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HORIZONTALLY HOMOTHETIC HARMONIC MORPHISMS AND STABILITY OF TOTALLY GEODESIC SUBMANIFOLDS

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ABSTRACT. In this article, we study the relations of horizontally homothetic harmonic morphisms with the stability of totally geodesic submanifolds. Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally homothetic harmonic morphism from a Riemannian manifold into a Riemannian manifold of non-positive sectional curvature and let T be the tensor measuring minimality or totally geodesics of fibers of φ. We prove that if T is parallel and the horizontal distribution is integrable, then for any totally geodesic submanifold P in N, the inverse set, $\varphi^{-1}(P)$, is volume-stable in M. In case that P is a totally geodesic hypersurface, the condition on the curvature can be weakened to Ricci curvature.

1. Introduction

A harmonic map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is a critical point of the energy functional defined on each compact domain of M. A harmonic morphism between Riemannian manifolds is a map preserving harmonic structure. In other words, a map $\varphi : (M^n, g) \to (N^m, h)$ is called a harmonic morphism if for any harmonic function f defined on an open subset $V \subset N$ such that $\varphi^{-1}(V) \neq \emptyset$, the composition $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{R}$ is also harmonic. Harmonic morphisms are characterized as harmonic maps which are horizontally (weakly) conformal ([4], [6]).

Let $\varphi : (M^n, g) \to (N^m, h)$ be a harmonic morphism between Riemannian manifolds. Then it is well-known ([1]) that if $\dim(N) = m = 2$, the regular fibers of φ are minimal submanifolds of M and if $\dim(N) = m \geq 3$, φ has minimal fibers if and only if it is horizontally homothetic.

A minimal submanifold of a Riemannian manifold is a submanifold whose mean curvature defined as the trace of the second fundamental form determined by a normal vector field is vanishing. Or, equivalently, a minimal submanifold is a critical point of the volume functional defined on the variation of each

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compact domain. A submanifold of a Riemannian manifold is called totally geodesic if the second fundamental form vanishes. On the other hand, a minimal submanifold of a Riemannian manifold is called stable (or volume-stable) if the second derivative of the volume functional is non-negative for any normal variation with compact support. Not much results for stable minimal submanifolds are known compared with minimal submanifolds.

Given a horizontally (weakly) conformal map $\varphi : (M^n, g) \to (N^m, h)$ with $n \geq m$, there is a $(2,1)$-tensor T defined originally by O'Neill ([11]) measuring whether the fibers of φ are minimal or totally geodesic (see section 2 for definition). In fact, it is easy observation that the fibers of φ are totally geodesic if and only if T vanishes. We say the tensor T is parallel if the covariant derivative of T with respect to any vector field is vanishing. Thus, if a horizontally (weakly) conformal map $\varphi : (M^n, g) \to (N^m, h)$ has totally geodesic fibers, then T is automatically parallel.

In [3], we studied the stability of minimal fibers of horizontally conformal submersions between Riemannian manifolds. We proved that if $\varphi : (M^n, g) \to (N^m, h)$ is a horizontally conformal submersion with integrable horizontal distribution, and T is parallel, then any minimal fiber is volume-stable. Consequently, if $\varphi : (M^n, g) \to (N^m, h)$ is a submersive harmonic morphism with totally geodesic fibers and the horizontal distribution is integrable, then all the fibers are volume-stable. In case the dimension of N is two, we could obtain the same result without the condition of minimality of fibers.

On the other hand, in [10], Montaldo proved that if a submersive harmonic morphism $\varphi : (M^n, g) \to (N^m, h)$ from a compact Riemannian manifold to a surface has volume-stable minimal fibers, then φ is energy-stable, i.e., the second derivative of the energy functional is non-negative for any variation. Applying this fact to our result, one can conclude that if $\varphi : (M^n, g) \to (N^m, h)$ is a submersive harmonic morphism from a compact Riemannian manifold M to a surface N with integrable horizontal distribution, and T is parallel, then φ is energy-stable.

In this paper, we studied the stability of minimal submanifolds given by inverse sets of horizontally homothetic harmonic morphisms between Riemannian manifolds.

Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally homothetic harmonic morphism and let P be a minimal submanifold of N. Then it is well-known ([2]) that the inverse set $\varphi^{-1}(P)$ is a minimal submanifold of M. However, it is not true in general that $\varphi^{-1}(P)$ is volume-stable. We proved that if P is totally geodesic and T is parallel, then $\varphi^{-1}(P)$ is volume-stable if the horizontal distribution determined by φ is integrable and the sectional curvature of N is non-positive. If the dimension of P is zero, i.e., P is a single set in N, then $\varphi^{-1}(P)$ is a regular fiber and by [1], $\varphi^{-1}(P)$ is minimal. In this case we proved ([3]) that the fiber $L = \varphi^{-1}(P)$ is volume-stable without the curvature condition on N. Thus we may assume that the dimension of P is greater than or equal to one and $\dim(N) = m \geq 2$.

2. Area formula and stability

In this section, we shall describe basic notions for horizontally (weakly) conformal maps and stability of minimal submanifolds and derive the area formula for the second derivative of the volume functional.

Let $\varphi : (M^n, g) \to (N^m, h)$ be a smooth map between Riemannian manifolds (M, g) and (N, h). For a point $x \in M$, we set $\mathcal{V}_x = \ker(d\varphi_x)$ which is called the vertical space of φ at x. Let $\mathcal{H}_x$ denote the orthogonal complement of $\mathcal{V}_x$ in the tangent space T_xM. For a tangent vector $X \in T_xM$, we denote $X^{\mathcal{V}}$ and $X^{\mathcal{H}}$, respectively, the vertical component and the horizontal component of X. Let $\mathcal{V}$ and $\mathcal{H}$ denote the corresponding vertical and horizontal distributions in the tangent bundle TM. We say that φ is horizontally (weakly) conformal if, for each point $x \in M$ at which $d\varphi_x \neq 0$, the restriction $d\varphi_x|_{\mathcal{H}_x} : \mathcal{H}_x \to T_{\varphi(x)}N$ is conformal and surjective. Thus there exists a non-negative function λ on M such that

$$h(d\varphi(X), d\varphi(Y)) = \lambda^2 g(X, Y)$$

for horizontal vectors X, Y. The function λ is called the dilation of φ. Note that λ^2 is smooth and is equal to $|d\varphi|^2/m$, where $m = \dim(N)$.

Let $\varphi : M^n \to N^m$ be a horizontally (weakly) conformal map between Riemannian manifolds (M, g) and (N, h). Denote the set of critical points of φ by $C_\varphi = \{x \in M : d\varphi_x = 0\}$ and let $M^* = M - C_\varphi$. We define two tensors T and A over M^* by

$$T_E F = (\overline{\nabla}_{E^{\mathcal{V}}} F^{\mathcal{V}})^{\mathcal{H}} + (\overline{\nabla}_{E^{\mathcal{V}}} F^{\mathcal{H}})^{\mathcal{V}}$$

and

$$A_E F = (\overline{\nabla}_{E^{\mathcal{H}}} F^{\mathcal{H}})^{\mathcal{V}} + (\overline{\nabla}_{E^{\mathcal{H}}} F^{\mathcal{V}})^{\mathcal{H}}$$

for vector fields E and F on M. Here $\overline{\nabla}$ denotes the Levi-Civita connection on M.

A smooth map $\varphi : (M^n, g) \to (N^m, h)$ between Riemannian manifolds M and N of dimensions n and m, respectively, is called a harmonic morphism if φ preserves the harmonic structures of (M, g) and (N, h). In other words, $\varphi : (M^n, g) \to (N^m, h)$ is a harmonic morphism if it pull backs local harmonic functions to harmonic functions. It is well-known ([4], [6]) that a smooth map $\varphi : M \to N$ is a harmonic morphism if and only if φ is both harmonic and horizontally (weakly) conformal.

Let $\varphi : (M^n, g) \to (N^m, h)$ be a submersion with the horizontal distribution $\mathcal{H}$ and vertical distribution $\mathcal{V}$, respectively. Let $P^l \subset N$ be an l-dimensional submanifold of N and define

$$L = \varphi^{-1}(P).$$

For each $x \in L = \varphi^{-1}(P)$, we define

$$\mathcal{W}_x = T_xL, \quad \mathcal{H}'_x = W_x \cap \mathcal{H}, \quad \mathcal{H}''_x = W_x^{\perp}$$

so that we have the following orthogonal decompositions

$$\mathcal{W} = \mathcal{V} \oplus \mathcal{H}', \quad \mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'', \quad TM = \mathcal{V} \oplus \mathcal{H} = \mathcal{W} \oplus \mathcal{H}''.$$

The second fundamental form B of $L = \varphi^{-1}(P)$ in M is defined by

$$B : \mathcal{W} \times \mathcal{W} \to \mathcal{H}'', \qquad B(X, Y) = \left(\overline{\nabla}_X Y\right)^{\mathcal{H}''}, \tag{2.1}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on M. On the other hand, the second fundamental form B_P of P in N is given by

$$B_P : TP \times TP \to TP^{\perp}, \quad B_P(E, F) = \left({}^N\nabla_E F\right)^{\perp},$$

where $TP^{\perp}$ denotes the normal bundle of P and ${}^N\nabla$ is the Levi-Civita connection on N.

Let $\{e_1, \ldots, e_k\}$ be a local orthonormal frame on $\mathcal{V}$, where $k = n - m$, and let $\{\check{X}_1, \ldots, \check{X}_l\}$ be a local orthonormal frame on P. Denote the horizontal lift of $\check{X}_i$ by X_i, i.e.,

$$\mathrm{d}\varphi(X_i) = \check{X}_i \quad (i = 1, \ldots, l)$$

so that

$$\{\lambda X_1, \ldots, \lambda X_l\}$$

forms a local orthonormal frame on $\mathcal{H}'$. Then

$$\{e_1, \ldots, e_k, \lambda X_1, \ldots, \lambda X_l\}$$

is a local orthonormal frame on $L = \varphi^{-1}(P)$.

Note that

$$\dim \varphi^{-1}(P) = k + l = n - m + l$$

and normal vector field $E \in \Gamma(\mathcal{H}'')$ corresponds to $\mathrm{d}\varphi(E) \in \Gamma(TP^{\perp})$.

Let E be a normal vector field on $L = \varphi^{-1}(P)$ with compact support, i.e., E is a component of $\mathcal{H}''$. Then the second derivative of the volume functional $\mathcal{A}$ in the direction E ([8]) is given by

$$\mathcal{A}''(0) = \int_L \langle -\Delta E + \overline{\mathcal{R}}(E) - \mathcal{B}(E), E\rangle. \tag{2.2}$$

Introducing a local orthonormal basis

$$\{e_1, \ldots, e_k, \lambda X_1, \ldots, \lambda X_l, \xi_{l+1}, \ldots, \xi_m\}$$

on TM such that $\{\xi_{l+1}, \ldots, \xi_m\}$ is a local orthonormal frame on $TL^{\perp} = T\varphi^{-1}(P)^{\perp}$, the equation (2.2) becomes

$$\mathcal{A}''(0) = \int_L \{|\nabla^{\perp} E|^2 - \sum_{i=1}^{k+l} \langle \overline{R}(e_i, E)E, e_i\rangle - \sum_{i,j=1}^{k+l} \langle B(e_i, e_j), E\rangle^2\}. \tag{2.3}$$

Here $e_{k+i} = \lambda X_i$ for $i \geq k + 1$ and $\nabla^{\perp}$ denotes the normal connection on $TL^{\perp} = T\varphi^{-1}(P)^{\perp}$.

We say a minimally immersed submanifold L of M is *stable* (or volume-stable) if, for any normal variation E with compact support, the second derivative of the volume functional in the direction E is non-negative, i.e.,

$$\mathcal{A}''(0) \geq 0.$$

From now on, we shall carry out some computations on covariant derivatives and derive the area formula for the second derivative of the volume functional from the equation (2.3).

First of all, denoting, for $i \geq k+1$,

$$e_{k+i} = \lambda X_i,$$

we have, from (2.1),

$$\begin{aligned} \langle \mathcal{B}(E), E\rangle &= \sum_{i,j=1}^{k+l} \langle B(e_i,e_j),E\rangle^2 = \sum_{i,j=1}^{k+l} \langle (\overline{\nabla}_{e_i} e_j)^{\mathcal{H}''}, E\rangle^2 = \sum_{i,j=1}^{k+l} \langle e_j, \overline{\nabla}_{e_i} E\rangle^2 \\ &= \sum_{i=1}^{k+l}\left(\sum_{j=1}^{k} \langle e_j, \overline{\nabla}_{e_i} E\rangle^2 + \sum_{\alpha=1}^{l} \langle \lambda X_\alpha, \overline{\nabla}_{e_i} E\rangle^2\right) \\ &= \sum_{i=1}^{k+l}\left(\left|(\overline{\nabla}_{e_i} E)^{\mathcal{V}}\right|^2 + \left|(\overline{\nabla}_{e_i} E)^{\mathcal{H}'}\right|^2\right) \\ &= \sum_{i=1}^{k}\left(\left|(\overline{\nabla}_{e_i} E)^{\mathcal{V}}\right|^2 + \left|(\overline{\nabla}_{e_i} E)^{\mathcal{H}'}\right|^2\right) \\ &\quad + \lambda^2 \sum_{j=1}^{l}\left(\left|(\overline{\nabla}_{X_j} E)^{\mathcal{V}}\right|^2 + \left|(\overline{\nabla}_{X_j} E)^{\mathcal{H}'}\right|^2\right). \end{aligned} \tag{2.4}$$

Note that if E is a normal vector field on $L = \varphi^{-1}(P)$,

$$\begin{aligned} \left|\nabla^{\perp} E\right|^2 &= \sum_{i=1}^{k+l} \left|\nabla^{\perp}_{e_i} E\right|^2 = \sum_{i=1}^{k+l}\left|(\overline{\nabla}_{e_i} E)^{\perp}\right|^2 = \sum_{i=1}^{k+l}\left|(\overline{\nabla}_{e_i} E)^{\mathcal{H}''}\right|^2 \\ &= \sum_{i=1}^{k}\left|(\overline{\nabla}_{e_i} E)^{\mathcal{H}''}\right|^2 + \lambda^2 \sum_{j=1}^{l}\left|(\overline{\nabla}_{X_j} E)^{\mathcal{H}''}\right|^2. \end{aligned} \tag{2.5}$$

Next, recall that

$$\begin{aligned} \langle \mathcal{R}(E), E\rangle &= \sum_{i=1}^{k+l} \langle \overline{R}(e_i,E)e_i, E\rangle = -\sum_{i=1}^{k+l} \langle \overline{R}(e_i,E)E, e_i\rangle \\ &= -|E|^2 \sum_{i=1}^{k} K_M\left(\frac{E}{|E|}\wedge e_i\right) - |E|^2 \sum_{j=1}^{l} K_M\left(\frac{E}{|E|}\wedge \lambda X_j\right), \end{aligned}$$

where $K_M\left(\frac{E}{|E|}\wedge e_i\right)$ is the sectional curvature of the plane spanned by $\frac{E}{|E|}$ and e_i on M.

By [5] or [7], the sectional curvatures are given by, for $i=1,\dots,k$,

$$\begin{aligned} K_M\left(\frac{E}{|E|}\wedge e_i\right) &= \left|A_{\frac{E}{|E|}}e_i\right|^2-\left|T_{e_i}\frac{E}{|E|}\right|^2+\left\langle\left(\overline{\nabla}_{\frac{E}{|E|}}T\right)_{e_i}e_i,\frac{E}{|E|}\right\rangle \\ &\quad -\frac{1}{2}\left\langle\nabla\log\lambda^2,e_i\right\rangle^2+\frac{1}{2}\left\langle\overline{\nabla}_{e_i}\left(\nabla\log\lambda^2\right)^{\mathcal{V}},e_i\right\rangle \end{aligned} \tag{2.6}$$

and, for $j=1,\dots,l$,

$$\begin{aligned} K_M\left(\frac{E}{|E|}\wedge\lambda X_j\right) &= \lambda^2K_N\left(d\varphi\left(\frac{E}{|E|}\right)\wedge d\varphi(\lambda X_j)\right)-3\left|A_{\frac{E}{|E|}}(\lambda X_j)\right|^2 \\ &\quad -\frac{1}{4}\left\{\left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2-\left|\left(\nabla\log\lambda^2\right)^{\mathcal{H}}\right|^2\right\} \\ &\quad -\frac{1}{4}\left\{\left\langle\nabla\log\lambda^2,\frac{E}{|E|}\right\rangle^2+\left\langle\nabla\log\lambda^2,\lambda X_j\right\rangle^2\right\} \\ &\quad +\frac{1}{2}\left\langle\overline{\nabla}_{\frac{E}{|E|}}\left(\nabla\log\lambda^2\right)^{\mathcal{H}},\frac{E}{|E|}\right\rangle \\ &\quad +\frac{1}{2}\left\langle\overline{\nabla}_{\lambda X_j}\left(\nabla\log\lambda^2\right)^{\mathcal{H}},\lambda X_j\right\rangle, \end{aligned} \tag{2.7}$$

where ∇ denotes the gradient on M. So the gradient on $\varphi^{-1}(P)$ is given by

$$\nabla_{\varphi^{-1}(P)}f=(\nabla f)^{\mathcal{V}}+(\nabla f)^{\mathcal{H}'}.$$

Thus from now on, we shall use the confused notation for the gradient on M and $\varphi^{-1}(P)$ since there are no ambiguities.

By definition of the tensor A, we obtain for $i=1,\dots,k$

$$A_{\frac{E}{|E|}}e_i=\left(\overline{\nabla}_{\frac{E}{|E|}}e_i\right)^{\mathcal{H}}=\frac{1}{|E|}\left(\overline{\nabla}_Ee_i\right)^{\mathcal{H}}. \tag{2.8}$$

So,

$$\left|A_{\frac{E}{|E|}}e_i\right|^2=\frac{1}{|E|^2}\left|\left(\overline{\nabla}_Ee_i\right)^{\mathcal{H}}\right|^2=\frac{1}{|E|^2}\left[\left|\left(\overline{\nabla}_Ee_i\right)^{\mathcal{H}'}\right|^2+\left|\left(\overline{\nabla}_Ee_i\right)^{\mathcal{H}''}\right|^2\right]. \tag{2.9}$$

For $i=k+1,\dots,k+l$ ($j=1,\dots,l$),

$$\left|A_{\frac{E}{|E|}}(\lambda X_j)\right|^2=\frac{1}{|E|^2}\left|\left(\overline{\nabla}_E(\lambda X_j)\right)^{\mathcal{V}}\right|^2=\frac{\lambda^2}{|E|^2}\left|\left(\overline{\nabla}_EX_j\right)^{\mathcal{V}}\right|^2. \tag{2.10}$$

By definition of the tensor T, for $i=1,\dots,k$,

$$\begin{aligned} T_{e_i}\frac{E}{|E|} &= \left(\overline{\nabla}_{e_i}\frac{E}{|E|}\right)^{\mathcal{V}}=\left(\frac{1}{|E|}\overline{\nabla}_{e_i}E+e_i\left(\frac{1}{|E|}\right)E\right)^{\mathcal{V}} \\ &= \frac{1}{|E|}\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{V}}. \end{aligned}$$

So,

$$\left|T_{e_i}\frac{E}{|E|}\right|^2 = \frac{1}{|E|^2}\left|\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{V}}\right|^2. \tag{2.11}$$

Lemma 2.1 ([3]). *For $i = 1, \dots, k$,*

$$\left(\overline{\nabla}_{\frac{E}{|E|}}T\right)_{e_i} e_i = \frac{1}{|E|}\left(\overline{\nabla}_E T\right)_{e_i} e_i. \tag{2.12}$$

The following lemma is well-known ([5], [7]).

Lemma 2.2. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion with dilation λ. Then for horizontal vector fields X, Y,*

$$A_X Y = (\overline{\nabla}_X Y)^{\mathcal{V}} = \frac{1}{2}\left\{[X,Y]^{\mathcal{V}} - \lambda^2\langle X, Y\rangle \nabla\left(\frac{1}{\lambda^2}\right)^{\mathcal{V}}\right\}.$$

Corollary 2.3. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion. If the horizontal distribution is integrable, then for horizontal vector fields X, Y which are orthogonal,*

$$(\overline{\nabla}_X Y)^{\mathcal{V}} = 0.$$

Therefore, it follows from Corollary 2.3 and the equation (2.10) that if the horizontal distribution of the horizontally conformal submersion φ is integrable, then

$$A_{\frac{E}{|E|}}(\lambda X_j) = 0. \tag{2.13}$$

Now, assume $\varphi : (M^n, g) \to (N^m, h)$ is horizontally homothetic so that it is horizontally conformal submersion and

$$\left(\nabla \log \lambda^2\right)^{\mathcal{H}} = 0.$$

Then by Corollary 2.3 and the equations (2.7) and (2.10), we have

$$K_M\left(\frac{E}{|E|} \wedge \lambda X_j\right) = \lambda^2 K_N\left(\mathrm{d}\varphi\left(\frac{E}{|E|}\right) \wedge \mathrm{d}\varphi(\lambda X_j)\right) - \frac{1}{4}\left|\left(\nabla \log \lambda^2\right)^{\mathcal{V}}\right|^2.$$

Hence

$$\begin{aligned}
&\langle \mathcal{R}(E), E\rangle \\
&= -\sum_{i=1}^{k}\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}}\right|^2 + \sum_{i=1}^{k}\left|\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{V}}\right|^2 - \sum_{i=1}^{k}\left\langle\left(\overline{\nabla}_E T\right)_{e_i} e_i, E\right\rangle \\
&\quad + \frac{|E|^2}{2}\sum_{i=1}^{k}\left\langle \nabla \log \lambda^2, e_i\right\rangle^2 - \frac{|E|^2}{2}\sum_{i=1}^{k}\left\langle \overline{\nabla}_{e_i}\left(\nabla \log \lambda^2\right)^{\mathcal{V}}, e_i\right\rangle \\
&\quad - \lambda^2|E|^2\sum_{j=1}^{l} K_N\left(\mathrm{d}\varphi\left(\frac{E}{|E|}\right) \wedge \mathrm{d}\varphi(\lambda X_j)\right) + \frac{l|E|^2}{4}\left|\left(\nabla \log \lambda^2\right)^{\mathcal{V}}\right|^2.
\end{aligned} \tag{2.14}$$

Therefore, from (2.3), (2.4), (2.5), and (2.14), we obtain the following area formula.

Proposition 2.4. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontal homothetic map with integrable horizontal distribution $\mathcal{H}$. Let $P^l \subset N$ be an l-dimensional submanifold of N and define $L := \varphi^{-1}(P)$. Let $\{e_i\}$ and $\{X_j\}$ be as above and denote by $\mathcal{V}$ the vertical distribution. Then for any normal vector field E to L with compact support*

$$\begin{aligned}
\mathcal{A}''(0) \;=\; & \sum_{i=1}^{k} \int_L \left(\left| (\overline{\nabla}_{e_i} E)^{\mathcal{H}''} \right|^2 - \left| (\overline{\nabla}_E e_i)^{\mathcal{H}} \right|^2 - \left| (\overline{\nabla}_{e_i} E)^{\mathcal{H}'} \right|^2 \right) \\
& + \sum_{j=1}^{l} \int_L \lambda^2 \left(\left| (\overline{\nabla}_{X_j} E)^{\mathcal{H}''} \right|^2 - \left| (\overline{\nabla}_{X_j} E)^{\mathcal{H}'} \right|^2 \right) \\
& - \sum_{i=1}^{k} \int_L \left\langle (\overline{\nabla}_E T)_{e_i} e_i, E \right\rangle + \frac{l}{4} \int_L |E|^2 \left| (\nabla \log \lambda^2)^{\mathcal{V}} \right|^2 \qquad (2.15) \\
& + \int_L \left\{ \frac{|E|^2}{2} \sum_{i=1}^{k} \langle \nabla \log \lambda^2, e_i \rangle^2 - \frac{|E|^2}{2} \sum_{i=1}^{k} \left\langle \overline{\nabla}_{e_i} (\nabla \log \lambda^2)^{\mathcal{V}}, e_i \right\rangle \right\} \\
& - \int_L \lambda^2 |E|^2 \sum_{j=1}^{l} K_N \left(d\varphi \left(\frac{E}{|E|} \right) \wedge d\varphi(\lambda X_j) \right).
\end{aligned}$$

3. Some properties for horizontally conformal submersions

In this section, we are going to prove some properties for horizontally conformal submersions and covariant derivatives of a horizontal vector field and a vertical vector field of a horizontally weakly conformal map.

Lemma 3.1. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion with dilation λ. Let P^l be an l-dimensional submanifold of N and let $L = \varphi^{-1}(P)$. Assume $l \leq m - 1$. Then*

$$\lambda^2 \Delta_L \left(\frac{1}{\lambda^2} \right) = \left| (\nabla \log \lambda^2)^{\mathsf{T}} \right|^2 - \operatorname{div}_L (\nabla \log \lambda^2)^{\mathsf{T}},$$

where Δ_L and div_L denote the Laplacian and divergence on L, respectively, ∇ denotes the gradient on M and T denotes the tangential component of L, i.e., $\mathcal{W}$-component.

Proof. Let $k = n - m$ and $\{e_1, \ldots, e_k, \lambda X_1, \ldots, \lambda X_l, \xi_1, \ldots, \xi_{m-l}\}$ be a local orthonormal frame on M so that $\{e_1, \ldots, e_k, \lambda X_1, \ldots, \lambda X_l\}$ is tangent to L and $\{\xi_1, \ldots, \xi_{m-l}\}$ is normal to L.

Define $Z_\alpha = \frac{1}{\lambda} \xi_\alpha$ $(\alpha = 1, \ldots, m - l)$ so that $\lambda^2 |Z_\alpha|^2 = |\xi_\alpha|^2 = 1$. Then the derivative in the direction e_i becomes

$$0 = e_i(\lambda^2 |Z_\alpha|^2)$$

and so

$$e_i(|Z_\alpha|^2) = -\frac{e_i(\lambda^2)}{\lambda^2}|Z_\alpha|^2 = -|Z_\alpha|^2 e_i(\log\lambda^2).$$

Here recall that

$$e_{k+j} = \lambda X_j \quad \text{for} \quad j = 1, \ldots, l.$$

Thus,

$$\left(\nabla|Z_\alpha|^2\right)^{\intercal} = -|Z_\alpha|^2 \left(\nabla \log\lambda^2\right)^{\intercal}.$$

In fact, the following identity holds between M and L:

$$(\nabla f)^{\intercal} = \nabla_L f = \sum_{i=1}^{k+l} e_i(f)e_i \qquad \text{(locally)}$$

for any function f defined on M.

Now by definition of the Laplacian,

$$\Delta_L|Z_\alpha|^2 = \mathrm{div}_P \left(\nabla|Z_\alpha|^2\right)^{\intercal} = -\langle\left(\nabla|Z_\alpha|^2\right)^{\intercal}, \left(\nabla\log\lambda^2\right)^{\intercal}\rangle - |Z_\alpha|^2 \mathrm{div}_L \left(\nabla\log\lambda^2\right)^{\intercal}.$$

Since $\lambda^2 \left(\nabla\left(\frac{1}{\lambda^2}\right)\right)^{\intercal} = -\left(\nabla\log\lambda^2\right)^{\intercal}$, we have

$$\left(\nabla|Z_\alpha|^2\right)^{\intercal} = \left(\nabla\left(\frac{1}{\lambda^2}\right)\right)^{\intercal} = -\frac{1}{\lambda^2}\left(\nabla\log\lambda^2\right)^{\intercal}. \tag{3.1}$$

Hence

$$\Delta_L\left(\frac{1}{\lambda^2}\right) = \Delta_P(|Z_\alpha|^2) = \frac{1}{\lambda^2}\left|\left(\nabla\log\lambda^2\right)^{\intercal}\right|^2 - \frac{1}{\lambda^2}\mathrm{div}_L\left(\nabla\log\lambda^2\right)^{\intercal}.$$

That is,

$$\lambda^2\Delta_L\left(\frac{1}{\lambda^2}\right) = \left|\left(\nabla\log\lambda^2\right)^{\intercal}\right|^2 - \mathrm{div}_L\left(\nabla\log\lambda^2\right)^{\intercal}.$$

□

Remark 3.2. Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally homothetic submersion with dilation λ. Then

$$\left(\nabla\log\lambda^2\right)^{\intercal} = \left(\nabla\log\lambda^2\right)^{\mathcal{V}} + \left(\nabla\log\lambda^2\right)^{\mathcal{H}'} = \left(\nabla\log\lambda^2\right)^{\mathcal{V}}$$

and so we have

$$\lambda^2\Delta_L\left(\frac{1}{\lambda^2}\right) = \left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2 - \mathrm{div}_L\left(\nabla\log\lambda^2\right)^{\mathcal{V}}.$$

Lemma 3.3. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion. If X is a basic vector field and V is a vertical vector field on M, then*

$$\left(\overline{\nabla}_X V\right)^{\mathcal{H}} = \left(\overline{\nabla}_V X\right)^{\mathcal{H}}$$

and in particular,

$$\left|\left(\overline{\nabla}_V X\right)^{\mathcal{H}}\right|^2 = \left|\left(\overline{\nabla}_X V\right)^{\mathcal{H}}\right|^2.$$

Proof. Since X is a basic vector field and V is a vertical vector field, it is easy to see that

$$[X, V] = \overline{\nabla}_X V - \overline{\nabla}_V X$$

is a vertical vector field and so

$$\left(\overline{\nabla}_X V - \overline{\nabla}_V X\right)^{\mathcal{H}} = 0.$$

□

Corollary 3.4. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally conformal submersion and P be a submanifold of N. If X is a basic vector field and V is a vertical vector field on M, then*

$$\left(\overline{\nabla}_X V\right)^{\mathcal{H}'} = \left(\overline{\nabla}_V X\right)^{\mathcal{H}'}$$

and

$$\left(\overline{\nabla}_X V\right)^{\mathcal{H}''} \approx \left(\overline{\nabla}_V X\right)^{\mathcal{H}''}.$$

In particular,

$$\left|\left(\overline{\nabla}_V X\right)^{\mathcal{H}'}\right|^2 = \left|\left(\overline{\nabla}_X V\right)^{\mathcal{H}'}\right|^2$$

and

$$\left|\left(\overline{\nabla}_V X\right)^{\mathcal{H}''}\right|^2 \approx \left|\left(\overline{\nabla}_X V\right)^{\mathcal{H}''}\right|^2.$$

Let $\varphi : M^n \to N^m$ be a horizontally homothetic harmonic morphism with integrable horizontal distribution $\mathcal{H}$ and let $\{e_1, \ldots, e_k\}$ be a local orthonormal frame on the vertical distribution $\mathcal{V}$. Let P^l be an l-dimensional submanifold of N^m and let $L = \varphi^{-1}(P)$. Note that the dimension of L is $n - m + l$ and the dimension of the vertical distribution is $k := n - m$.

Let $\{\check{X}_1, \ldots, \check{X}_l\}$ be an (local) orthonormal frame for TP and $\{\check{\xi}_1, \ldots, \check{\xi}_{m-l}\}$ be an orthonormal frame for $TP^\perp$. Let X_j and $\widetilde{\xi_\alpha}$ be the horizontal lifts of $\check{X}_j$ and $\check{\xi}_\alpha$, respectively, so that $\{\lambda X_1, \ldots, \lambda X_l\}$ together with $\{e_1, \ldots, e_k\}$ forms a local orthonormal frame on TL and $\xi_\alpha := \lambda\widetilde{\xi_\alpha}$ is a local orthonormal frame on the normal space $TL^\perp$.

Lemma 3.5. *Let $\{e_i\}$ be a local orthonormal frame on the vertical distribution $\mathcal{V}$, and $\{X_j\}$ and $\{\xi_1, \ldots, \xi_{m-l}\}$ be as above. Then*

$$\begin{aligned}\lambda^2 \Delta_L\left(\frac{1}{\lambda^2}\right) &= \sum_{i=1}^{k} \langle \nabla \log \lambda^2, e_i\rangle^2 - \sum_{i=1}^{k} \left\langle \overline{\nabla}_{e_i} \left(\nabla \log \lambda^2\right)^{\mathcal{V}}, e_i \right\rangle \\ &\quad + \frac{l}{2} \left|\left(\nabla \log \lambda^2\right)^{\mathcal{V}}\right|^2.\end{aligned}$$

Proof. By Lemma 3.1

$$\lambda^2 \Delta_L\left(\frac{1}{\lambda^2}\right) = \left|\left(\nabla \log \lambda^2\right)^{\top}\right|^2 - \mathrm{div}_L(\nabla \log \lambda^2)^{\top}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k}\langle\nabla\log\lambda^2, e_i\rangle^2 - \sum_{i=1}^{k}\left\langle\overline{\nabla}_{e_i}\left(\nabla\log\lambda^2\right)^{\mathcal{V}}, e_i\right\rangle \\
&\quad -\lambda^2\sum_{j=1}^{l}\left\langle\overline{\nabla}_{X_j}\left(\nabla\log\lambda^2\right)^{\mathcal{V}}, X_j\right\rangle.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sum_{j=1}^{l}\left\langle\overline{\nabla}_{X_j}\left(\nabla\log\lambda^2\right)^{\mathcal{V}}, X_j\right\rangle &= -\sum_{j=1}^{l}\left\langle\left(\nabla\log\lambda^2\right)^{\mathcal{V}}, \overline{\nabla}_{X_j}X_j\right\rangle \\
&= -\sum_{j=1}^{l}\left\langle\left(\nabla\log\lambda^2\right)^{\mathcal{V}}, \left(\overline{\nabla}_{X_j}X_j\right)^{\mathcal{V}}\right\rangle.
\end{aligned}
$$

Since, by Lemma 2.2,

$$
\left(\overline{\nabla}_{X_j}X_j\right)^{\mathcal{V}} = -\frac{1}{2}\left(\nabla\left(\frac{1}{\lambda^2}\right)\right)^{\mathcal{V}} \qquad \text{and} \qquad \nabla\log\lambda^2 = -\lambda^2\nabla\left(\frac{1}{\lambda^2}\right),
$$

we have

$$
\lambda^2\sum_{j=1}^{l}\left\langle\overline{\nabla}_{X_j}\left(\nabla\log\lambda^2\right)^{\mathcal{V}}, X_j\right\rangle = \frac{l}{2}\left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2. \tag{3.2}
$$

Therefore

$$
\begin{aligned}
\lambda^2\Delta_L\left(\frac{1}{\lambda^2}\right) &= \sum_{i=1}^{k}\langle\nabla\log\lambda^2, e_i\rangle^2 - \sum_{i=1}^{k}\left\langle\overline{\nabla}_{e_i}\left(\nabla\log\lambda^2\right)^{\mathcal{V}}, e_i\right\rangle \\
&\quad +\frac{l}{2}\left|\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right|^2.
\end{aligned}
$$

□

Next, we shall consider the local orthonormal frame $\{\xi_1, \ldots, \xi_{m-l}\}$ of $TL^{\perp}$ and derive some relations of the covariant derivatives of ξ_α with the covariant derivative $\overline{\nabla}_{e_i}E$ for a normal vector field E whose values in $\mathcal{H}''$-distribution.

First of all, the following lemma follows directly from Lemma 2.2.

Lemma 3.6. *If $\alpha \neq \beta$,*

$$
\left(\overline{\nabla}_{\xi_\alpha}\xi_\beta\right)^{\mathcal{V}} = \frac{1}{2}[\xi_\alpha, \xi_\beta]^{\mathcal{V}}.
$$

Lemma 3.7.

$$
\left\langle\overline{\nabla}_{e_i}\xi_\alpha, \xi_\beta\right\rangle = -\frac{1}{2}\left\langle[\xi_\alpha, \xi_\beta]^{\mathcal{V}}, e_i\right\rangle.
$$

Proof. If $\alpha = \beta$, it is obvious since $|\xi_\alpha| = 1$. So, we may assume $\alpha \neq \beta$.

Since $\widetilde{\xi_\alpha}$ is basic and e_i is vertical, by Corollary 3.4,

$$
\left(\overline{\nabla}_{e_i}\widetilde{\xi_\alpha}\right)^{\mathcal{H}''} = \left(\overline{\nabla}_{\widetilde{\xi_\alpha}}e_i\right)^{\mathcal{H}''}. \tag{3.3}
$$

Since $\xi_\alpha = \lambda\widetilde{\xi_\alpha}$, by Lemma 3.6,

$$\begin{aligned}
\langle \overline{\nabla}_{e_i}\xi_\alpha, \xi_\beta\rangle &= \left\langle e_i(\lambda)\widetilde{\xi_\alpha} + \lambda\overline{\nabla}_{e_i}\widetilde{\xi_\alpha}, \xi_\beta\right\rangle \\
&= \lambda\left\langle \left(\overline{\nabla}_{e_i}\widetilde{\xi_\alpha}\right)^{\mathcal{H}''}, \xi_\beta\right\rangle = \lambda\left\langle \left(\overline{\nabla}_{\widetilde{\xi_\alpha}}e_i\right)^{\mathcal{H}''}, \xi_\beta\right\rangle \\
&= \langle \overline{\nabla}_{\xi_\alpha}e_i, \xi_\beta\rangle = -\left\langle \left(\overline{\nabla}_{\xi_\alpha}\xi_\beta\right)^{\mathcal{V}}, e_i\right\rangle = -\frac{1}{2}\left\langle [\xi_\alpha, \xi_\beta]^{\mathcal{V}}, e_i\right\rangle.
\end{aligned}$$

□

Corollary 3.8. *If the horizontal distribution $\mathcal{H}$ is integrable, then*

$$\langle \overline{\nabla}_{e_i}\xi_\alpha, \xi_\beta\rangle = 0 \ \textit{and so} \ \left(\overline{\nabla}_{e_i}\xi_\alpha\right)^{\mathcal{H}''} = 0.$$

Also by Lemma 2.2 and Corollary 2.3, we have

$$\left(\overline{\nabla}_{\xi_\alpha}\xi_\beta\right)^{\mathcal{V}} = -\frac{1}{2}\lambda^2\left(\nabla\left(\frac{1}{\lambda^2}\right)\right)^{\mathcal{V}}\delta_{\alpha\beta} = -\frac{1}{2}\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\delta_{\alpha\beta}. \tag{3.4}$$

Next, since it follows from Corollary 2.3 that $\left(\overline{\nabla}_{\xi_\alpha}X_j\right)^{\mathcal{V}} = 0$, we get

$$\begin{aligned}
\left(\overline{\nabla}_{\xi_\alpha}e_i\right)^{\mathcal{H}'} &= \sum_{j=1}^{l}\langle \overline{\nabla}_{\xi_\alpha}e_i, \lambda X_j\rangle \lambda X_j = -\lambda^2\sum_{j=1}^{l}\langle e_i, \overline{\nabla}_{\xi_\alpha}X_j\rangle X_j \\
&= -\lambda^2\sum_{j=1}^{l}\left\langle e_i, \left(\overline{\nabla}_{\xi_\alpha}X_j\right)^{\mathcal{V}}\right\rangle X_j = 0.
\end{aligned}$$

Thus,

$$\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}'} = \sum_{\alpha=1}^{m-l} f_\alpha\left(\overline{\nabla}_{\xi_\alpha}e_i\right)^{\mathcal{H}'} = 0. \tag{3.5}$$

On the other hand, note that, by Lemma 2.2,

$$\left(\overline{\nabla}_{\xi_\alpha}\xi_\beta\right)^{\mathcal{V}} = -\frac{1}{2}\lambda^2\left(\nabla\left(\frac{1}{\lambda^2}\right)\right)^{\mathcal{V}}\delta_{\alpha\beta} = -\frac{1}{2}\left(\nabla\log\lambda^2\right)^{\mathcal{V}}\delta_{\alpha\beta}.$$

So,

$$\begin{aligned}
\left(\overline{\nabla}_{\xi_\alpha}e_i\right)^{\mathcal{H}''} &= \sum_{\beta=1}^{m-l}\langle \overline{\nabla}_{\xi_\alpha}e_i, \xi_\beta\rangle \xi_\beta = -\sum_{\beta=1}^{m-l}\langle e_i, \overline{\nabla}_{\xi_\alpha}\xi_\beta\rangle \xi_\beta \\
&= \sum_{\beta=1}^{m-l}\left\langle e_i, \left(\overline{\nabla}_{\xi_\alpha}\xi_\beta\right)^{\mathcal{V}}\right\rangle \xi_\beta = \left\langle e_i, \left(\overline{\nabla}_{\xi_\alpha}\xi_\alpha\right)^{\mathcal{V}}\right\rangle \xi_\alpha \\
&= -\frac{1}{2}\left\langle e_i, \left(\nabla\log\lambda^2\right)^{\mathcal{V}}\right\rangle \xi_\alpha.
\end{aligned}$$

Hence,

$$\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}''} = \sum_{\alpha=1}^{m-l} f_\alpha \left(\overline{\nabla}_{\xi_\alpha} e_i\right)^{\mathcal{H}''} = -\frac{1}{2}\left\langle e_i, \left(\nabla \log \lambda^2\right)^{\mathcal{V}}\right\rangle E$$

and so

$$\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}''}\right|^2 = \frac{1}{4}\left\langle e_i, \left(\nabla \log \lambda^2\right)^{\mathcal{V}}\right\rangle^2 |E|^2.$$

Therefore, by (3.5),

$$\begin{aligned}\sum_{i=1}^{k}\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}}\right|^2 &= \sum_{i=1}^{k}\left|\left(\overline{\nabla}_E e_i\right)^{\mathcal{H}''}\right|^2 = \frac{1}{4}\sum_{i=1}^{k}\left\langle e_i, \left(\nabla \log \lambda^2\right)^{\mathcal{V}}\right\rangle^2 |E|^2 \\ &= \frac{1}{4}\left|\left(\nabla \log \lambda^2\right)^{\mathcal{V}}\right|^2 |E|^2 = \frac{1}{4}\sum_{\alpha=1}^{m-l} f_\alpha^2 \left|\left(\nabla \log \lambda^2\right)^{\mathcal{V}}\right|^2. \qquad (3.6)\end{aligned}$$

Next, since ξ_α is obtained from the basic normal vector field $\widetilde{\xi}_\alpha$, we have the following property.

Lemma 3.9. *If E is a normal vector field to L and the horizontal distribution $\mathcal{H}$ is integrable, then*

$$\left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}'} = 0.$$

Proof. Let ξ be an unit normal vector field obtained from a basic normal vector field. By linearity, it suffices to show that only for $E = f\xi$,

$$\left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}'} = 0.$$

First, note that for $E = f\xi$

$$\left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}'} = f\left(\overline{\nabla}_{e_i} \xi\right)^{\mathcal{H}'} = f\lambda^2 \sum_{j=1}^{m-1} \langle \overline{\nabla}_{e_i}\xi, X_j\rangle X_j.$$

On the other hand, for any vector fields X, Y, Z on M, the covariant derivative $\langle \overline{\nabla}_X Y, Z\rangle$ is given by

$$\begin{aligned}2\langle \overline{\nabla}_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &\quad + \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle. \qquad (3.7)\end{aligned}$$

Applying the equation (3.7) to $\langle \overline{\nabla}_{e_i}\xi, X_j\rangle$, we get

$$\langle \overline{\nabla}_{e_i}\xi, X_j\rangle = \langle [e_i, \xi], X_j\rangle - \langle [\xi, X_j], e_i\rangle.$$

Since the horizontal distribution $\mathcal{H}$ is integrable, $[\xi, X_j]$ is a horizontal vector field and so

$$\langle \overline{\nabla}_{e_i}\xi, X_j\rangle = \langle [e_i, \xi], X_j\rangle.$$

Now let $\widetilde{\xi}$ be the basic vector field so that $d\varphi(\widetilde{\xi}) = \check{\xi}$ and

$$\xi = \lambda\widetilde{\xi},$$

where $\breve{\xi}$ is an unit normal vector field on $P \subset N$. Then since $[e_i, \widetilde{\xi}]$ is a vertical vector field,

$$\langle [e_i, \xi], X_j \rangle = \langle e_i(\lambda)\xi + \lambda[e_i, \widetilde{\xi}], X_j \rangle = 0.$$

Hence

$$\left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}'} = 0.$$

□

Proposition 3.10. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally homothetic map with integrable horizontal distribution $\mathcal{H}$. Let $P^l \subset N$ be an l-dimensional submanifold of N and define $L = \varphi^{-1}(P)$. Let $\{e_i\}$ and $\{X_j\}$ be as above and denote by $\mathcal{V}$ the vertical distribution. Then for any normal vector field E to L with compact support*

$$(3.8)\qquad \begin{aligned} \mathcal{A}''(0) \; = \; & \sum_{i=1}^{k} \int_L \left| \left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}''} \right|^2 - \frac{1}{4} \int_L |E|^2 \left| \left(\nabla \log \lambda^2\right)^{\mathcal{V}} \right|^2 \\ & + \frac{1}{2} \int_L |E|^2 \lambda^2 \Delta \left(\frac{1}{\lambda^2}\right) - \sum_{i=1}^{k} \int_L \left\langle \left(\overline{\nabla}_E T\right)_{e_i} e_i, E \right\rangle \\ & + \sum_{j=1}^{l} \int_L \lambda^2 \left(\left| \left(\overline{\nabla}_{X_j} E\right)^{\mathcal{H}''} \right|^2 - \left| \left(\overline{\nabla}_{X_j} E\right)^{\mathcal{H}'} \right|^2 \right) \\ & - \int_L \lambda^2 |E|^2 \sum_{j=1}^{l} K_N \left(d\varphi\left(\frac{E}{|E|}\right) \wedge d\varphi(\lambda X_j) \right). \end{aligned}$$

Proof. The proof follows from Lemma 3.5, Lemma 3.9 and the equations (2.15), (3.6). □

4. Stability of totally geodesic submanifolds

In this section, we assume that $\varphi : (M^n, g) \to (N^m, h)$ is a horizontal homothetic map with integrable horizontal distribution $\mathcal{H}$ and $P^l \subset N$ is an l-dimensional totally geodesic submanifold of N and define $L = \varphi^{-1}(P)$. We are going to use the same notations as in the previous sections for $\{e_i\}$, $\{X_j\}$ and ξ_α, and denote by $\mathcal{V}$ the vertical distribution.

Let E be a (local) normal vector field to TL, i.e., a section of $TL^\perp$ with compact support. Then we can write

$$E = \sum_{\alpha=1}^{m-l} f_\alpha \xi_\alpha,$$

where f_α is a local function with compact support. Then, by Corollary 3.8,

$$\left(\overline{\nabla}_{e_i} E\right)^{\mathcal{H}''} = \sum_{\alpha=1}^{m-l} e_i(f_\alpha)\xi_\alpha + f\alpha \left(\overline{\nabla}_{e_i} \xi_\alpha\right)^{\mathcal{H}''} = \sum_{\alpha=1}^{m-l} e_i(f_\alpha)\xi_\alpha$$

and so

$$\left|\left(\overline{\nabla}_{e_i}E\right)^{\mathcal{H}''}\right|^2=\sum_{\alpha=1}^{m-l}e_i(f_\alpha)^2. \tag{4.1}$$

Lemma 4.1. *Let $\varphi : M^n \to N^m$ be a horizontally homothetic map and let P^l be an l-dimensional totally geodesic submanifold of N^m and let $L=\varphi^{-1}(P)$. Then for any normal vector field E to TL,*

$$\left(\overline{\nabla}_X E\right)^{\mathcal{H}'}=0$$

for any basic vector field X on M.

Proof. Let $\{\check{X}_1,\dots,\check{X}_l\}$ be an (local) orthonormal frame for TP and $\{\check{\xi}_1,\dots,\check{\xi}_{m-l}\}$ be an orthonormal frame for $TP^\perp$. Let X_j and $\widetilde{\xi_\alpha}$ be the horizontal lifts of $\check{X}_j$ and $\check{\xi}_\alpha$, respectively, so that $\{\lambda X_1,\dots,\lambda X_l\}$ together with $\{e_1,\dots,e_k\}$ forms a local orthonormal frame on TL and $\xi_\alpha := \lambda\widetilde{\xi_\alpha}$ is a local orthonormal frame on the normal space $TL^\perp$. Let $d\varphi(X)=\check{X}$. Since P is totally geodesic, it follows from definition that

$$\left({}^N\nabla_{\check{X}}\check{X}_j\right)^\perp=0$$

and so by [7]

$$d\varphi\left(\left(\overline{\nabla}_X X_k\right)^{\mathcal{H}''}\right)=\left({}^N\nabla_{\check{X}}\check{X}_j\right)^\perp=0.$$

Thus,

$$\left(\overline{\nabla}_X X_k\right)^{\mathcal{H}''}=0.$$

Also since φ is horizontally homothetic, the derivative of the conformal factor λ in the direction X is vanishing, i.e., $X(\lambda)=\langle\nabla\lambda, X\rangle=0$. Hence, for each $\alpha=1,\dots,m-l$

$$\begin{aligned}\left(\overline{\nabla}_X\widetilde{\xi_\alpha}\right)^{\mathcal{H}''} &= \left\langle\overline{\nabla}_X\widetilde{\xi_\alpha},\lambda X_k\right\rangle\lambda X_k=-\left\langle\widetilde{\xi_\alpha},\overline{\nabla}_X\lambda X_k\right\rangle\lambda X_k\\ &= -\lambda^2\left\langle\widetilde{\xi_\alpha},\overline{\nabla}_X X_k\right\rangle X_k=-\lambda^2\left\langle\widetilde{\xi_\alpha},\left(\overline{\nabla}_X X_k\right)^{\mathcal{H}''}\right\rangle X_k\\ &= 0.\end{aligned}$$

Therefore, letting $E=\sum_{\alpha=1}^{m-l}f_\alpha\xi_\alpha$,

$$\left(\overline{\nabla}_X E\right)^{\mathcal{H}'}=\sum_{\alpha=1}^{m-l}f_\alpha\left(\overline{\nabla}_X\xi_\alpha\right)^{\mathcal{H}'}=\sum_{\alpha=1}^{m-l}\lambda f_\alpha\left(\overline{\nabla}_X\widetilde{\xi_\alpha}\right)^{\mathcal{H}'}=0.$$

□

We are ready to prove our main result.

Theorem 4.2. *Let $\varphi:(M^n,g)\to(N^m,h)$ be a horizontally homothetic harmonic morphism with dilation λ to a Riemannian manifold N of non-positive sectional curvature. Let P be a totally geodesic submanifold of N and let*

$L = \varphi^{-1}(P)$. *If T is parallel and the horizontal distribution of φ is integrable, then L is a stable minimal submanifold of M.*

Proof. First of all, note that L is a minimal submanifold of M by [2]. By Proposition 3.10, the equation (4.1) and Lemma 4.1 together with our assumptions, we obtain for any normal vector field $E = \sum_{\alpha=1}^{m-l} f_\alpha \xi_\alpha$ with compact support,

$$\begin{aligned} \mathcal{A}''(0) &= \sum_{\alpha=1}^{m-l}\left\{\int_L \left|(\nabla f_\alpha)^{\mathcal{V}}\right|^2 - \frac{1}{4}\int_L f_\alpha^2 \left|(\nabla \log \lambda^2)^{\mathcal{V}}\right|^2\right\} \\ &\quad + \sum_{\alpha=1}^{m-l}\frac{1}{2}\int_L f_\alpha^2 \lambda^2 \Delta\left(\frac{1}{\lambda^2}\right) + \sum_{j=1}^{l}\int_L \lambda^2 \left|(\overline{\nabla}_{X_j} E)^{\mathcal{H}''}\right|^2 \\ &\quad - \sum_{\alpha=1}^{m-l}\int_L \lambda^2 f_\alpha^2 \sum_{j=1}^{l} K_N\left(\mathrm{d}\varphi\left(\frac{E}{|E|}\right)\wedge \mathrm{d}\varphi(\lambda X_j)\right). \end{aligned}$$

Applying the integration by parts to each $f_\alpha = f$, we have

$$\begin{aligned} \frac{1}{2}\int_L f^2\lambda^2\Delta\left(\frac{1}{\lambda^2}\right) &= -\frac{1}{2}\int_L \left\langle \nabla(f^2\lambda^2), \nabla\left(\frac{1}{\lambda^2}\right)\right\rangle \\ &= -\frac{1}{2}\int_L \left\langle \lambda^2\nabla f^2 + f^2\nabla\lambda^2, \nabla\left(\frac{1}{\lambda^2}\right)\right\rangle \\ &= \frac{1}{2}\int_L \left\langle \nabla f^2, (\nabla\log\lambda^2)^{\mathcal{V}}\right\rangle - \frac{1}{2}\int_L f^2\left\langle \nabla\lambda^2, \nabla\left(\frac{1}{\lambda^2}\right)\right\rangle \\ &= \int_L f\left\langle \nabla f, (\nabla\log\lambda^2)^{\mathcal{V}}\right\rangle + 2\int_L f^2\frac{|\nabla\lambda|^2}{\lambda^2} \\ &= \int_L f\left\langle \nabla f, (\nabla\log\lambda^2)^{\mathcal{V}}\right\rangle + \frac{1}{2}\int_L f^2\left|(\nabla\log\lambda^2)^{\mathcal{V}}\right|^2. \end{aligned}$$

Here ∇ denotes the gradient on P (Actually, we used a confused notation for gradients on P and M, but there are no ambiguities). Thus,

$$\begin{aligned} \mathcal{A}''(0) &= \sum_{\alpha=1}^{m-l}\left\{\int_L \left|(\nabla f_\alpha)^{\mathcal{V}}\right|^2 + \frac{1}{4}\int_L f_\alpha^2 \left|(\nabla \log \lambda^2)^{\mathcal{V}}\right|^2\right\} \\ &\quad + \int_L f_\alpha\left\langle \nabla f_\alpha, (\nabla\log\lambda^2)^{\mathcal{V}}\right\rangle + \sum_{j=1}^{l}\int_L \lambda^2 \left|(\overline{\nabla}_{X_j} E)^{\mathcal{H}''}\right|^2 \\ &\quad - \sum_{\alpha=1}^{m-l}\int_L \lambda^2 f_\alpha^2 \sum_{j=1}^{l} K_N\left(\mathrm{d}\varphi\left(\frac{E}{|E|}\right)\wedge \mathrm{d}\varphi(\lambda X_j)\right). \end{aligned} \tag{4.2}$$

Since, for each $f_\alpha = f$,

$$\begin{aligned} \left|f\left\langle \nabla f, (\nabla\log\lambda^2)^{\mathcal{V}}\right\rangle\right| &= 2\cdot\frac{1}{2}\left|f\left\langle (\nabla f)^{\mathcal{V}}, (\nabla\log\lambda^2)^{\mathcal{V}}\right\rangle\right| \\ &\le |(\nabla f)^{\mathcal{V}}|^2 + \frac{1}{4}f^2\left|(\nabla\log\lambda^2)^{\mathcal{V}}\right|^2. \end{aligned}$$

Hence

$$\begin{aligned}\mathcal{A}''(0) &\geq -\sum_{\alpha=1}^{m-l}\int_L \lambda^2 f_\alpha^2 \sum_{j=1}^{l} K_N\left(\mathrm{d}\varphi\left(\frac{E}{|E|}\right)\wedge \mathrm{d}\varphi(\lambda X_j)\right)\\ &\quad +\sum_{j=1}^{l}\int_L \lambda^2\left|(\overline{\nabla}_{X_j}E)^{\mathcal{H}''}\right|^2 \geq 0\end{aligned}$$

by curvature assumption on N. □

As a direct application, we can obtain the following result.

Corollary 4.3. *Let $\varphi:(M^n,g)\to(N^m,h)$ be a horizontally homothetic harmonic morphism with totally geodesic fibers to a Riemannian manifold N of non-positive sectional curvature. Let P be a totally geodesic submanifold of N and let $L=\varphi^{-1}(P)$. If the horizontal distribution of φ is integrable, then L is a stable minimal submanifold of M.*

Example 4.4. Let $\varphi:\mathbb{R}^n\to\mathbb{R}^m$ $(n\geq m)$ be the canonical projection defined by

$$\varphi(x^1,x^2,\ldots,x^m,x^{m+1},\ldots,x^n)=(x^1,x^2,\ldots,x^m).$$

Then it is easy to see that the map φ is a horizontally homothetic harmonic morphism and the fiber $\varphi^{-1}(p)$, of a point $p\in\mathbb{R}^m$ is $\mathbb{R}^{n-m}$ which is totally geodesic. Also the horizontal space of φ is integrable and so the map $\varphi:\mathbb{R}^n\to\mathbb{R}^m$ $(n\geq m)$ satisfies the conditions in Theorem 4.2. Therefore, for any totally geodesic submanifold P of $\mathbb{R}^m$, the inverse set $\varphi^{-1}(P)$ is volume-stable minimal submanifold of $\mathbb{R}^n$.

In fact, it is well-known that any totally geodesic submanifold of $\mathbb{R}^m$ is the Euclid subspace $\mathbb{R}^k$ $(k\leq m)$ and we have

$$\varphi^{-1}(\mathbb{R}^k)=\mathbb{R}^{n-m}\times\mathbb{R}^k$$

which is both totally geodesic and volume-stable.

Example 4.5. Let $\mathbb{H}^n=\mathbb{R}^{n-1}\times\mathbb{R}^+$ be the upper-half space with the standard hyperbolic metric $g=\frac{1}{(x^n)^2}\langle\ ,\ \rangle_{\mathbb{R}^n}$. Let $\pi:\mathbb{H}^n\to\mathbb{R}^{n-1}$ be the projection defined by $\pi(p,x)=p$. For $p\in\mathbb{R}^{n-1}$ the fiber of π over p is parametrized with respect to arclength by $\gamma_p(s)=(p,e^s)$. Along the fibers we have $\lambda^2(s)=e^{2s}$ and so the map π is horizontally homothetic and it has totally geodesic fibers. The level hypersurfaces corresponding to the horizontal space are parallel affine subspaces

$$\mathbb{R}^{n-1}_{e^s}=\{(p,e^s)\in\mathbb{R}^{n-1}\times\mathbb{R}^+\ :\ p\in\mathbb{R}^{n-1}\}$$

with constant sectional curvature $K_{\mathbb{R}^{n-1}_{e^s}}=0$. Thus the horizontal space is integrable. Consequently, the projection $\pi:\mathbb{H}^n\to\mathbb{R}^{n-1}$ satisfies the conditions in Theorem 4.2. Therefore, for any totally geodesic submanifold P of $\mathbb{R}^{n-1}$, the inverse set $\pi^{-1}(P)$ is volume-stable. In fact, for $P=\mathbb{R}^k$, we have

$$\pi^{-1}(\mathbb{R}^k)=\mathbb{R}^k\times\mathbb{R}^+$$

which is totally geodesic. It is well-known that for any Riemannian manifold of non-positive Ricci curvature, any totally geodesic submanifold is automatically volume-stable.

Remark 4.6. In view of examples, the integrability of the horizontal distribution of a harmonic morphism is deeply related with the curvature of manifolds. For example, let $S^3 = \{(z,w) \in \mathbb{C}^2 \; : \; |z|^2 + |w|^2 = 1\}$ and consider the Hopf map $\varphi : S^3 \to S^2$ defined by

$$\varphi(z,w) = (2z\bar{w}, |z|^2 - |w|^2)$$

in complex variables. Then φ is a harmonic morphism and the fibers of φ are circles S^1 which is totally geodesic. However, the horizontal distributions are nowhere integrable. Lack of examples, the authors expect that for any horizontally homothetic harmonic morphism $\varphi : M \to N$ between Riemannian manifolds, the horizontal distribution is integrable if N has non-positive sectional curvature. It is known that the sectional curvature of the horizontal distribution is non-negative and the sectional curvature of N is non-positive, then the horizontal distribution is integrable.

In case of hypersurfaces, the assumption on the sectional curvature condition in Theorem 4.2 can be weakened to non-positive Ricci curvature.

Corollary 4.7. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a horizontally homothetic harmonic morphism to a Riemannian manifold N of non-positive Ricci curvature. Let P be a totally geodesic hypersurface of N and let $L = \varphi^{-1}(P)$. If T is parallel and the horizontal distribution of φ is integrable, then L is a stable minimal hypersurface of M.*

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