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COMPLETE MOMENT CONVERGENCE OF MOVING AVERAGE PROCESSES WITH DEPENDENT INNOVATIONS

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Abstract. Let $\{Y_i; -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed and ϕ-mixing random variables with zero means and finite variances and $\{a_i; -\infty < i < \infty\}$ an absolutely summable sequence of real numbers. In this paper, we prove the complete moment convergence of $\{\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_{i+k}Y_i/n^{1/p}; n \geq 1\}$ under some suitable conditions.

1. Introduction

We assume that $\{Y_i; -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed random variables with zero means and finite variances. Let $\{a_i; -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and

$$X_k = \sum_{i=-\infty}^{\infty} a_{i+k}Y_i, k \geq 1. \tag{1.1}$$

Under independence assumptions, i,e., $\{Y_i; -\infty < i < \infty\}$ is a sequence of independent random variables, many limiting results have been obtained for moving average process $\{X_k; k \geq 1\}$. For examples, Ibragimov [4] has established the central limit theorem for $\{X_k; k \geq 1\}$, Burton and Dehling [2] have obtained a large deviation principle for $\{X_k; k \geq 1\}$ assuming $E\exp(tY_1) < \infty$ for all t, and Li et al. [5] have obtained the following result on complete convergence.

Theorem A. *Suppose $\{Y_i; -\infty < i < \infty\}$ is a sequence of independent and identically distributed random variables. Let $\{X_k; k \geq 1\}$ be defined as (1.1) and $1 \leq p < 2$. Then $EY_1 = 0$ and $E|Y_1|^{2p} < \infty$ imply*

$$\sum_{n=1}^{\infty} P\{|\sum_{k=1}^{n} X_k| \geq n^{1/p}\epsilon\} < \infty \textit{ for all } \epsilon > 0.$$

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Zhang [9] extended Theorem A to the case of dependence as follows:

Theorem B. *Suppose $\{Y_i; -\infty < i < \infty\}$ is a sequence of identically distributed and ϕ-mixing random variables with $\sum_{m=1}^{\infty} \phi^{\frac{1}{2}}(m) < \infty$ and $\{X_k; k \geq 1\}$ is defined as (1.1). Let $h(x) > 0 (x > 0)$ be a slowly varying function and $1 \leq p < 2, r \geq 1$. Then $EY_1 = 0$ and $E|Y_1|^r h(|Y_1|^p) < \infty$ imply*

$$\sum_{n=1}^{\infty} n^{r/p-2} h(n) P\{|\sum_{k=1}^{n} X_k| \geq \epsilon n^{1/p}\} < \infty \text{ for all } \epsilon > 0.$$

Moreover, Baek, Kim and Liang [1] discussed the complete convergence of moving average processes under negative dependence assumptions and Liang [7] obtained some general results on the complete convergence of weighted sums of negatively associated random variables, including moving average processes.

When $\{X_k; k \geq 1\}$ is a sequence of i.i.d random variables with mean zeros and positive finite variances, Chow [3] obtained the following result on the complete moment convergence:

Theorem C. *Suppose that $\{X_k; k \geq 1\}$ is a sequence of i.i.d random variables with $EX_1 = 0$. For $1 \leq p < 2$ and $r > p$, if $E\{|X_1|^r + |X_1| \log(1 + |X_1|)\} < \infty$, then for any $\epsilon > 0$, we have*

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p} E\{|\sum_{k=1}^{\infty} X_k| - \epsilon n^{1/p}\}^+ < \infty.$$

Recently Li and Zhang [6] showed that this kind of result also holds for moving average processes under negative association as follows:

Theorem D. *Suppose $\{X_k; k \geq 1\}$ is defined as (1.1), where $\{a_i; -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$ and $\{Y_i; -\infty < i < \infty\}$ is a sequence of identically distributed and negatively associated random variables with $EY_1 = 0, EY_1^2 < \infty$. Let $h(x) > 0 (x > 0)$ be a slowly varying function and $1 \leq p < 2, r > 1 + p/2$. Then $E|Y_1|^r h(|Y_1|^p) < \infty$ implies $\sum_{n=1}^{\infty} n^{r/p-2-1/p} h(n) E\{|S_n| - \epsilon n^{1/p}\}^+ < \infty$, where $S_n = \sum_{k=1}^{n} X_k, n \geq 1$.*

In this paper we shall extend Theorem D to the ϕ-mixing case.

2. Results

We suppose $\{Y_i; -\infty < i < \infty\}$ is a sequence of identically distributed and ϕ-mixing random variables, i.e.,

$$\phi(m) = \sup_k \phi(\mathcal{F}_{\infty}^{k}, \mathcal{F}_{k+m}^{\infty}) \to 0, \text{ as } m \to \infty,$$

where $\mathcal{F}_n^m = \sigma(Y_k; n \leq k \leq m)$ and $\phi(A, B) = \sup(P(B|A) - P(B))$ $A \in \mathcal{A}$, $B \in \mathcal{B}, P(A) > 0$.

Throughout the sequel, C will represent a positive constant although its value may change from one appearance to the next and $[x]$ will indicate the maximum integer not larger than x.

The following lemma comes from Burton and Dehling [2].

Lemma 2.1. *Let $\sum_{-\infty}^{\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{-\infty}^{\infty} a_i$ and $k \geq 1$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} \Big| \sum_{j=i+1}^{i+n} a_j \Big|^k = |a|^k. \tag{2.1}$$

The following lemma will be useful. A proof appears in Shao [8].

Lemma 2.2. *Let $\{Y_n; n \geq 1\}$ be a sequence of ϕ-mixing random variables and let $S_n = \sum_{k=1}^{n} Y_k, n \geq 1$. Suppose that there exists a sequence $\{C_n\}$ of positive numbers such that*

$$\max_{1\leq i\leq n} ES_i^2 \leq C_n. \tag{2.2}$$

Then, for any $q \geq 2$, there exists $\mathrm{C} = C(q, \phi(\cdot))$ such that

$$E \max_{1\leq i\leq n} |S_i|^q \leq \mathrm{C}(C_n^{q/2} + E \max_{1\leq i\leq n} |Y_i|^q). \tag{2.3}$$

Our main result is as follows:

Theorem 2.3. *Set $S_n = \sum_{k=1}^{n} X_k, n \geq 1$, where $\{X_k; k \geq 1\}$ is defined as (1.1). Suppose $\{Y_i; -\infty < i < \infty\}$ is a sequence of identically distributed and ϕ-mixing random variables with $EY_1 = 0$, $EY_1^2 < \infty$ and $\sum_{m=1}^{\infty} \phi^{\frac{1}{2}}(m) < \infty$ and $\{X_k; k \geq 1\}$ is defined as (1.1), where $\{a_i; -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{i=-\infty}^{\infty} |a_i| < \infty$. Let $h(x) > 0 (x > 0)$ be a slowly varying function and $1 \leq p < 2, r > p$. Then $E|Y_1|^r h(|Y_1|^p) < \infty$ implies*

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p} h(n) E\{|S_n| - \epsilon n^{1/p}\}^+ < \infty \text{ for all } \epsilon > 0. \tag{2.4}$$

Remark. Let $a_{i+k} = 1, i = k; a_{i+k} = 0, i \neq k, 1 \leq k \leq n$. Then $X_k = Y_k$, $S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} Y_k$. Hence Theorem 2.3 holds when $\{X_k; k \geq 1\}$ is a sequence of identically distributed and ϕ-mixing random variables.

Corollary 2.4. *Under the conditions of Theorem 2.3, $E|Y_1|^r h(|Y_1|^p) < \infty$ implies*

$$\sum_{n=1}^{\infty} n^{r/p-2} h(n) P\{|S_n| > \epsilon n^{1/p}\} < \infty \text{ for all } \epsilon > 0. \tag{2.5}$$

Proof. By Theorem 2.3 we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r/p-2-1/p} h(n) E\{|S_n| - \epsilon n^{1/p}\}^+ \\ &= \sum_{n=1}^{\infty} n^{r/p-2-1/p} h(n) \int_0^{\infty} P\{|S_n| - \epsilon n^{\frac{1}{p}} > x\} dx \end{aligned} \tag{2.6}$$

$$= \int_0^\infty \sum_{n=1}^\infty n^{r/p-2-1/p} h(n) P\{|S_n| > (\epsilon + y) n^{\frac{1}{p}}\} n^{\frac{1}{p}} dy$$
$$= \int_0^\infty \sum_{n=1}^\infty n^{r/p-2} h(n) P\{|S_n| > (\epsilon + y) n^{\frac{1}{p}}\} dy < \infty.$$

Hence from (2.6) the result (2.5) follows. □

Remark. Note that Corollary 2.4 is Theorem B for $1 \le p < 2, r > 1 + \frac{p}{2}$.

3. Proof of Theorem 2.3

Recall that

$$\sum_{k=1}^n X_k = \sum_{i=-\infty}^\infty \sum_{k=1}^n a_{i+k} Y_i = \sum_{i=-\infty}^\infty a_{ni} Y_i,$$

where $a_{ni} = \sum_{k=1}^n a_{i+k}$.

From Lemma 2.1, we can assume, without loss of generality, that $\sum_{i=-\infty}^\infty |a_{ni}| \le n,\ n \ge 1$ and $\tilde{a} = \sum_{i=-\infty}^\infty |a_i| \le 1$.

Let $S_n = \sum_{-\infty}^\infty a_{ni} Y_i I\{|a_{ni} Y_i| \le x\}$.

First note that for $x > n^{\frac{1}{p}}$,

$$\begin{aligned}
x^{-1} E|S_n| &= x^{-1} |\sum_{i=-\infty}^\infty a_{ni} E Y_i I\{|a_{ni} Y_i| > x\}| \\
&\le x^{-1} \sum_{i=-\infty}^\infty |a_{ni}| E|Y_1| I\{|a_{ni} Y_1| > x\} \\
&\le x^{-1} n E|Y_1| I\{\tilde{a}|Y_1| > x\} \\
&\le x^{-1} n E|Y_1| I\{|Y_1| > x\} \\
&\le x^{-1} x^p E|Y_1| I\{\tilde{a}|Y_1| > x\} \\
&\le E|Y_1|^p I\{|Y_1| > x\} \to 0 \text{ as } x \to \infty.
\end{aligned}$$

So, for x large enough we have $x^{-1} E|S_n| < \epsilon/2$. Then

$$\begin{aligned}
&\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}} h(n) E\{|\sum_{k=1}^n X_k| - \epsilon n^{\frac{1}{p}}\}^+ \\
= &\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{\epsilon n^{\frac{1}{p}}}^\infty P\{|\sum_{k=1}^n X_k| \ge x\} dx \text{ (letting } x = \epsilon x') \\
= &\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \epsilon \int_{n^{\frac{1}{p}}}^\infty P\{|\sum_{k=1}^n X_k| \ge \epsilon x'\} dx'
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n)\epsilon \int_{n^{\frac{1}{p}}}^{\infty} (P\{\sup_i |a_{ni}Y_i| \ge x\} \\
&\quad + P\{|S_n - ES_n| \ge x\frac{\epsilon}{2}\})dx \\
&= C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n)\epsilon \int_{n^{\frac{1}{p}}}^{\infty} (I_1 + I_2)dx,
\end{aligned}$$

where $I_1 = P\{\sup_i |a_{ni}Y_i| > x\}$ and $I_2 = P\{|S_n - ES_n| \ge x\frac{\epsilon}{2}\}$.

Set $I_{nj} = \{i \in \mathcal{I}; (j+1)^{-\frac{1}{p}} < |a_{ni}| \le j^{-\frac{1}{p}}\}, j = 1, 2, \ldots$. Then $\cup_{j\ge 1} I_{nj} = \mathcal{I}$. Note that (cf. Li et al., [5])

$$\sum_{j=1}^{k} \#I_{nj} \le n(k+1)^{\frac{1}{p}}.$$

For I_1 and $1 \le p < 2, r > p$ noting that $E|Y_1|^r h(|Y_1|^p) < \infty$, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} I_1 dx \\
&\le C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{i=-\infty}^{\infty} P\{|a_{ni}Y_i| > x\}dx \\
&= C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{i=-\infty}^{\infty} P\{|a_{ni}Y_1| > x\}dx \\
&\le C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{j=1}^{\infty} \sum_{i\in I_{nj}} P\{|Y_1| > j^{\frac{1}{p}} x\}dx \\
&\le C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{j=1}^{\infty} (\#I_{nj}) \sum_{k\ge jx^p} P\{k \le |Y_1|^p < k+1\}dx \\
&\le C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{k=[x^p]}^{\infty} \sum_{j=1}^{[k/x^p]} (\#I_{nj}) P\{k \le |Y_1|^p < k+1\}dx \\
&\le C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{k=[x^p]}^{\infty} n(\frac{k}{x^p}+1)^{\frac{1}{p}} P\{k \le |Y_1|^p < k+1\}dx \\
&\le C\sum_{n=1}^{\infty} n^{\frac{r}{p}-1-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} \sum_{k=[x^p]}^{\infty} k^{\frac{1}{p}} x^{-1} P\{k \le |Y_1|^p < k+1\}dx \\
&\le C\int_{1}^{\infty} t^{\frac{r}{p}-1-\frac{1}{p}} h(t) \int_{t^{\frac{1}{p}}}^{\infty} \sum_{k=[x^p]}^{\infty} k^{\frac{1}{p}} x^{-1} P\{k \le |Y_1|^p < k+1\}dxdt
\end{aligned}$$

(letting $y = t^{\frac{1}{p}}$)

$$
\begin{aligned}
&\le C\int_1^\infty y^{r-2}h(y^p)\int_y^\infty x^{-1}\sum_{k=[x^p]}^\infty k^{\frac{1}{p}}P\{k\le |Y_1|^p<k+1\}dxdy\\
&\le C\int_1^\infty (\int_1^x y^{r-2}h(y^p)dy)x^{-1}\sum_{k=[x^p]}^\infty k^{\frac{1}{p}}P\{k\le |Y_1|^p<k+1\}dx\\
&\le C\int_1^\infty x^{r-2}h(x^p)\sum_{k=[x^p]}^\infty k^{\frac{1}{p}}P\{k\le |Y_1|^p<k+1\}dx\\
&\le C\sum_{k=1}^\infty k^{\frac{1}{p}}P\{k\le |Y_1|^p<k+1\}\int_1^{(k+1)^{\frac{1}{p}}} x^{r-2}h(x^p)dx\\
&\le C\sum_{k=1}^\infty k^{\frac{1}{p}}P\{k\le |Y_1|^p<k+1\}(k+1)^{\frac{r-1}{p}}h(k+1)\\
&\le C\sum_{k=0}^\infty (k+1)^{\frac{r}{p}}h(k+1)P\{k\le |Y_1|^p<k+1\}\\
&\le CE|Y_1|^r h(|Y_1|^p)+1<\infty.
\end{aligned}
$$

Now we estimate I_2, for $r>p$. For I_2, note that $\sum_{m=1}^\infty \phi^{\frac{1}{2}}(m)<\infty$, we have

$$
\begin{aligned}
&\sup_{-\infty<l\le m<\infty} E(\sum_{i=l}^m a_{ni}Y_iI\{|a_{ni}Y_i|\le x\}-E\sum_{i=l}^m a_{ni}Y_iI\{|a_{ni}Y_i|\le x\})^2\\
\le\ & C\sum_{i=-\infty}^\infty E(a_{ni}Y_i)^2I\{|a_{ni}Y_i|\le x\}\\
=\ & C\sum_{i=-\infty}^\infty E(a_{ni}Y_1)^2I\{|a_{ni}Y_i|\le x\}.
\end{aligned}
$$

By Lemma 2.2, we have, for $q\ge 2$,

$$
\begin{aligned}
&P\{|S_n-ES_n|\ge \frac{\epsilon}{2}x\}\le Cx^{-q}E|S_n-ES_n|^q\\
\le\ & Cx^{-q}((\sum_{i=-\infty}^\infty a_{ni}^2EY_1^2I\{|a_{ni}Y_1|\le x\})^{q/2}\\
&+E\max_i|a_{ni}Y_i|^qI\{|a_{ni}Y_i|\le x\})\\
\le\ & Cx^{-q}((\sum_{i=-\infty}^\infty a_{ni}^2EY_1^2I\{|a_{ni}Y_1|\le x\})^{q/2}\\
&+\sum_{i=-\infty}^\infty E|a_{ni}Y_1|^qI\{|a_{ni}Y_1|\le x\}).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} I_2 dx \\
&\leq C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \Big(\sum_{i=-\infty}^{\infty} a_{ni}^2 E Y_1^2 I\{|a_{ni}Y_1| \leq x\} \Big)^{\frac{q}{2}} dx \\
&\quad + \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{i=-\infty}^{\infty} E|a_{ni}Y_1|^q I\{|a_{ni}Y_1| \leq x\} dx \\
&= C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} (I_3 + I_4) dx.
\end{aligned} \tag{3.1}
$$

If $r \geq 2$, by choosing q large enough such that $q(\frac{1}{p} - \frac{1}{2}) > \frac{r}{p} - 1$, for I_3 we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} I_3 dx \\
&= \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \Big(\sum_{i=-\infty}^{\infty} a_{ni}^2 E Y_1^2 I\{|a_{ni}Y_1| \leq x\} \Big)^{\frac{q}{2}} dx \\
&\leq C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}+\frac{q}{2}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} dx \\
&= C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-q(\frac{1}{p}-\frac{1}{2})} h(n) < \infty.
\end{aligned} \tag{3.2}
$$

If $p < r < 2$ by choosing $q = 2$ (3.2) still holds and $q > r$.

Note that $r \geq 2$ and $q(\frac{1}{p} - \frac{1}{2}) > \frac{r}{p} - 1$ imply $q > r$ because $1 \leq p < 2$.

For I_4, choosing q such that $q(\frac{1}{p} - \frac{1}{2}) > \frac{r}{p} - 1$ and $r \geq 2$

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} I_4 dx \\
&= \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{i=-\infty}^{\infty} E|a_{ni}Y_1|^q I\{|a_{ni}Y_1| \leq x\} dx \\
&\leq C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} E|a_{ni}Y_1|^q I\{|a_{ni}Y_1| \leq x\} dx \\
&\leq C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{j=1}^{\infty} (\# I_{nj}) j^{-\frac{q}{p}} E|Y_1|^q I\{|Y_1|^p \leq x^p (j+1)\} dx \\
&\leq C \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}} h(n) \int_{n^{\frac{1}{p}}}^{\infty} x^{-q} \sum_{j=1}^{\infty} (\# I_{nj}) j^{-\frac{q}{p}}
\end{aligned}
$$

$$\times \sum_{0\le k\le (j+1)x^p} E|Y_1|^q I\{k \le |Y_1|^p < k+1\}dx.$$

$$\le C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)[\int_{n^{\frac{1}{p}}}^{\infty} x^{-q}\sum_{j=1}^{\infty}(\#I_{nj})j^{-\frac{q}{p}}$$

$$\times \sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \le |Y_1|^p < k+1\}dx$$

$$+\int_{n^{\frac{1}{p}}}^{\infty} x^{-q}\sum_{j=1}^{\infty}(\#I_{nj})j^{-\frac{q}{p}} \sum_{k=[2x^p]+1}^{[(j+1)x^p]} E|Y_1|^q I\{k \le |Y_1|^p < k+1\}dx]$$

$$= C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^{\infty}(I_5+I_6)dx.$$

Note that for $q \ge 1$ and $m \ge 1$, we have

$$n \ge \sum_{i=-\infty}^{\infty}|a_{ni}| = \sum_{j=1}^{\infty}\sum_{i\in I_{nj}}|a_{ni}| \ge \sum_{j=1}^{\infty}(\#I_{nj})(j+1)^{-\frac{1}{p}}$$

$$\ge \sum_{j=m}^{\infty}(\#I_{nj})(j+1)^{-\frac{1}{p}} \ge \sum_{j=m}^{\infty}(\#I_{nj})(j+1)^{-\frac{q}{p}}(m+1)^{\frac{q}{p}-\frac{1}{p}}.$$

So

$$\sum_{j=m}^{\infty}(\#I_{nj})j^{-q/p} \le Cnm^{-(q-1)/p}.$$

Then, for I_5 choosing q such that $q(\frac{1}{p}-\frac{1}{2}) > \frac{r}{p}-1$ and $r \ge 2$

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^{\infty} I_5 dx$$

$$= \sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^{\infty} x^{-q}\sum_{j=1}^{\infty}(\#I_{nj})j^{-\frac{q}{p}}\sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \le |Y_1|^p < k+1\}dx$$

$$\le C\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)n\int_{n^{\frac{1}{p}}}^{\infty} x^{-q}\sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \le |Y_1|^p < k+1\}dx$$

$$\le C\int_1^{\infty} t^{\frac{r}{p}-1-\frac{1}{p}}h(t)\int_{t^{\frac{1}{p}}}^{\infty} x^{-q}\sum_{k=0}^{[2x^p]} E|Y_1|^q I\{k \le |Y_1|^p < k+1\}dxdt$$

(letting $t = y^p$)

$$
\begin{aligned}
&\leq C\int_1^\infty y^{r-2}h(y^p)\int_y^\infty x^{-q}\sum_{k=0}^{[2x^p]}E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}dxdy\\
&\leq C\int_1^\infty (\int_1^x y^{r-2}h(y^p)dy)x^{-q}\sum_{k=0}^{[2x^p]}E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}dx\\
&\leq C\int_1^\infty x^{r-1-q}h(x^p)\sum_{k=0}^{[2x^p]}E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}dx\\
&\leq C\sum_{k=1}^\infty E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}\int_{(\frac{k}{2})^{\frac{1}{p}}}^\infty x^{r-1-q}h(x^p)dx\\
&\leq C\sum_{k=1}^\infty E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}k^{\frac{r-q}{p}}h(k)\\
&\leq C\sum_{k=1}^\infty (k+1)^{\frac{r}{p}}h(k+1)P\{k\leq|Y_1|^p<k+1\}\\
&\leq CE|Y_1|^rh(|Y_1|^p)+1<\infty.
\end{aligned}
$$

For I_6, by choosing q such that $q(\frac{1}{p}-\frac{1}{2})>\frac{r}{p}-1$ and $r\geq 2$ we also get

$$
\begin{aligned}
&\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty I_6dx\\
=&\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty x^{-q}\sum_{j=1}^\infty(\#I_{nj})j^{-\frac{q}{p}}\\
&\quad\times\sum_{k=[2x^p]+1}^{[(j+1)x^p]}E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}dx\\
\leq& C\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty x^{-q}\sum_{k=[2x^p]+1}^\infty\\
&\quad\times\sum_{j\geq[\frac{k}{x^p}]-1}(\sharp I_{nj})j^{-\frac{q}{p}}E|Y_1|^qI\{k\leq|Y_1|^q<k+1\}dx\\
\leq& C\sum_{n=1}^\infty n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty x^{-q}\sum_{k=[2x^p]+1}^\infty n(\frac{k}{x^p})^{-\frac{q-1}{p}}E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}dx\\
\leq& C\int_1^\infty t^{\frac{r}{p}-1-\frac{1}{p}}h(t)\int_{t^{\frac{1}{p}}}^\infty x^{-1}\sum_{k=[2x^p]+1}^\infty k^{-\frac{q-1}{p}}E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}dxdt
\end{aligned}
$$

(letting $t=y^p$)

$$\leq C\int_1^\infty y^{r-2}h(y^p)\int_y^\infty x^{-1}\sum_{k=[2x^p]+1}^{\infty} k^{-\frac{q-1}{p}}E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}dxdy$$

$$\leq C\int_1^\infty(\int_1^x y^{r-2}h(y^p)dy)x^{-1}\sum_{k=[2x^p]+1}^{\infty} k^{-\frac{q-1}{p}}E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}dx$$

$$\leq C\int_1^\infty x^{r-2}h(x^p)\sum_{k=[2x^p]+1}^{\infty} k^{-\frac{q-1}{p}}E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}dx$$

$$\leq C\sum_{k=1}^{\infty} k^{-\frac{q-1}{p}}E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}\int_0^{(\frac{k}{2})^{\frac{1}{p}}} x^{r-2}h(x^p)dx$$

$$\leq C\sum_{k=1}^{\infty} k^{\frac{r-q}{p}}h(k)E|Y_1|^qI\{k\leq|Y_1|^p<k+1\}$$

$$\leq C\sum_{k=1}^{\infty}(k+1)^{\frac{r}{p}}h(k+1)P\{k\leq|Y_1|^p<k+1\}$$

$$\leq CE|Y_1|^rh(|Y_1|^p)+1<\infty.$$

So, for $r\geq 2$ and q such that $q(\frac{1}{p}-\frac{1}{2})>\frac{r}{p}-1$

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty I_4dx<\infty, \tag{3.3}$$

and then (3.1), (3.2) and (3.3) yield

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)\int_{n^{\frac{1}{p}}}^\infty I_2dx<\infty. \tag{3.4}$$

If $p<r<2$, by choosing $q=2$, (3.2) still holds and (3.4) follows from (3.1) and (3.2) since $I_3=I_4$. Thus we have $\sum_{n=1}^{\infty} n^{\frac{r}{p}-2-\frac{1}{p}}h(n)E\{|S_n|-\epsilon n^{\frac{1}{p}}\}^+<\infty$ for all $\epsilon>0$.

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