J. Korean Math. Soc. **45** (2008), No. 2, pp. 367–376

DIAGONAL LIFTS OF TENSOR FIELDS OF TYPE (1,1) ON CROSS-SECTIONS IN TENSOR BUNDLES AND ITS APPLICATIONS

Aydin Gezer and Arif Salimov

Abstract. The main purpose of this paper is to investigate diagonal lift of tensor fields of type $(1,1)$ from manifold to its tensor bundle of type (p,q) and to prove that when a manifold M_n admits a Kählerian structure (φ, g), its tensor bundle of type (p,q) admits an complex structure.

1. Introduction

Let M_n be n-dimensional differentiable manifold of class C^∞, $T_q^p(M_n)$ its tensor bundle of type (p,q), and π the natural projection $T_q^p(M_n) \to M_n$. Let x^j, $j = 1, \ldots, n$ be local coordinates in neighborhood U of a point x of M_n. Then a tensor t of type (p,q) at $x \in M_n$ which is an element of $T_q^p(M_n)$ is expressible in the form

$$(x^j, t_{j_1 \cdots j_q}^{i_1 \cdots i_p}) = (x^j, x^{\bar{j}}),\ x^{\bar{j}} = t_{j_1 \cdots j_q}^{i_1 \cdots i_p},\ \bar{j} = n+1, \ldots, n+n^{p+q},$$

whose $t_{j_1 \cdots j_q}^{i_1 \cdots i_p}$ are components of t with respect to the natural frame ∂_j. We may consider $(x^j, x^{\bar{j}})$ as local coordinates in a neighborhood $\pi^{-1}(U)$ of $T_q^p(M_n)$. To a transformation of local coordinates of $M_n : x^{j'} = x^{j'}(x^j)$, there corresponds in $T_q^p(M_n)$ the coordinates transformation

$$(1.1) \qquad \begin{cases} x^{j'} = x^{j'}(x^j) \\ x^{\bar{j}'} = t_{j'_1 \cdots j'_q}^{i'_1 \cdots i'_p} = A_{i_1}^{i'_1} \cdots A_{i_p}^{i'_p} A_{j'_1}^{j_1} \cdots A_{j'_q}^{j_q} t_{j_1 \cdots j_q}^{i_1 \cdots i_p} = A_{(i)}^{(i')} A_{(j')}^{(j)} x^{\bar{j}}, \end{cases}$$

where

$$A_{(i)}^{(i')} A_{(j')}^{(j)} = A_{i_1}^{i'_1} \cdots A_{i_p}^{i'_p} A_{j'_1}^{j_1} \cdots A_{j'_q}^{j_q},\ A_{i_1}^{i'_1} = \frac{\partial x^{i'}}{\partial x^i},\ A_{j'_1}^{j_1} = \frac{\partial x^j}{\partial x^{j'}}.$$

Received July 24, 2006.

2000 *Mathematics Subject Classification.* 53C07, 53B05, 55R10, 53C55.

Key words and phrases. tensor bundle, diagonal lift, complex structure, Nijenhuis tensor, Kählerian structure.

The Jacobian of (1.1) is given by the matrix

$$(1.2)\quad \left(\frac{\partial x^{J'}}{\partial x^{J}}\right) = \begin{pmatrix} \dfrac{\partial x^{j'}}{\partial x^{j}} & \dfrac{\partial x^{j'}}{\partial x^{\bar{j}}} \\ \dfrac{\partial x^{\bar{j}'}}{\partial x^{j}} & \dfrac{\partial x^{\bar{j}'}}{\partial x^{\bar{j}}} \end{pmatrix} = \begin{pmatrix} A_{j}^{j'} & 0 \\ t_{(k)}^{(i)}\partial_j(A_{(i)}^{(i')}A_{(j')}^{(k)}) & A_{(i)}^{(i')}A_{(j')}^{(j)} \end{pmatrix},$$

where

$$J = (j, \bar{j}),\ J = 1, \ldots, n + n^{p+q},\ t_{(k)}^{(i)} = t_{k_1 \cdots k_q}^{i_1 \cdots i_p}.$$

We denote by $\Im_q^p(M_n)$ the module over $F(M_n)$ of C^∞ tensor fields of type (p,q)($F(M_n)$ is ring of real-valued C^∞ functions on M_n). If $\alpha \in \Im_p^q(M_n)$, it is regarded, in a natural way, by contraction, as a function in $T_q^p(M_n)$, which we denote by $\imath\alpha$. If α has the local expression

$$\alpha = \alpha_{i_1 \cdots i_p}^{j_1 \cdots j_q} \partial_{j_1} \otimes \cdots \otimes \partial_{j_q} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_p}$$

in a coordinate neighborhood $U(x^i) \subset M_n$, then $\imath\alpha = \alpha(t)$ has the local expression $\imath\alpha = \alpha_{i_1 \cdots i_p}^{j_1 \cdots j_q} t_{j_1 \cdots j_q}^{i_1 \cdots i_p}$ with respect to the coordinates $(x^i, x^{\bar{i}})$ in $\pi^{-1}(U)$.

Let $A \in \Im_q^p(M_n)$. Then there is a unique vector field ${}^VA \in \Im_0^1(T_q^p(M_n))$ such that for $\alpha \in \Im_p^q(M_n)$

$${}^VA(\imath\alpha) = \alpha(A) \circ \pi = {}^V(\alpha(A)),$$

where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in F(M_n)$. We call ${}^VA$ the vertical lift of $A \in \Im_q^p(M_n)$ to $T_q^p(M_n)$ (see [2]). The vertical lift ${}^VA$ has components of the form

$${}^VA = \begin{pmatrix} {}^VA^j \\ {}^VA^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_{j_1 \cdots j_q}^{i_1 \cdots i_p} \end{pmatrix}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T_q^p(M_n)$.

Let ∇ be a symmetric affine connection on M_n. We define the horizontal lift ${}^H\nabla = \tilde{\nabla}_V \in \Im_0^1(T_q^p(M_n))$ of $V \in \Im_0^1(M_n)$ to $T_q^p(M_n)$ [2] by

$${}^HV(\imath\alpha) = \imath(\nabla_V\alpha), \alpha \in \Im_p^q(M_n).$$

The horizontal lift ${}^HV$ of $V \in \Im_0^1(M_n)$ to $T_q^p(M_n)$ has components

$$(1.3)\quad {}^HV = \begin{pmatrix} V^j \\ V^s\left(\sum\limits_{\mu=1}^{q} \Gamma_{sj_\mu}^{m} t_{j_1 \cdots m \cdots j_q}^{i_1 \cdots i_p} - \sum\limits_{\lambda=1}^{p} \Gamma_{sm}^{i_\lambda} t_{j_1 \cdots j_q}^{i_1 \cdots m \cdots i_p}\right) \end{pmatrix}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T_q^p(M_n)$ [1], where Γ_{ij}^k are local components of ∇ in M_n.

Suppose that there is given a tensor field $\xi \in \Im_q^p(M_n)$. Then the correspondence $x \to \xi_x$, ξ_x being the value of ξ at $x \in M_n$, determines a mapping $\sigma_\xi : M_n \to T_q^p(M_n)$, such that $\pi \circ \sigma_\xi = id_{M_n}$, and the n dimensional submanifold $\sigma_\xi(M_n)$ of $T_q^p(M_n)$ is called the cross-section determined by ξ. If the tensor field ξ has the local components $\xi_{k_1\cdots k_q}^{l_1\cdots l_p}(x^k)$, the cross-section $\sigma_\xi(M_n)$ is locally expressed by

$$(1.4)\qquad \begin{cases} x^k = x^k \\ x^{\bar{k}} = \xi_{k_1\cdots k_q}^{l_1\cdots l_p}(x^k) \end{cases}$$

with respect to the coordinates $(x^k, x^{\bar{k}})$ in $T_q^p(M_n)$. Differentiating (1.4) by x^j, we see that n tangent vector fields B_j to $\sigma_\xi(M_n)$ have components

$$(1.5)\qquad (B_j^K) = (\frac{\partial x^K}{\partial x^j}) = \begin{pmatrix} \delta_j^k \\ \partial_j \xi_{k_1\cdots k_q}^{l_1\cdots l_p} \end{pmatrix}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T_q^p(M_n)$.

On the other hand, the fibre is locally expressed by

$$\begin{cases} x^k = const, \\ t_{k_1\cdots k_q}^{l_1\cdots l_p} = t_{k_1\cdots k_q}^{l_1\cdots l_p}, \end{cases}$$

$t_{k_1\cdots k_q}^{l_1\cdots l_p}$ being considered as parameters. Thus, on differentiating with respect to $x^{\bar{j}} = t_{j_1\cdots j_q}^{i_1\cdots i_p}$, we see that n^{p+q} tangent vector fields $C_{\bar{j}}$ to the fibre have components

$$(1.6)\qquad (C_{\bar{j}}^K) = (\frac{\partial x^K}{\partial x^{\bar{j}}}) = \begin{pmatrix} 0 \\ \delta_{k_1}^{j_1}\cdots\delta_{k_q}^{j_q}\delta_{i_1}^{l_1}\cdots\delta_{i_p}^{l_p} \end{pmatrix}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T_q^p(M_n)$, where δ is the Kronecker symbol.

We consider in $\pi^{-1}(U) \subset T_q^p(M_n)$, $n + n^{p+q}$ local vector fields B_j and $C_{\bar{j}}$ along $\sigma_\xi(M_n)$. They form a local family of frames $\{B_j, C_{\bar{j}}\}$ along $\sigma_\xi(M_n)$, which is called the adapted (B,C)- frame of $\sigma_\xi(M_n)$ in $\pi^{-1}(U)$. Taking account of (1.3) and also (1.5) and (1.6), we can easily prove that, the lifts ${}^VA$ and ${}^HV$ have along $\sigma_\xi(M_n)$ components of the form

$$(1.7)\qquad {}^VA = \begin{pmatrix} {}^V\tilde{A}^j \\ {}^V\tilde{A}^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_{j_1\cdots j_q}^{i_1\cdots i_p} \end{pmatrix},$$

$$(1.8)\qquad {}^HV = \begin{pmatrix} {}^H\tilde{V}^j \\ {}^H\tilde{V}^{\bar{j}} \end{pmatrix} = \begin{pmatrix} V^j \\ -(\nabla_V\xi)_{j_1\cdots j_q}^{i_1\cdots i_p} \end{pmatrix}$$

with respect to the adapted (B,C)-frame, where $(\nabla_V\xi)_{j_1\cdots j_q}^{i_1\cdots i_p}$ are local components of $\nabla_V\xi$ in M_n.

Let $A, B \in \Im^p_q(M_n)$, $V, W \in \Im^1_0(M_n)$ and $\varphi \in \Im^1_1(M_n)$. Let R denotes the curvature tensor field of the connection ∇. Then (see [1])

(1.9)
$$\begin{cases} [{}^V A, {}^V B] = 0 \\ [{}^H V, {}^V A] = {}^V (\nabla_V A) \\ [{}^H V, \tilde{\gamma}\varphi - \gamma\varphi] = \tilde{\gamma}(L_V\varphi + (\nabla V)\varphi - \varphi(\nabla V)) - \gamma(L_V\varphi + (\nabla V)\varphi - \varphi(\nabla V)) \\ [{}^H V, {}^H W] = {}^H [V, W] + (\tilde{\gamma} - \gamma)R(V, W), \end{cases}$$

where $\tilde{\gamma}\varphi - \gamma\varphi$ is a vector field in $T^p_q(M_n)$ defined by

$$\tilde{\gamma}\varphi - \gamma\varphi = \begin{pmatrix} 0 \\ \sum_{\mu=1}^q t^{i_1\cdots i_p}_{j_1\cdots m\cdots j_q}\varphi^m_{j_\mu} - \sum_{\lambda=1}^p t^{i_1\cdots m\cdots i_p}_{j_1\cdots j_q}\varphi^{i_\lambda}_m \end{pmatrix}. \tag{1.10}$$

2. Diagonal lifts of affinor fields on a cross-section

Let $\varphi \in \Im^1_1(M_n)$. We define a tensor field ${}^D\varphi \in \Im^1_1(T^p_q(M_n)$ along the cross-section $\sigma_\xi(M_n)$ by

$$\begin{cases} {}^D\varphi({}^H V) = {}^H (\varphi(V)), \ \forall V \in \Im^1_0(M_n) \\ {}^D\varphi({}^V A) = \ -{}^V (\varphi(A)), \ \forall A \in \Im^p_q(M_n), \end{cases} \tag{2.1}$$

where $\varphi(A) = C(\varphi \otimes A) \in \Im^p_q(M_n)$ and call ${}^D\varphi$ the diagonal lift of $\varphi \in \Im^1_1(M_n)$ to $T^p_q(M_n)$ along $\sigma_\xi(M_n)$. Then, from (2.1) we have

$$\begin{cases} \text{(i)} \ {}^D\tilde{\varphi}^K_L \ {}^H\tilde{V}^L = {}^H (\tilde{\varphi(V)})^K, \\ \text{(ii)} \ {}^D\varphi^K_L \ {}^V\tilde{A}^L = -{}^V (\tilde{\varphi(A)})^K, \end{cases} \tag{2.2}$$

where ${}^V(\tilde{\varphi(A)}) = \begin{pmatrix} 0 \\ {}^V(\tilde{\varphi(A)})^{\bar{k}} \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi^{l_1}_m A^{ml_2\cdots l_p}_{k_1\cdots k_q} \end{pmatrix}$.

First, consider the case where $K = k$. In the case, (i) of (2.2) reduces to

$$ {}^D\tilde{\varphi}^k_l \ {}^H\tilde{V}^l + {}^D\tilde{\varphi}^k_{\bar{l}} \ {}^D\tilde{V}^{\bar{l}} = {}^H (\tilde{\varphi(V)})^k = (\varphi(V))^k = \varphi^k_l V^l. \tag{2.3}$$

Since the right-hand side of (2.3) are functions depending only on the base coordinates x^i, the left-hand side of (2.3) are too. Then, since ${}^H\tilde{V}^{\bar{l}}$ depend on fibre coordinates, from (2.3) we obtain

$$ {}^D\tilde{\varphi}^k_{\bar{l}} = 0, \tag{2.4}$$

this implies

$$ {}^D\tilde{\varphi}^k_l = \varphi^k_l. \tag{2.5}$$

When $K = k$, (ii) of (2.2) can be rewritten, by virtue of (1.7), (2.4) and (2.5), as $0 = 0$.

When $K = \bar{k}$, (ii) of (2.2) reduces to

$$ {}^D\tilde{\varphi}^{\bar{k}}_l \ {}^V\tilde{A}^l + {}^D\tilde{\varphi}^{\bar{k}}_{\bar{l}} \ {}^V A^{\bar{l}} = -{}^V (\tilde{\varphi(A)})^{\bar{k}}$$

or

$$^{D}\tilde{\varphi}_{\bar{l}}^{\bar{k}} A_{r_1\cdots r_q}^{s_1\cdots s_p} = -\varphi_m^{l_1} A_{k_1\cdots k_q}^{ml_2\cdots l_p} = -\varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\cdots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\cdots\delta_{k_q}^{r_q} A_{r_1\cdots r_q}^{s_1\cdots s_p}$$

for all $A \in \Im_q^p(M_n)$, which implies

$$^{D}\tilde{\varphi}_{\bar{l}}^{\bar{k}} = -\varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\cdots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\cdots\delta_{k_q}^{r_q},\ p \geq 1, \tag{2.6}$$

where $x^{\bar{l}} = t_{r_1\cdots r_q}^{s_1\cdots s_p}$, $x^{\bar{k}} = t_{k_1\cdots k_q}^{l_1\cdots l_p}$.

Similarly, we have

$$^{D}\tilde{\varphi}_{\bar{l}}^{\bar{k}} = -\delta_{s_1}^{l_1}\cdots\delta_{s_p}^{l_p}\varphi_{k_1}^{r_1}\delta_{k_2}^{r_2}\cdots\delta_{k_q}^{r_q},\ q \geq 1. \tag{2.7}$$

When $K = \bar{k}$, (i) of (2.2) reduces to

$$^{D}\tilde{\varphi}_{l}^{\bar{k}}\ {}^{H}\tilde{V}^{l} + {}^{D}\tilde{\varphi}_{\bar{l}}^{\bar{k}}\ {}^{H}\tilde{V}^{\bar{l}} = {}^{H}(\tilde{\varphi(V)})^{\bar{k}}. \tag{2.8}$$

We will investigate components $^{D}\tilde{\varphi}_{l}^{\bar{k}}$.
Let $\xi \in \Im_q^p(M_n)$. We consider a new Φ-operator

$$(\Phi_\varphi\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p} = \varphi_l^m\nabla_m\xi_{k_1\cdots k_q}^{l_1\cdots l_p} + \begin{cases} \varphi_m^{l_1}\nabla_l\xi_{k_1\cdots k_q}^{ml_2\cdots l_p}, & p \geq 1 \\ \varphi_{k_1}^{m}\nabla_l\xi_{mk_2\cdots k_q}^{l_1\cdots l_p}, & q \geq 1. \end{cases} \tag{2.9}$$

From (2.9), we have

$$V^l(\Phi_\varphi\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p} = (V^l\varphi_l^m)\nabla_m\xi_{k_1\cdots k_q}^{l_1\cdots l_p} + \varphi_m^{l_1}V^l(\nabla_l\xi_{k_1\cdots k_q}^{ml_2\cdots l_p}),\ p \geq 1$$

and

$$V^l(\Phi_\varphi\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p} - \varphi_m^{l_1}(\nabla_V\xi_{k_1\cdots k_q}^{ml_2\cdots l_p}) = \nabla_{\varphi(V)}\xi_{k_1\cdots k_q}^{l_1\cdots l_p}. \tag{2.10}$$

Using (1.8), from (2.10) we have

$$(\Phi_\varphi\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p}\ {}^{H}V^l + \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\cdots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\cdots\delta_{s_q}^{r_q}\ {}^{H}V^{\bar{l}} = -{}^{H}(\varphi(V))^{\bar{k}}$$

and

$$-(\Phi_\varphi\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p}\ {}^{H}V^l - \varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\cdots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\cdots\delta_{s_q}^{r_q}\ {}^{H}V^{\bar{l}} = {}^{H}(\varphi(V))^{\bar{k}}. \tag{2.11}$$

Comparing (2.8) and (2.11) and making use of (2.6), we get

$$^{D}\tilde{\varphi}_{l}^{\bar{k}} = -(\Phi_\varphi\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p},\ p \geq 1.$$

Similarly, we obtain

$$^{D}\tilde{\varphi}_{l}^{\bar{k}} = -(\Phi_\varphi\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p},\ q \geq 1.$$

Thus, the diagonal lift $^{D}\varphi$ of φ has along the cross-section $\sigma_\xi(M_n)$ components

$$^{D}\tilde{\varphi}_{l}^{k} = \varphi_l^k,$$

$$^{D}\tilde{\varphi}_{\bar{l}}^{k} = 0,$$

$$^{D}\tilde{\varphi}_{\bar{l}}^{\bar{k}} = \begin{cases} -\varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\cdots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\cdots\delta_{k_q}^{r_q}, & p \geq 1 \\ -\delta_{s_1}^{l_1}\cdots\delta_{s_p}^{l_p}\varphi_{k_1}^{r_1}\delta_{k_2}^{r_2}\cdots\delta_{k_q}^{r_q}, & q \geq 1, \end{cases} \tag{2.12}$$

$$ {}^{D}\tilde{\varphi}_{l}^{\bar{k}} = -(\Phi_{\varphi}\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p}. $$

Now, on putting $B_{\bar{j}} = C_{\bar{j}}$, we write the adapted (B, C)-frame of $\sigma_{\xi}(M_n)$ as $B_J = \{B_j, B_{\bar{j}}\}$. We define a coframe $\tilde{B}^J$ of $\sigma_{\xi}^{\varphi}(M_n)$ by $\tilde{B}^I(B_J) = \delta_J^I$. From (1.5), (1.6) and $B_J^K \tilde{B}_K^I = \delta_J^I$ we see that covector fields $\tilde{B}^I$ have components

$$ \begin{aligned} &\tilde{B}^{i} = (\tilde{B}_{K}^{i}) = (\delta_{k}^{i}, 0) \\ &\tilde{B}^{\bar{i}} = (\tilde{B}_{K}^{\bar{i}}) = (-\partial_k \xi_{i_1\cdots i_q}^{j_1\cdots j_p},\ \delta_{i_1}^{k_1}\cdots\delta_{i_q}^{k_q}\delta_{l_1}^{j_1}\cdots\delta_{l_p}^{j_p}) \end{aligned} \tag{2.13} $$

with respect to the natural coframe $(dx^k, dx^{\bar{k}})$. Taking account of

$$ \begin{aligned} {}^{D}\varphi_{L}^{K} &= {}^{D}\varphi(dx^K, \partial_L) = \tilde{\varphi}_I^J B_J \otimes (dx^K, \partial_L) \\ &= \tilde{\varphi}_I^J dx^K(B_J)\tilde{B}^I(\partial_L) = \tilde{\varphi}_I^J dx^K(B_J^H \partial_H)\tilde{B}_L^I \\ &= {}^{D}\tilde{\varphi}_I^J B_J^H \delta_H^K \tilde{B}_L^I = {}^{D}\tilde{\varphi}_I^J B_J^K \tilde{B}_L^I \end{aligned} $$

and also (1.5), (1.6), (2.12) and (2.13), we see that ${}^{D}\varphi$ has along the cross-section $\sigma_{\xi}(M_n)$ components of the form

$$ \begin{aligned} {}^{D}\varphi_{l}^{k} &= \varphi_{l}^{k}, \\ {}^{D}\varphi_{\bar{l}}^{k} &= 0, \end{aligned} $$

$$ {}^{D}\varphi_{\bar{l}}^{\bar{k}} = \begin{cases} -\varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\cdots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\cdots\delta_{k_q}^{r_q}, & p \geq 1 \\ -\delta_{s_1}^{l_1}\cdots\delta_{s_p}^{l_p}\varphi_{k_1}^{r_1}\delta_{k_2}^{r_2}\cdots\delta_{k_q}^{r_q}, & q \geq 1, \end{cases} $$

$$ {}^{D}\varphi_{l}^{\bar{k}} = -(\Phi_{\varphi}\xi)_{lk_1\cdots k_q}^{l_1\cdots l_p} + \varphi_l^m \partial_m \xi_{k_1\cdots k_q}^{l_1\cdots l_p} + \begin{cases} \varphi_m^{l_1}\partial_l \xi_{k_1\cdots k_q}^{ml_2\cdots l_p}, & p \geq 1 \\ \varphi_{k_1}^{m}\partial_l \xi_{mk_2\cdots k_q}^{l_1\cdots l_p}, & q \geq 1. \end{cases} $$

Thus, ${}^{D}\varphi$ has along the cross-section $\sigma_{\xi}(M_n)$ components of the form

(2.14)

$$ \begin{cases} {}^{D}\varphi_{l}^{k} = \varphi_{l}^{k},\ {}^{D}\varphi_{\bar{l}}^{k} = 0,\ {}^{D}\varphi_{\bar{l}}^{\bar{k}} = \begin{cases} -\varphi_{s_1}^{l_1}\delta_{s_2}^{l_2}\cdots\delta_{s_p}^{l_p}\delta_{k_1}^{r_1}\cdots\delta_{k_q}^{r_q}, & p \geq 1 \\ -\delta_{s_1}^{l_1}\cdots\delta_{s_p}^{l_p}\varphi_{k_1}^{r_1}\delta_{k_2}^{r_2}\cdots\delta_{k_q}^{r_q}, & q \geq 1 \end{cases} \\ {}^{D}\varphi_{l}^{\bar{k}} = \begin{cases} \varphi_l^m(-\sum\limits_{\lambda=1}^{p}\Gamma_{ms}^{l_\lambda}\xi_{k_1\cdots k_q}^{l_1\cdots s\cdots l_p} + \sum\limits_{\mu=1}^{q}\Gamma_{mk_\mu}^{s}\xi_{k_1\cdots s\cdots k_q}^{l_1\cdots l_p}) & \\ +\varphi_m^{l_1}(\sum\limits_{\mu=1}^{q}\Gamma_{lk_\mu}^{s}\xi_{k_1\cdots s\cdots k_q}^{ml_2\cdots l_p} - \sum\limits_{\lambda=2}^{p}\Gamma_{ls}^{l_\lambda}\xi_{k_1\cdots k_q}^{ml_2\cdots s\cdots l_p} - \Gamma_{ls}^{m}\xi_{k_1\cdots k_q}^{s\cdots l_p}), & p \geq 1, \\ \varphi_l^m(-\sum\limits_{\lambda=1}^{p}\Gamma_{ms}^{l_\lambda}\xi_{k_1\cdots k_q}^{l_1\cdots s\cdots l_p} + \sum\limits_{\mu=1}^{q}\Gamma_{mk_\mu}^{s}\xi_{k_1\cdots s\cdots k_q}^{l_1\cdots l_p}) & \\ +\varphi_{k_1}^{m}(\sum\limits_{\mu=2}^{q}\Gamma_{lk_\mu}^{s}\xi_{mk_2\cdots s\cdots k_q}^{l_1\cdots l_p} + \Gamma_{lm}^{s}\xi_{s\cdots k_q}^{l_1\cdots l_p} - \sum\limits_{\lambda=1}^{p}\Gamma_{ls}^{l_\lambda}\xi_{mk_2\cdots k_q}^{l_1\cdots s\cdots l_p}), & q \geq 1 \end{cases} \end{cases} $$

with respect to the natural frame $\{\partial_h, \partial_{\bar{h}}\}$ of $\sigma_{\xi}^{\varphi}(M_n)$ in $\pi^{-1}(U) \subset T_q^p(M_n)$ [5].

In particular, if we put $p = 0$, $q = 1$ in (2.14), then ${}^{D}\varphi_{L}^{K}$ are the components of the diagonal lift of φ from manifold to its cotangent bundle with respect to the natural frame $\{\partial_h, \partial_{\bar{h}}\}$ of $\sigma_{\xi}(M_n)$ [6, p.291].

3. Complex structure in $T_q^p(M_n)$

We shall first state the following lemma for later use.

Lemma ([3]). *Let $\tilde{S}$ and $\tilde{T}$ be tensor fields in $T_q^p(M_n)$ of type $(1,q)$, where $q > 0$, such that*

$$\tilde{S}(\tilde{V}_1,\dots,\tilde{V}_q) = \tilde{T}(\tilde{V}_1,\dots,\tilde{V}_q)$$

for all vector fields $\tilde{V}_1,\dots,\tilde{V}_q$ which are of the form ${}^VA$ or ${}^HV$, where $A \in \Im_q^p(M_n)$ and $V \in \Im_0^1(M_n)$. Then $\tilde{S} = \tilde{T}$.

Let $\varphi \in \Im_1^1(M_n)$. We define ${}^H\varphi \in \Im_1^1(T_q^p(M_n))$ along $\sigma_\xi(M_n)$ by

$$\begin{cases} {}^H\varphi({}^HV) = {}^H(\varphi(V)), \ \forall V \in \Im_0^1(M_n) \\ {}^H\varphi({}^VA) = {}^V(\varphi(A)), \ \forall A \in \Im_q^p(M_n), \end{cases} \tag{3.1}$$

where $\varphi(A) = C(\varphi \otimes A) \in \Im_q^p(M_n)$ ([4]).

Theorem 3.1. *If φ, $\phi \in \Im_1^1(M_n)$, then with respect to symmetric affine connection ∇ in M_n, we have*

$$ {}^D\varphi^D\phi +^D \phi^D\varphi =^H (\varphi\phi + \phi\varphi), \tag{3.2}$$

$$ {}^D\varphi^H\phi +^D \phi^H\varphi =^H \varphi^D\phi +^H \phi^D\varphi =^D (\varphi\phi + \phi\varphi). \tag{3.3}$$

Proof. If $V \in \Im_0^1(M_n)$ and $A \in \Im_q^p(M_n)$, then we have by using (2.1) and (3.1)

$$\begin{aligned}
&({}^D\varphi^D\phi +^D \phi^D\varphi)({}^HX) \\
&= {}^D\varphi^H(\phi(X) +^D \phi^H(\varphi(X) =^H (\varphi(\phi(X)) +^H (\phi(\varphi(X)) \\
&= {}^H(\varphi\phi)({}^HX) +^H (\phi\varphi)({}^HX) =^H (\varphi\phi + \phi\varphi)({}^HX), \\
&({}^D\varphi^D\phi +^D \phi^D\varphi)({}^VA) \\
&= \ -^D \varphi^V(\phi(A) -^D \phi^V(\varphi(A) =^V (\varphi(\phi(A)) +^V (\phi(\varphi(A)) \\
&= {}^H(\varphi\phi)({}^VA) +^H (\phi\varphi)({}^VA) =^H (\varphi\phi + \phi\varphi)({}^VA).
\end{aligned}$$

In a way similar to that of the proof of (3.3), we can prove by using similar device easily. □

Putting $\varphi = \phi$ in (3.2), we obtain

$$ {}^D\varphi^D\varphi =^H (\varphi\varphi), \quad ({}^D\varphi)^2 =^H (\varphi^2). \tag{3.4}$$

Since ${}^H(id_{M_n}) = id_{\Im_q^p(M_n)}$, using (3.4), we have

Theorem 3.2. *If φ is almost complex structure in M_n, then the diagonal lift ${}^D\varphi$ of φ to $T_q^p(M_n)$ along $\sigma_\xi(M_n)$ is an almost complex structure in $T_q^p(M_n)$.*

Theorem 3.3. *If φ, $\phi \in \Im_1^1(M_n)$, then*

$$ {}^D\varphi(\tilde{\gamma} - \gamma)\phi = (\gamma - \tilde{\gamma})(\varphi\phi). \tag{3.5}$$

Proof. (3.5) can be prove by using local expressions of ${}^D\varphi$ and $(\tilde{\gamma} - \gamma)\phi$. □

Let $\varphi \in \Im^1_1(M_n)$ and N_φ be the Nijenhuis tensor of φ:

$$N_\varphi(V,W) = [\varphi V, \varphi W] - \varphi[\varphi V, W] - \varphi[V, \varphi W] + \varphi^2[V,W], \ V,W \in \Im^1_0(M_n).$$

Let now $\tilde{N}_{^D\varphi}$ be the Nijenhuis tensor of $^D\varphi$ in $T^p_q(M_n)$. Then by (1.9) and (3.5), if $V,W \in \Im^1_0(M_n)$ and $A,B \in \Im^p_q(M_n)$, we have

$$\begin{aligned}
&\tilde{N}_{^D\varphi}(^VA,^VB)\\
&= [^D\varphi^VA,^D\varphi^VB] -^D\varphi[^D\varphi^VA,^VB] -^D\varphi[^VA,^D\varphi^VB] + (^D\varphi)^2[^VA,^VB]\\
&= [^V(\varphi(A)),^V(\varphi(B))] +^D\varphi[^V(\varphi(A)),^VB] +^D\varphi[^VA,^V(\varphi(B))] +\\
&\quad + (^D\varphi)^2[^VA,^VB]\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
&\tilde{N}_{^D\varphi}(^HV,^VB)\\
&= [^D\varphi^HV,^D\varphi^VB] -^D\varphi[^D\varphi^HV,^VB] -^D\varphi[^HV,^D\varphi^VB] + (^D\varphi)^2[^HV,^VB]\\
&= -[^H(\varphi(V)),^V(\varphi(B))] -^D\varphi[^H(\varphi(V)),^VB] +^D\varphi[^HV,^V(\varphi(B))] +\\
&\quad + (^D\varphi)^2[^HV,^VB]\\
&= -^V(\nabla_{\varphi(V)}\varphi(B)) -^D\varphi^V(\nabla_{\varphi(V)}B) +^D\varphi^V(\nabla_V\varphi(B)) + (^D\varphi)^2\,^V(\nabla_VB)\\
&= -^V(\nabla_{\varphi(V)}\varphi(B)) +^V(\varphi(\nabla_{\varphi(V)}B)) -^V(\varphi(\nabla_V\varphi(B))) +^V(\varphi^2(\nabla_VB))\\
&= {}^V(-\nabla_{\varphi(V)}\varphi(B)) + \varphi(\nabla_{\varphi(V)}B) - \varphi(\nabla_V\varphi(B)) + \varphi^2(\nabla_VB))\\
&= {}^V(-(\nabla_{\varphi(V)}\varphi)B - \varphi(\nabla_{\varphi(V)}B) + \varphi(\nabla_{\varphi(V)}B) - \varphi(\nabla_V\varphi)B - \varphi^2(\nabla_VB)+\\
&\quad + \varphi^2(\nabla_VB))\\
&= {}^V(-(\nabla_{\varphi(V)}\varphi)B - \varphi(\nabla_V\varphi)B),
\end{aligned}$$

$$\begin{aligned}
&\tilde{N}_{^D\varphi}(^HV,^HW)\\
&= [^D\varphi^HV,^D\varphi^HW] -^D\varphi[^D\varphi^HV,^HW] -^D\varphi[^HV,^D\varphi^HW] + (^D\varphi)^2[^HV,^HW]\\
&= [^H(\varphi(V)),^H(\varphi(W))] -^D\varphi[^H(\varphi(V)),^HW] -^D\varphi[^HV,^H(\varphi(W))] +\\
&\quad + (^D\varphi)^2[^HV,^HW]\\
&= {}^H[\varphi(V),\varphi(W)] + (\tilde{\gamma}-\gamma)R(\varphi(V),\varphi(W)) -^D\varphi(^H[\varphi(V),W] +\\
&\quad + (\tilde{\gamma}-\gamma)R(\varphi(V),W)) -^D\varphi(^H[V,\varphi(W)] + (\tilde{\gamma}-\gamma)R(V,\varphi(W)))+\\
&\quad + (^D\varphi)^2(^H[V,W] + (\tilde{\gamma}-\gamma)R(V,W))\\
&= {}^H[(\varphi(V)),\varphi(W)] + (\tilde{\gamma}-\gamma)R((\varphi(V)),\varphi(W)) -^H(\varphi[(\varphi(V)),W])+\\
&\quad + (\gamma-\tilde{\gamma})\varphi R((\varphi(V)),W)) -^H(\varphi[V,\varphi(W)]) + (\gamma-\tilde{\gamma})\varphi R(V,\varphi(W)))+\\
&\quad + (\varphi^2[V,W]) + (\tilde{\gamma}-\gamma)\varphi^2R(V,W))\\
&= {}^H(N_\varphi) + (\tilde{\gamma}-\gamma)(R(\varphi(V),\varphi(W)) - \varphi R(V,\varphi(W) - \varphi R(\varphi V,W)+\\
&\quad + \varphi^2R(V,W)).
\end{aligned}$$

Summing up, we have the following formulas;

$$(3.6)\quad \begin{cases} \tilde{N}_{^D\varphi}(^VA,^VB) = 0 \\ \tilde{N}_{^D\varphi}(^HV,^VB) = {}^V(-(\nabla_{\varphi(V)}\varphi)B - \varphi(\nabla_V\varphi)B) \\ \tilde{N}_{^D\varphi}(^HV,^HW) = {}^H(N_\varphi) + (\tilde{\gamma} - \gamma)(R(\varphi(V),\varphi(W)) - \varphi R(V,\varphi(W)) \\ \qquad\qquad -\varphi R(\varphi V, W) + \varphi^2 R(V,W). \end{cases}$$

We now suppose that (φ, g) is a Kählerian structure in M_n and ∇ the Riemannian connection determined by the metric g. Then we see that

(i) φ is an almost complex structure in M_n. i.e., $\varphi^2 = -I$;
(ii) $\nabla\varphi = 0$;
(iii) The curvature tensor R of ∇ satisfies $R(\varphi V, \varphi W) = R(V,W)$ for any $V, W \in \Im_0^1(M_n)$ [7, p.129].

Thus, from (iii), we get $R(\varphi V, W) = -R(V, \varphi W)$, since $\varphi^2 = -I$. Hence again using $\varphi^2 = -I$, we find

$$R(\varphi(V), \varphi(W)) - \varphi R(V, \varphi(W) - \varphi R(\varphi V, W) + \varphi^2 R(V,W) = 0.$$

Therefore it follows, from (3.6) and (ii), that

$$\begin{cases} \tilde{N}_{^D\varphi}(^VA,^VB) = 0 \\ \tilde{N}_{^D\varphi}(^HV,^VB) = 0 \\ \tilde{N}_{^D\varphi}(^HV,^HW) = 0 \end{cases}$$

for any $V, W \in \Im_0^1(M_n)$ and $A, B \in \Im_q^p(M_n)$. Hence, by Lemma, $\tilde{N}_{^D\varphi}$ is zero, since N is skew-symmetric. Thus, $^D\varphi$ is necessary integrable. Summing up, we have

Theorem 3.4. *Let (φ, g) be a Kählerian structure in M_n and ∇ the Riemannian connection determined by the metric g. Then the diagonal lift $^D\varphi$ of φ to $T_q^p(M_n)$ along $\sigma_\xi(M_n)$ is an complex structure in $T_q^p(M_n)$.*

Acknowledgment. This paper is supported by The Scientific & Technological Council of Turkey with number (105T551 TBAG-HD/112).

References

[1] N. Cengiz and A. A. Salimov, *Complete lifts of derivations to tensor bundles*, Bol. Soc. Mat. Mexicana (3) **8** (2002), no. 1, 75–82.

[2] A. Ledger and K. Yano, *Almost complex structures on tensor bundles*, J. Differential Geometry **1** (1967), 355–368.

[3] A. Mağden, N. Cengiz, and A. A. Salimov, *Horizontal lift of affinor structures and its applications*, Appl. Math. Comput. **156** (2004), no. 2, 455–461.

[4] A. Mağden and A. A. Salimov, *Horizontal lifts of tensor fields to sections of the tangent bundle*, Izv. Vyssh. Uchebn. Zaved. Mat. (2001), no. 3, 77–80; translation in Russian Math. (Iz. VUZ) **45** (2001), no. 3, 73–76

[5] A. A. Salimov, *A new method in the theory of liftings of tensor fields in a tensor bundle*, Izv. Vyssh. Uchebn. Zaved. Mat. (1994), no. 3, 69–75; translation in Russian Math. (Iz. VUZ) **38** (1994), no. 3, 67–73.

[6] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles: Differential Geometry*, Pure and Applied Mathematics, No. 16. Marcel Dekker, Inc., New York, 1973.

[7] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, 3. World Scientific Publishing Co., Singapore, 1984.

Aydin Gezer
Department of Mathematics
Faculty of Arts and Sciences
Atatürk University
25240, Erzurum, Turkey
E-mail address: agezer@atauni.edu.tr, gezeraydin@hotmail.com

Arif Salimov
Department of Mathematics
Faculty of Arts and Sciences
Atatürk University
25240, Erzurum, Turkey
E-mail address: asalimov@atauni.edu.tr