_0'\} = r_0\}$ is constructed. We prove below that the hypotheses of Theorem 1 are verified by the spaces and mappings, we have just defined.

The Metric Space $(M', |\cdot|)$ will be considered as the parameter space of the hypothesis (i) of Continuous Dependence Theorem 1. M will be considered as the closed and non-empty set and $(X, \|\cdot\|)$ as the complete metric space considered in hypothesis (ii) of Theorem 1, which is verified as we will see in the two following paragraphs.

Owing to Equation (9), for any fixed $t \in M'$, and for all $x \in M$,

$$\|T_t(x)\| \leq \|H_X(x_a, t_a)^{-1}\|(o(1)\|x\| + o'(1)) \leq C(\frac{1}{2C}r + \frac{1}{2C}r) \leq r,$$

therefore $T_t(x) \in M$, and hence $T_t : M \to M$. That is, T_t maps the closed non-empty set M of the Banach space X into itself.

Due to Equation (8), for any $x, x' \in M$ and all fixed $t \in M'$

$$\begin{aligned}\|T_t(x) - T_t(x')\| &\leq \|H_X(x_a, t_a)^{-1}(h(x,t) - h(x',t))\| \\ &\leq \|H_X(x_a, t_a)^{-1}\|\frac{1}{2C}\|x - x'\| \leq \frac{1}{2}\|x - x'\|,\end{aligned}$$

therefore T_t is half-contractive for any $t \in M'$ which has been fixed. Hence hypothesis (ii) of Theorem 1 is verified.

For any fixed $t_0 \in M'$ and for all $x \in M$,

$$\begin{aligned}\lim_{t\to t_0, t\in M'} T_t(x) &= \lim_{t\to t_0, t\in M'} H_X(x_a, t_a)^{-1}(H_X(x_a, t_a)(x) - G(x,t)) \\ &= H_X(x_a, t_a)^{-1}(H_X(x_a, t_a)(x) - G(x, t_0)) = T_{t_0}(x),\end{aligned}$$

and hence hypothesis (iii) of Theorem 1 is also verified.

Thus, Theorem 1 implies, for any $t \in M'$, that T_t has a unique fixed point $T_t(x) = x := x(t)$, and it is verified that $x(t) \to x(t_0)$ as $t \to t_0$, $t, t_0 \in M'$, that is, $x(\cdot)$ is a continuous mapping. Thus for each $t \in M'$ there is only one $x(t) \in M \subset X$ such that $G(x(t), t) = 0$, and hence

$$H(x_a + x(t), t) = 0. \tag{10}$$

Let us also observe that $T_{t_a}(0) = 0, x(t_a) = 0$. Equation (10) can be written in the following way: $H(\alpha(t), t) = 0, \forall t \in M'$, where α is the following continuous mapping $\alpha : M' \to Y$, $\alpha(t) := x_a + x(t)$.

(a6) We will now prove that f and g share exactly N values in the open ball $B(x_0, \rho)$, using iteratively the process of the previous section a finite number of times, to find each shared value. To this end we have to prove that it is possible to select the same r_0 for each iteration of the process of the previous section.

Let us define the mapping

$$\varphi : V'' \times [0,1] \times V'' \times [0,1] \subset X \times [0,1] \times X \times [0,1] \to Y,$$

$$\varphi(x_a,t_a;x,t) := H_X(x_a,t_a)x - H(x_a+x,t),$$

which, as a composition of continuous mappings, is uniformly continuous on the compact set $V'' \times [0,1] \times V'' \times [0,1]$ of the product topological space $X \times \mathbb{R} \times X \times \mathbb{R}$. Therefore for any fixed $r > 0$, there is $\delta(\frac{r}{2C}) > 0$ such that, if $(x_a,t_a;x,t),(x_{a'},t_{a'};x',t') \in V'' \times [0,1] \times V'' \times [0,1]$, with $\|(x_a,t_a;x,t) - (x_{a'},t_{a'};x',t')\| < \delta(\frac{r}{2C})$, then $\|\varphi(x_a,t_a;x,t) - \varphi(x_{a'},t_{a'};x',t')\| < \frac{r}{2C}$.

If we restrict the domain of mapping φ by fixing any $(x_a,t_a) \in V'' \times [0,1]$ such that $H(x_a,t_a) = 0$, we obtain mapping h considered in the previous section, that is

$$h : (H^{-1}\{0\}) \cap (V'' \times [0,1]) \subset X \times \mathbb{R} \to Y,$$
$$h(x,t) = \varphi(x_a,t_a;x,t) = H_X(x_a,t_a)(x) - H(x_a+x,t).$$

We are now able to fix r'_0 considered in the previous section by taking $r'_0 = \delta(\frac{r}{2C})$, where r will be established later in this section.

On the other hand the mapping $\varphi_X : V'' \times [0,1] \times V'' \times [0,1] \to \mathcal{L}(X,Y)$, $\varphi_X(x_a,t_a;x,t) = H_X(x_a,t_a) - H_X(x_a+x,t)$, is uniformly continuous on the compact set $V'' \times [0,1] \times V'' \times [0,1]$, and therefore there is $\delta(\frac{1}{2C}) > 0$ such that,

$$\forall (x_a,t_a;x,t),(x_{a'},t_{a'};x',t') \in V'' \times [0,1] \times V'' \times [0,1],$$
$$\|(x_a,t_a;x,t) - (x_{a'},t_{a'};x',t')\| < \delta(\frac{1}{2C})$$
$$\Rightarrow \|\varphi_X(x_a,t_a;x,t) - \varphi_X(x_{a'},t_{a'};x',t')\| < \frac{1}{2C}.$$

Let us observe that the mapping h_X considered in the previous section is the mapping φ_X, when (x_a,t_a) is fixed: $h_X : V'' \times [0,1] \to \mathcal{L}(X,Y)$,

$$h_X(x,t) = \varphi_X(x_a,t_a;x,t) = H_X(x_a,t_a) - H_X(x_a+x,t).$$

At this point we determine the previously mentioned r by taking $r = \delta(\frac{1}{2C})$. Since r and r'_0 have been fixed, we are now able to fix r_0 in the same way as in the previous section, that is, $r_0 = \min\{r, r'_0\}$.

The previous section implies that if $H(x_a,t_a) = 0, (x_a,t_a) \in B(x_0,\rho) \times [0,1]$ then there is a continuous mapping $x(\cdot) : [t_a - r_0, t_a + r_0] \cap [0,1] \to X$, which verifies

$$H(x_a + x(t), t) = 0, \forall t \in [t_a - r_0, t_a + r_0] \cap [0,1].$$

This lets us construct the continuous mapping $\alpha : [t_a, t_a + r_0] \to Y$, $\alpha(t) = x_a + x(t)$ with $H(\alpha(t),t) = 0$, $\forall t \in [t_a, t_a + r_0], \alpha(t_a) = x_a$.

We repeat this process of (a5) by taking $(\alpha(t_a + r_0), t_a + r_0)$ as an initial point in each iteration, where (x_a,t_a) is the previous initial point, and $(x_i, 0) \in B(x_0,\rho) \times [0,1], i \in 1,\ldots,N$ as the initial point of the first iteration with $\alpha : [0, r_0] \to Y, \alpha(t) = x_i + x(t), \alpha(0) = x_i$ to be extended in successive iterations of the process. A point $(x'_i, 1) \in B(x_0,\rho) \times [0,1]$ which verifies $H(x'_i,1) = 0, i \in 1,\ldots,N$ is reached in a finite number of iterations, since $[0,1]$ is a compact set, and from the frontier condition established in hypothesis (iv) of the theorem. In an identical way, but starting at in a value shared by f and g, and by the same process but taking initial successive points conveniently, we reach a zero of f on the ball $B(x_0, \rho)$. Therefore f has the same number of zeros that shared values by f and g. □

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Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Sevilla
Aptdo. 1160, Sevilla 41080, Spain
E-mail address: soriano@us.es