J. Korean Math. Soc. **45** (2008), No. 2, pp. 479–491

# THE GENERALIZED PASCAL MATRIX VIA THE GENERALIZED FIBONACCI MATRIX AND THE GENERALIZED PELL MATRIX

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Abstract. In [4], the authors studied the Pascal matrix and the Stirling matrices of the first kind and the second kind via the Fibonacci matrix. In this paper, we consider generalizations of Pascal matrix, Fibonacci matrix and Pell matrix. And, by using Riordan method, we have factorizations of them. We, also, consider some combinatorial identities.

## 1. Introduction

The Pascal numbers, the Fibonacci number and the Pell number are very interesting in combinatorial analysis.

For integers $n, i$ and $j$, $n \geq i, j \geq 1$, the $n \times n$ *Pascal matrix* $P_n = [p_{ij}]$ is defined by

$$p_{ij} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

In [1], the authors gave matrices representations of the Pascal triangle. More generally, for nonzero real variable $x$, the Pascal matrix was generalized in $P[x]_n$ *and* $Q[x]_n$, *respectively which are defined in [8], and these generalized* Pascal matrices were also extended in $\Phi[x,y]_n = [\varphi[x,y]_{ij}]$ (see [9]) for any two nonzero real variables $x$ and $y$ where

$$\varphi[x,y]_{ij} = \begin{cases} \binom{i-1}{j-1} x^{i-j} y^{i+j-2} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper, we call $\Phi[x,y]_n$ the *GP matrix*. By the definition, we have

$$\begin{aligned} P[x]_n &= \Phi[x,1]_n, \ Q[y]_n = \Phi[1,y]_n, \ P_n = P[1]_n = Q[1]_n = \Phi[1,1]_n, \\ P[x]_n^{-1} &= P[-x]_n = \left[(-1)^{i-j}\binom{i-1}{j-1}x^{i-j}\right]. \end{aligned}$$

Received August 29, 2006; Revised October 30, 2006.
2000 *Mathematics Subject Classification.* 05A19, 05B20, 11B39.
*Key words and phrases.* Pascal matrix, Fibonacci matrix, Pell matrix, Riordan matrix.
This paper was supported by Hanseo University, 2006.

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In [8] and [9], the factorizations of $P[x]_n$, $Q[x]_n$, and $\Phi[x,y]_n$ are obtained, respectively.

The Fibonacci sequence has been discussed in so many articles and books. In [5], the authors introduced the *Fibonacci matrix* $\mathcal{F}_n = [f_{ij}]$ of order $n$ as follows

$$\mathcal{F}_n = [f_{ij}] = \begin{cases} F_{i-j+1} & i-j+1 \geq 0, \\ 0 & i-j+1 < 0, \end{cases}$$

where $F_k$ is the $k$th Fibonacci number.

Now, we introduce a generalization of the Fibonacci matrix. For any two nonzero real numbers $x$ and $y$ and positive integer $n$, the $n \times n$ *GF matrix* $\mathcal{F}[x,y]_n = [f[x,y]_{ij}]$ defined by

$$f[x,y]_{ij} = \begin{cases} F_{i-j+1}x^{i-j}y^{i+j-2} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition, we have $\mathcal{F}[1,1]_n = \mathcal{F}_n$.

In [5], the authors gave the Cholesky factorization of the Fibonacci matrix $\mathcal{F}_n$ and they also discussed eigenvalues of the symmetric Fibonacci matrix. In [4], the authors studied the Pascal matrix and the Stirling matrix of the first kind and of the second kind via the Fibonacci matrix and some combinatorial identities are obtained from the matrices representations of the Pascal matrix, the Stirling matrices and the Fibonacci matrix.

The Pell sequence $\{a_n\}$ is defined recursively by the equation

$$a_{n+1} = 2a_n + a_{n-1}$$

for $n \geq 2$, where $a_1 = 1$, $a_2 = 2$. The Pell sequence is

$$1, 2, 5, 12, 29, 70, 169, 408, \ldots.$$

In [2] and [3], the authors gave well-known Pell identities as follows, for arbitrary integers $q$ and $r$

$$a_{n+q}a_{n+r} - a_n a_{n+q+r} = a_q a_r(-1)^n, a_{2n+1} = a_n^2 + a_{n+1}^2,$$

and for $n$th Pell number $a_n$,

$$a_n = \sum_{r=0}^{[(n-1)/2]} \binom{n-1-r}{r} 2^{n-1-2r}.$$

We define $n \times n$ *Pell matrix* $S_n = [s_{ij}]$ as follows

$$s_{ij} = \begin{cases} a_{i-j+1}, & i-j+1 \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_n$ is the $n$th Pell number.

Now, we consider a generalization of the Pell matrix. For any two nonzero real numbers $x$ and $y$ and positivie integer $n$, the $n \times n$ *generalized Pell matrix*,

$S[x,y]_n = [s[x,y]_{ij}]$ defined by

$$s[x,y]_{ij} = \begin{cases} a_{i-j+1}x^{i-j}y^{i+j-2}, & i-j+1 \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_n$ is the $n$th Pell number. By the definition, we have $S[1,1]_n = S_n$.

In [7], the authors introduced a group which called the Riordan group, and they gave some applications in the group. In [6], the authors introduced Riordan matrix, and they proved that each Riordan matrix $R$ in the group can be factorized by Pascal matrix $P$, Catalan matrix $C$ and Fibonacci matrix $F$ as $R = PCF$.

In [7], the Riordan group was defined as follows:

Let $R = [r_{ij}]_{i,j\geq 0}$ be an infinite matrix with entries in the complex numbers. Let $c_i(t) = \sum_{n\geq 0}^{\infty} r_{n,i}t^n$ be the generating function of the $i$th column of $R$. We call $R$ a *Riordan matrix* if $c_i(t) = g(t)[f(t)]^i$, where

$$g(t) = 1 + g_1 t + g_2 t^2 + g_3 t^3 + \cdots, \text{ and } f(t) = t + f_2 t^2 + f_3 t^3 + \cdots.$$

In this case, we write $R = (g(t), f(t))$. We denote by $\Re$ the set of Riordan matrices. Then the set $\Re$ is a group under matrix multiplication, $*$, with the following properties:

(R1) $(g(t), f(t)) * (h(t), l(t)) = (g(t)h(f(t)), l(f(t)))$.
(R2) $I = (1, t)$ is the identity element.
(R3) The inverse of $R$ is given by $R^{-1} = \left(\frac{1}{g(\bar{f}(t))}, \bar{f}(t)\right)$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$, i.e., $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.
(R4) If $(a_0, a_1, a_2, \ldots)^T$ is a column vector with generating function $A(t)$, then multiplying $R = (g(t), f(t))$ on the right by this column vector yields a column vector with generating function $B(t) = g(t)A(f(t))$.

We call $\Re$ a *Riordan group*. From the definition of the Riordan matrix, we know that the matrices in the Riordan group are infinite and lower triangular.

Here are three examples about the Riordan matrices.

The first example of element in $\Re$ is the Pascal matrix, and the following representation is well-known.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 2 & 1 & 0 & 0 & 0 & \vdots \\ 1 & 3 & 3 & 1 & 0 & 0 & \\ 1 & 4 & 6 & 4 & 1 & 0 & \\ & & & \cdots & & & \end{bmatrix} = (g_P(t), f_P(t)) = \left(\frac{1}{1-t}, \frac{t}{1-t}\right).$$

The next example is the Fibonacci matrix. We consider the infinite Fibonacci matrix $\mathcal{F} = [F_{ij}]$ as follows;

$$\mathcal{F} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & \\ 2 & 1 & 1 & 0 & 0 & 0 & \vdots \\ 3 & 2 & 1 & 1 & 0 & 0 & \\ 5 & 3 & 2 & 1 & 1 & 0 & \\ & & & \cdots & & & \end{bmatrix} = (g_{\mathcal{F}}(t), f_{\mathcal{F}}(t)).$$

Since the first column of $\mathcal{F}$ is $(1,1,2,3,5,\ldots)^T$, it is obvious that $g_{\mathcal{F}}(t) = \sum_{n=0}^{\infty} F_{n+1}t^n = \frac{1}{1-t-t^2}$. The rule of formation in $\mathcal{F}$ is that each entry is the sum of the elements in the upper two rows. In other words, $F_{n+1,j} = F_{n,j} + F_{n-1,j}$, $j \geq 1$. So, we have $f_{\mathcal{F}}(t) = t$ because $c_j(t) = g_{\mathcal{F}}(t) \cdot [f_{\mathcal{F}}(t)]^j = \frac{1}{1-t-t^2}t^j$, i.e.,

$$\mathcal{F} = (g_{\mathcal{F}}(t), f_{\mathcal{F}}(t)) = \left(\frac{1}{1-t-t^2}, t\right)$$

and hence $\mathcal{F}$ is in $\Re$.

Finally, we consider the infinite Pell matrix $S = [s_{ij}]$ as follows;

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & 0 & 0 & \\ 5 & 2 & 1 & 0 & 0 & 0 & \vdots \\ 12 & 5 & 2 & 1 & 0 & 0 & \\ 29 & 12 & 5 & 2 & 1 & 0 & \\ & & & \cdots & & & \end{bmatrix} = (g_S(t), f_S(t)).$$

Since the first column of $S$ is $(1,2,5,12,29,\ldots)^T$, we have $g_S(t) = \frac{1}{1-2t-t^2}$. By the rule of formation in $S$, it is obvious that $f_S(t) = t$. That is,

$$S = (g_S(t), f_S(t)) = \left(\frac{1}{1-2t-t^2}, t\right)$$

and hence $S$ is also in $\Re$.

In this paper, we consider the relationships between GP matrix and GF matrix and generalized Pell matrix $S[x,y]$. Also, we give some interesting combinatorial identities.

## 2. Main theorems

To begin with, we define a matrix. For any two nonzero real variables $x$ and $y$, an infinite matrix $L[x,y] = [\ell[x,y]_{ij}]$ is defined as follows:

$$(1) \qquad \ell[x,y]_{ij} = \left(\binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-3}{j-1}\right) x^{i-j}y^{j-i}.$$

From the definition of $L[x,y]$, we see that $\ell[x,y]_{11} = 1$, $\ell[x,y]_{1j} = 0$ for $j \geq 2$, $\ell[x,y]_{21} = 0$, $\ell[x,y]_{22} = 1$ and $\ell[x,y]_{2j} = 0$ for $j \geq 3$. Also, we see that $\ell[x,y]_{i1} = -x^{i-1}y^{1-i}$ for $i \geq 3$, and, for $i,j \geq 2$,

$$\ell[x,y]_{ij} = \ell[x,y]_{i-1,j-1} + \ell[x,y]_{i-1,j}xy^{-1}. \tag{2}$$

From (2), we know that $\ell_{ij}$ satisfy the Pascal-like recurrence relation. Using the definitions of $\Phi[x,y]$, $\mathcal{F}[x,y]$ and $L[x,y]$, we can derive the following theorem.

**Theorem 1.** *Let $L[x,y]$ be the infinite matrix as in* (1). *For the infinite GP matrix $\Phi[x,y]$ and the infinite GF matrix $\mathcal{F}[x,y]$, we have*

$$\Phi[x,y] = \mathcal{F}[x,y] * L[x,y].$$

*Proof.* From the definitions of the GP matrix and GF matrix, we have the following Riordan representations

$$\Phi[x,y] = \left(\frac{1}{1-xyt}, \frac{y^2t}{1-xyt}\right), \ \mathcal{F}[x,y] = \left(\frac{1}{1-xyt-(xyt)^2}, y^2t\right). \tag{3}$$

We know that if $L[x,y]$ is in $\Re$, then we may assume $L[x,y] = (g_L(t), f_L(t))$. From (2), we have the infinite matrix $L[x,y] = [\ell[x,y]_{ij}]$ as follows:

$$L[x,y] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ -x^2y^{-2} & xy^{-1} & 1 & 0 & 0 & 0 & \cdots \\ -x^3y^{-3} & 0 & 2xy^{-1} & 1 & 0 & 0 & \\ -x^4y^{-4} & -x^3y^{-3} & 2x^2y^{-2} & 3xy^{-1} & 1 & 0 & \\ & & & \cdots & & & \end{bmatrix}.$$

Since the first column vector of $L[x,y]$ is $(1,0,-x^2y^{-2},-x^3y^{-3},\ldots)^T$, it is obvious that

$$g_L(t) = 1 + 0\cdot t + (-x^2y^{-2})t^2 + (-x^3y^{-3})t^3 + \cdots = \frac{1-xy^{-1}t-(xy^{-1}t)^2}{1-xy^{-1}t}.$$

The rule of formation in $L[x,y]$ is that each entry is the sum of the elements to the left and above in the row above. In other words, for $j \geq 1$,

$$\ell[x,y]_{n+1,j} = \ell[x,y]_{n,j-1} + \ell[x,y]_{n,j}xy^{-1}.$$

Thus

$$c_j(t) = tc_{j-1}(t) + xy^{-1}tc_j(t),$$

i.e.,

$$g_L(t)[f_L(t)]^j = tg_L(t)[f_L(t)]^{j-1} + xy^{-1}tg_L(t)[f_L(t)]^j$$

or $f_L(t) = t + xy^{-1}tf_L(t)$. Solving for $f_L(t)$, we have $f_L(t) = \frac{t}{1-xy^{-1}t}$. Thus,

$$L[x,y] = \left(\frac{1-xy^{-1}t-(xy^{-1}t)^2}{1-xy^{-1}t}, \frac{t}{1-xy^{-1}t}\right).$$

Therefore,

$$\begin{aligned}\mathcal{F}[x,y] &* L[x,y] \\ &= \left(\frac{1}{1-xyt-(xyt)^2}, y^2t\right) * \left(\frac{1-xy^{-1}t-(xy^{-1})^2}{1-xy^{-1}t}, \frac{t}{1-xy^{-1}t}\right) \\ &= \left(\frac{1}{1-xyt-(xyt)^2}\cdot\frac{1-xyt-(xyt)^2}{1-xyt}, \frac{y^2t}{1-xy^{-1}(y^2t)}\right) \\ &= \left(\frac{1}{1-xyt}, \frac{y^2t}{1-xyt}\right) \\ &= \Phi[x,y],\end{aligned}$$

the proof is completed. □

Since $x$ and $y$ are nonzero real variables, from Theorem 1, we have, for $1 \le j \le n$,

$$\binom{n-1}{j-1} = \sum_{k=j}^{n} F_{n-j+1}\left(\binom{k-1}{j-1} - \binom{k-2}{j-1} - \binom{k-3}{j-1}\right). \tag{4}$$

By (4), we have $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$, and this identity is the sum of the first $n$ terms of the Fibonacci sequence. From (4), we have, for $k \ge j+2$,

$$\binom{n-1}{j-1} = \sum_{k=j}^{n} F_{n-k+1}\frac{(k-3)!(j(k-1)-2(j-1)-(k-j)^2)}{(j-1)!(k-j)!}.$$

Now, we consider the relationship between GP matrix and generalized Pell matrix.

For any two nonzero real variables $x$ and $y$, we define an infinite matrix $M[x,y] = [m[x,y]_{ij}]$, $1 \le i,j$ as follows:

$$m[x,y]_{ij} = \left(\binom{i-1}{j-1} - 2\binom{i-2}{j-1} - \binom{i-3}{j-1}\right) x^{i-j}y^{j-i}. \tag{5}$$

From the definition of $M[x,y]$, we have $m[x,y]_{11} = 1, m[x,y]_{1j} = 0$ for $j \ge 2$, $m[x,y]_{21} = xy^{-1}, m[x,y]_{22} = 1$ and $m[x,y]_{2j} = 0$ for $j \ge 3$. Also, we have $m[x,y]_{i1} = -2x^{i-1}y^{1-i}$ for $i \ge 3$, and, for $i,j \ge 2$,

$$m[x,y]_{ij} = m[x,y]_{i-1,j-1} + m[x,y]_{i-1,j}xy^{-1}. \tag{6}$$

From (6), we know that $m[x,y]_{ij}$ satisfy the Pascal-like recurrence relation. From the definition of $\Phi[x,y]$, $S[x,y]$ and $M[x,y]$, we can derive the following theorem.

**Theorem 2.** *Let $M[x,y]$ be the infinitie matrix as in* (5). *For the infinite GP matrix $\Phi[x,y]$ and the infinite generalized Pell matrix $S[x,y]$, we have*

$$\Phi[x,y] = S[x,y] * M[x,y].$$

*Proof.* From the definition of $S[x,y]$,

$$S[x,y] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 2xy & y^2 & 0 & 0 & 0 & 0 & \\ 5x^2y^2 & 2xy^3 & y^4 & 0 & 0 & 0 & \cdots \\ 12x^3y^3 & 5x^2y^4 & 2xy^5 & y^6 & 0 & 0 & \\ 29x^4y^4 & 12x^3y^5 & 5x^2y^6 & 2xy^7 & y^8 & 0 & \\ & & & \cdots & & & \end{bmatrix},$$

and we know that $S[x,y]$ is in $\Re$. So, we have the following representation

$$S[x,y] = \left(\frac{1}{1-2xyt-(xyt)^2}, y^2t\right).$$

We know that if $M[x,y]$ is in $\Re$, then we may assume $M[x,y] = (g_M(t), f_M(t))$. From (5), we have

$$M[x,y] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ -xy^{-1} & 1 & 0 & 0 & 0 & 0 & \\ -2x^2y^{-2} & 0 & 1 & 0 & 0 & 0 & \cdots \\ -2x^3y^{-3} & -2x^2y^{-2} & xy^{-1} & 1 & 0 & 0 & \\ -2x^4y^{-4} & -4x^3y^{-3} & -x^2y^{-2} & 2xy^{-1} & 1 & 0 & \\ & & & \cdots & & & \end{bmatrix}.$$

Since the first column vector of $M[x,y]$ is

$$(1, -xy^{-1}, -2x^2y^{-2}, -2x^3y^{-3}, -2x^4y^{-4}, \ldots)^T,$$

it is obvious that

$$\begin{aligned} g_M(t) &= 1 - xy^{-1}t - 2x^2y^{-2}t^2 - 2x^3y^{-3}t^3 - 2x^4y^{-4}t^4 - \cdots \\ &= 1 - (xy^{-1}t) - 2(xy^{-1}t)^2 - 2(xy^{-1}t)^3 - 2(xy^{-1}t)^4 - \cdots \\ &= \frac{1-2xy^{-1}t-(xy^{-1}t)^2}{1-xy^{-1}t}. \end{aligned}$$

The rule of formation in $M[x,y]$ is that each entry is the sum of the elements to the left and above in the row above. That is, for $j \geq 1$

$$m[x,y]_{ij} = m[x,y]_{i-1,j-1} + m[x,y]_{i-j,j}xy^{-1}.$$

Thus $c_j(t) = tc_{j-1}(t) + xy^{-1}tc_j(t)$, and hence

$$g_M(t)[f_M(t)]^j = tg_M(t)[f_M(t)]^{j-1} + xy^{-1}tg_M(t)[f_M(t)]^j.$$

Thus, we have $f_M(t) = \frac{t}{1-xy^{-1}t}$ and

$$M[x,y] = \left(\frac{1-2xy^{-1}t-(xy^{-1}t)^2}{1-xy^{-1}t}, \frac{t}{1-xy^{-1}t}\right).$$

Therefore, we have

$$\begin{aligned} S[x,y]*M[x,y] &= \left(\frac{1}{1-2xyt-(xyt)^2}\cdot\frac{1-2xyt-(xyt)^2}{1-xyt}, \frac{y^2t}{1-xyt}\right) \\ &= \left(\frac{1}{1-xyt}, \frac{y^2t}{1-xyt}\right) \\ &= \Phi[x,y]. \end{aligned}$$

The proof is completed. □

By Theorem 1 and Theorem 2, we know that for positive integer $n$, $\mathcal{F}[x,y]_n = \Phi[x,y]_n L[x,y]_n^{-1}$ and $S[x,y]_n = \Phi[x,y]_n M[x,y]_n^{-1}$. Thus we consider inverse matrices.

Let $L[x,y]_n$ be an $n\times n$ matrix as in (1). From the definition of $L[x,y]_n$, the inverse matrix $L[x,y]_n^{-1}$ of $L[x,y]_n$ is of the form $L[x,y]_n^{-1} = [\tilde{\ell}[x,y]_{ij}]$ with

$$\tilde{\ell}[x,y]_{ij} = (-1)^{i+j}\left[\binom{i-1}{j-1} - \binom{i-2}{j-1} + \binom{i-3}{j-1}\right] x^{i-j}y^{j-i},$$

and hence we have, for $j \geq 2$,

$$\tilde{\ell}[x,y]_{ij} = \tilde{\ell}[x,y]_{i-1,j-1} - \tilde{\ell}[x,y]_{i-1,j}xy^{-1}.$$

For the matrix $\mathcal{F}[x,y]$, $\bar{f}_{\mathcal{F}}(t) = y^{-2}t$ because $f_{\mathcal{F}}(t) = y^2t$. So, we have

$$\frac{1}{g_{\mathcal{F}}(\bar{f}_{\mathcal{F}}(t))} = 1 - xy^{-1}t - (xy^{-1}t)^2.$$

Thus,

$$\mathcal{F}[x,y]^{-1} = (1 - xy^{-1}t - (xy^{-1}t)^2, y^{-2}t).$$

Also, for the matrix $L[x,y]$, we see that $\bar{f}_L(t) = \frac{t}{1+xy^{-1}t}$ and

$$\begin{aligned} \frac{1}{g_L(\bar{f}_L(t))} &= \frac{1 - xy^{-1}\left(\frac{t}{1+xy^{-1}t}\right)}{1 - xy^{-1}\left(\frac{t}{1+xy^{-1}t}\right) - \left(xy^{-1}\left(\frac{t}{1+xy^{-1}t}\right)\right)^2} \\ &= \frac{1+xy^{-1}t}{1+xy^{-1}t-(xy^{-1}t)^2}. \end{aligned}$$

Hence, we have the following lemma.

**Lemma 3.** *Let $\mathcal{F}[x,y]$ be the infinite GF matrix and let $L[x,y]$ be the matrix as in* (1). *Then we have*

$$\begin{aligned} \mathcal{F}[x,y]^{-1} &= (1 - xy^{-1}t - (xy^{-1}t)^2, y^{-2}t) \\ L[x,y]^{-1} &= \left(\frac{1+xy^{-1}t}{1+xy^{-1}t-(xy^{-1}t)^2}, \frac{t}{1+xy^{-1}t}\right). \end{aligned}$$

From the definition of $S[x,y]$ and $M[x,y]$, we consider the inverse matrices. For the matrix $S[x,y]$, $\bar{f}_S(t) = y^{-2}t$ because $f_S(t) = y^2 t$. Thus we have

$$\frac{1}{g_S(\bar{f}_S(t))} = 1 - 2xy^{-1}t - (xy^{-1}t)^2,$$

and hence, we have

$$S[x,y]^{-1} = (1 - 2xy^{-1}t - (xy^{-1}t)^2, y^{-2}t).$$

For the matrix $M[x,y]$, we have $\bar{f}_M(t) = \frac{t}{1+xy^{-1}t}$ and

$$\begin{aligned}\frac{1}{g_M(\bar{f}_M(t))} &= \frac{1 - xy^{-1}\left(\frac{t}{1+xy^{-1}t}\right)}{1 - 2xy^{-1}\left(\frac{t}{1+xy^{-1}t}\right) - \left(xy^{-1}\left(\frac{t}{1+xy^{-1}t}\right)\right)^2} \\ &= \frac{1+xy^{-1}}{1-2(xy^{-1}t)^2}.\end{aligned}$$

Hence, we have the following lemma.

**Lemma 4.** *Let $S[x,y]$ be the infinite generalized Pell matrix and let $M[x,y]$ be the matrix as in (5). Then we have*

$$\begin{aligned}S[x,y]^{-1} &= (1 - 2xy^{-1}t - (xy^{-1}t)^2, y^{-2}t) \\ M[x,y]^{-1} &= \left(\frac{1+xy^{-1}}{1-2(xy^{-1}t)^2}, \frac{t}{1+xy^{-1}t}\right).\end{aligned}$$

From Theorem 1, Theorem 2, Lemma 3 and Lemma 4, we have the following corollaries.

**Corollary 5.** *For the infinite GP matrix $\Phi[x,y]$,*

$$\begin{aligned}\Phi[x,y]^{-1} &= L[x,y]^{-1} * \mathcal{F}[x,y]^{-1} \\ &= M[x,y]^{-1} * S[x,y]^{-1} \\ &= \left(\frac{1}{1+xy^{-1}t}, \frac{t}{y^2+xyt}\right).\end{aligned}$$

*Proof.* From Theorem 1 and Theorem 4, we know that

$$\Phi[x,y]^{-1} = L[x,y]^{-1} * \mathcal{F}[x,y]^{-1} = M[x,y]^{-1} * S[x,y]^{-1}.$$

Thus, we have

$$\begin{aligned}\Phi[x,y]^{-1} &= L[x,y]^{-1} * \mathcal{F}[x,y]^{-1} \\ &= \left(\frac{1+xy^{-1}t}{1+xy^{-1}t-(xy^{-1}t)^2}, \frac{t}{1+xy^{-1}t}\right) \\ &\quad *(1 - xy^{-1}t - (xy^{-1}t)^2, y^{-2}t) \\ &= \left(\frac{1}{1+xy^{-1}t}, \frac{t}{y^2+xyt}\right).\end{aligned}$$

□

**Corollary 6.** *Let* $u_n = xy + xy^3 + \cdots + xy^{2n-1}$. *For positive integer* $n \geq 1$, *we have*

(i) $\Phi[x,y]^n = \left(\frac{1}{1-u_n t}, \frac{y^{2n}t}{1-u_n t}\right)$, $\Phi[x,y]^{-n} = \left(\frac{y^{2n}}{y^{2n}+u_n t}, \frac{t}{y^{2n}+u_n t}\right)$.

(ii) $\mathcal{F}[x,y]^n = \left(\prod_{k=1}^n \frac{1}{1-xy^{2k-1}t-(xy^{2k-1}t)^2}, y^{2n}t\right)$,
$\mathcal{F}[x,y]^{-n} = \left(\prod_{k=1}^n (1-xy^{-2k+1}t-(xy^{-2k+1})^2), y^{-2n}t\right)$.

(iii) $S[x,y]^n = \left(\prod_{k=1}^n \frac{1}{1-2xy^{2k-1}t-(xy^{2k-1}t)^2}, y^{2n}t\right)$,
$S[x,y]^{-n} = \left(\prod_{k=1}^n (1-2xy^{-2k+1}t-(xy^{-2k+1}t)^2), y^{-2n}t\right)$.

*Proof.* (i) From (3), we have

$$\Phi[x,y]^2 = \left(\frac{1}{1-(xy+xy^3)t}, \frac{y^4 t}{1-(xy+xy^3)t}\right).$$

By induction on $n$ for $n \geq 1$, we can get the $\Phi[x,y]^n$ as follows:

$$\begin{aligned}\Phi[x,y]^n &= \Phi[x,y]^{n-1} * \Phi[x,y] \\ &= \left(\frac{1}{1-u_{n-1}t}, \frac{y^{2(n-1)}t}{1-u_{n-1}t}\right) * \left(\frac{1}{1-xyt}, \frac{y^2 t}{1-xyt}\right) \\ &= \left(\frac{1}{1-u_n t}, \frac{y^{2n}t}{1-u_n t}\right).\end{aligned}$$

Since $\Phi[x,y]^n = \left(\frac{1}{1-u_n t}, \frac{y^{2n}t}{1-u_n t}\right)$, we have $\bar{f}(t) = \frac{t}{y^{2n}+u_n t}$, and $\frac{1}{g(\bar{f}(t))} = \frac{y^{2n}}{y^{2n}+u_n t}$.

Thus,

$$\Phi[x,y]^{-n} = \left(\frac{y^{2n}}{y^{2n}+u_n t}, \frac{t}{y^{2n}+u_n t}\right).$$

The proofs of (ii) and (iii) are similar to (i).

Therefore, the proof is completed. □

**Corollary 7.** *Let* $S[x,y]$ *be the infinite generalized Pell matrix. Then we have*

$$\mathcal{F}[x,y] = S[x,y] * \left(1 - \frac{xy^{-1}t}{1-xy^{-1}t-(xy^{-1}t)^2}, t\right).$$

*Proof.* From Theorem 1, Theorem 2 and Lemma 3, we have

$$\begin{aligned}&\mathcal{F}[x,y] \\ =\ & S[x,y] * M[x,y] * L[x,y]^{-1} \\ =\ & S[x,y] * \left(\frac{1-2xy^{-1}t-(xy^{-1}t)^2}{1-xy^{-1}t} \cdot \frac{1+xy^{-1}T}{1+xy^{-1}T-(xy^{-1}T)^2}, \frac{T}{1+xy^{-1}T}\right) \\ =\ & S[x,y] * \left(1 - \frac{xy^{-1}t}{1-xy^{-1}t-(xy^{-1}t)^2}, t\right),\end{aligned}$$

where $T = \frac{t}{1-xy^{-1}t}$. □

**Corollary 8.** *For $\Phi[x,y]$, $\mathcal{F}[x,y]$, $S[x,y]$, and positive integer $n \geq 1$, the following results hold:*

(i) $\Phi[-x,y]^n = \Phi[x,-y]^n$, $\Phi[x,y]^n = \Phi[-x,-y]^n$.
(ii) $\mathcal{F}[-x,y]^n = \mathcal{F}[x,-y]^n$, $\mathcal{F}[x,y]^n = \mathcal{F}[-x,-y]^n$.
(iii) $S[-x,y]^n = S[x,-y]^n$, $S[x,y]^n = S[-x,-y]^n$.

**Example 1.** If $x = y = 1$, then, from Theorem 1, we have

$$\begin{aligned} \Phi[1,1] = P &= \left(\frac{1}{1-t}, \frac{t}{1-t}\right), \ \mathcal{F}[1,1] = \mathcal{F} = \left(\frac{1}{1-t-t^2}, t\right), \text{ and} \\ L[1,1] = L &= \left(\frac{1-t-t^2}{1-t}, \frac{t}{1-t}\right). \end{aligned}$$

Then, we have

$$\begin{aligned} \mathcal{F} * L &= \left(\frac{1}{1-t-t^2}, t\right) * \left(\frac{1-t-t^2}{1-t}, \frac{t}{1-t}\right) \\ &= \left(\frac{1}{1-t-t^2} \cdot \frac{1-t-t^2}{1-t}, \frac{t}{1-t}\right) = \left(\frac{1}{1-t}, \frac{t}{1-t}\right) \\ &= P. \end{aligned}$$

From (i) of Corollary 6, we have

$$P^n = \left(\frac{1}{1-nt}, \frac{t}{1-nt}\right). \tag{7}$$

Let $P^n = [p_{ij}^{(n)}]$ for $n \geq 1$. From (7) and the definition of the Pascal matrix, we have the interesting result as follows:

$$p_{ij}^{(n)} = n^{i-j}\binom{i-1}{j-1} = n^{i-j}p_{ij}.$$

From the Corollary 5, the $n$th column of $P^{-1}$ has

$$\frac{1}{1+t} \cdot \left(\frac{t}{1+t}\right)^n = \frac{t^n}{(1+t)^{n+1}}$$

as its generating function. And, we have, for $n \geq 1$,

$$P^{-n} = \left(\frac{1}{1+nt}, \frac{t}{1+nt}\right).$$

If $x = 1$ and $y = -1$, then, from (3), we have

$$\Phi[1,-1] = \left(\frac{1}{1+t}, \frac{t}{1+t}\right) = P^{-1}.$$

More generally, for positive integer $n$, we have,

$$\Phi[1,-1]^n = P^{-n}.$$

Also, from (ii) of Corollary 6 and by induction on $k$ for $k \geq 1$, we have

$$\mathcal{F}^k = \left(\left(\frac{1}{1-t-t^2}\right)^{k-1}, t\right) * \left(\frac{1}{1-t-t^2}, t\right) = \left(\frac{1}{(1-t-t^2)^k}, t\right).$$

**Example 2.** If $x = y = 1$, then, from Theorem 4, we have

$$\Phi[1,1] = P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right), \; S[1,1] = S = \left(\frac{1}{1-2t-t^2}, t\right),$$

$$M[1,1] = \left(\frac{1-2t-t^2}{1-t}, \frac{t}{1-t}\right).$$

So, we have

$$P = \left(\frac{1}{1-2t-t^2}, t\right) * \left(\frac{1-2t-t^2}{1-t}, \frac{t}{1-t}\right),$$

and from (iii) of Corollary 6 and by induction on $k$ for $k \geq 1$, we have

$$S^k = \left(\frac{1}{(1-2t-t^2)^{k-1}}, t\right) * \left(\frac{1}{1-2t-t^2}, t\right) = \left(\frac{1}{(1-2t-t^2)^k}, t\right).$$

We consider the $7 \times 7$ matrices $P_7 = S_7 \cdot M_7$. That is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 & 0 & 0 \\ 12 & 5 & 2 & 1 & 0 & 0 & 0 \\ 29 & 12 & 5 & 2 & 1 & 0 & 0 \\ 70 & 29 & 12 & 5 & 2 & 1 & 0 \\ 169 & 70 & 29 & 12 & 5 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & -2 & 1 & 1 & 0 & 0 & 0 \\ -2 & -4 & -1 & 2 & 1 & 0 & 0 \\ -2 & -6 & -5 & 1 & 3 & 1 & 0 \\ -2 & -8 & -11 & -4 & 4 & 4 & 1 \end{bmatrix}$$

From this multiplication, we can get many identities. For example,

$$\begin{aligned} p_{6,3} &= \binom{6-1}{3-1} = \binom{5}{2} = 10 \\ &= a_4 \cdot 1 + a_3 \cdot 1 + a_2 \cdot (-1) + a_1 \cdot (-5). \end{aligned}$$

More generally, we have

$$\begin{aligned} p_{ij} = \binom{i-1}{j-1} &= s_{i1}m_{1j} + s_{i2}m_{2j} + \cdots \\ &= a_i m_{1j} + a_{i-1}m_{2j} + \cdots + a_2 m_{i-1,j} + a_1 m_{ij}, \end{aligned}$$

where $a_n$ is the $n$th Pell number. For example, for $j = 2$,

$$\binom{i-1}{2-1} = i - 1 = a_{i-1} - 2a_{i-3} - 4a_{i-4} - \cdots - 2(i-4)a_2 - 2(i-3)a_1,$$

where $a_n$ is the $n$th Pell number.

In particular, from the above identity, we have, for $j = 1$, $1 = a_n - a_{n-1} - 2(a_{n-2} + \cdots + a_1)$, i.e.,

$$a_n = a_{n-1} + 2(a_{n-2} + \cdots + a_1) + 1.$$

Also, we have, for positive integer $n$ and Pell sequence $\{a_n\}$,

$$a_1 + a_2 + \cdots + a_n = \frac{1}{2}(a_{n+2} - a_{n+1} - 1) = \frac{1}{2}(a_{n+1} + a_n - 1).$$

The above identity is the sum of the first $n$ terms of the Pell sequence.

From Corollary 7, we have the following identity

$$F_n = a_n - (a_{n-1}F_1 + a_{n-2}F_2 + \cdots + a_1F_{n-1}),$$

where $a_n$ is the $n$th Pell number.

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