

LINEAR PRESERVERS OF SPANNING COLUMN RANK OF MATRIX SUMS OVER SEMIRINGS

SEOK-ZUN SONG

ABSTRACT. The spanning column rank of an $m \times n$ matrix A over a semiring is the minimal number of columns that span all columns of A . We characterize linear operators that preserve the sets of matrix pairs which satisfy additive properties with respect to spanning column rank of matrices over semirings.

1. Introduction

In the last few decades a lot of work has been done on the problems of determining the linear maps on the $n \times n$ matrix algebra $M_n(\mathbb{F})$ over a field \mathbb{F} that leave certain matrix relations, subsets or properties invariant. For a survey of these type of problems see [6]. Although the linear preservers concerned are mostly linear operators on matrix spaces over fields or rings, the same problem has been extended to matrices over various semirings.

Marsaglia and Styan [5] studied on the inequalities for rank of matrices. Recently, Beasley and Guterman [1] investigated the rank inequalities of matrices over semirings. And they characterized the equality cases for some inequalities in [2]. This characterization problems are open even over fields (see [5]). The structure of matrix varieties which arise as extremal cases in these inequalities is far from being understood over fields, as well as over semirings. A usual way to generate elements of such a variety is to find a tuple of matrices which belongs to it and to act on this tuple by various linear operators that preserve this variety. The investigation of the corresponding problems over semirings for the factor rank function was done in [2]. The complete classification of linear operators that preserve equality cases in matrix inequalities over fields was obtained in [3]. Song and Hwang characterized the linear operators that preserve spanning column ranks of matrices over nonnegative reals in [7]. For

Received May 3, 2006.

2000 *Mathematics Subject Classification.* Primary 15A03, 15A04, 15A45.

Key words and phrases. spanning column rank, (P, Q, B) -operator, (U, V) -operator, rank inequality.

This work was supported by Korean Science and Engineering Foundation (F01-2007-000-10047-0).

the details on linear operators preserving matrix invariants one can see [6] and references therein.

In this paper, we characterize linear operators that preserve the sets of matrix pairs which satisfy additive properties with respect to spanning column rank of matrices over semirings.

2. Preliminaries

A *semiring* \mathbb{S} is essentially a ring in which only the zero is required to have an additive inverse([8]). Thus all rings are semirings. A semiring is called *antinegative* if the zero element is the only element with an additive inverse. The set of nonnegative integers is an example of antinegative semiring but it is not a ring.

A semiring \mathbb{S} is called *Boolean* if \mathbb{S} is equivalent to a set of subsets of a given set M , the sum of two subsets is their union, and the product is their intersection. The zero element is the empty set and the identity element is the whole set M .

It is straightforward to see that a Boolean semiring is commutative and antinegative. If \mathbb{S} consists of only the empty subset and M then it is called a binary Boolean semiring (or $\{0, 1\}$ -semiring) and is denoted by \mathbb{B} .

A semiring is called *chain* if the set \mathbb{S} is totally ordered with universal lower and upper bounds and the operations are defined by $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$.

It is straightforward to see that any chain semiring \mathbb{S} is a Boolean semiring on the set of appropriate subsets of \mathbb{S} .

Let $\mathbb{M}_{m,n}(\mathbb{S})$ denote the set of $m \times n$ matrices with entries from the semiring \mathbb{S} . If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{S})$ instead of $\mathbb{M}_{n,n}(\mathbb{S})$.

A vector space is usually only defined over fields or division rings, and modules are generalizations of vector spaces defined over rings. We generalize the concept of vector spaces to *semiring vector spaces* defined over arbitrary semirings.

Given a semiring \mathbb{S} , we define a *semiring vector space*, $V(\mathbb{S})$, to be a non-empty set with two operations, addition and scalar multiplication such that $V(\mathbb{S})$ is closed under addition and scalar multiplication, addition is associative and commutative, and such that for all \mathbf{u} and \mathbf{v} in $V(\mathbb{S})$ and $r, s \in \mathbb{S}$:

1. There exists a $\mathbf{0}$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$,
2. $\mathbf{1}\mathbf{v} = \mathbf{v} = \mathbf{v}\mathbf{1}$,
3. $r\mathbf{sv} = r(\mathbf{sv})$,
4. $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$, and
5. $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.

A set \mathbb{W} of vectors from a semiring vector space $V(\mathbb{S})$, is called *linearly independent* if there is no vector in \mathbb{W} that can be expressed as a nontrivial linear combination of the others. The set is *linearly dependent* if it is not independent.

A set \mathbb{B} of linearly independent vectors is said to be a *basis* of the semiring vector space $V(\mathbb{S})$ if its linear span is $V(\mathbb{S})$. The *dimension* of $V(\mathbb{S})$ is a minimal number of vectors in any basis of $V(\mathbb{S})$.

The following rank functions are usual in the semiring context.

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *factor rank* k ($\text{rank}(A) = k$) if there exist matrices $B \in \mathbb{M}_{m,k}(\mathbb{S})$ and $C \in \mathbb{M}_{k,n}(\mathbb{S})$ such that $A = BC$ and k is the smallest positive integer for which such factorization exists. By definition, the only matrix with factor rank 0 is the zero matrix, O .

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *column rank* k ($c(A) = k$) if the dimension of the linear span of the columns of A is equal to k .

The matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *spanning column rank* k ($sc(A) = k$) if the minimal number of columns that span all columns of A is k .

It follows that

$$1 \leq \text{rank}(A) \leq c(A) \leq sc(A) \leq n$$

for all nonzero matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ (see [1, 4, 7]). These inequalities may be strict: let

$$A = [\ 3 \ 4 \], \ B = [\ 3 - \sqrt{7} \ \ \sqrt{7} - 2 \] \in \mathbb{M}_{1,2}(\mathbb{S}),$$

where $\mathbb{S} = (\mathbb{Z} [\sqrt{7}])^+$ is the semiring of nonnegative elements of the ring $\mathbb{Z} [\sqrt{7}]$. Then $\text{rank}(A) = 1 < 2 = c(A)$, and $c(B) = 1 < 2 = sc(B)$ since $(3 - \sqrt{7}) + (\sqrt{7} - 2) = 1$ but any one column of B does not span the other column over $\mathbb{S} = (\mathbb{Z} [\sqrt{7}])^+$.

A *line* of a matrix A is a row or a column of A .

If \mathbb{S} is a subsemiring of a field then there is a usual rank function $\rho(A)$ over field for any matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$. Easy examples show that over semirings these functions are not equal in general. However, the inequalities

$$\rho(A) \leq \text{rank}(A) \leq c(A) \leq sc(A)$$

always hold. The behavior of the function ρ with respect to matrix multiplication and addition is given by well-known Frobenius, Schwartz and Sylvester inequalities (see [1]). Arithmetic properties of spanning column ranks depend on the structure of semiring of entries.

For matrices $X = [x_{i,j}]$ and $Y = [y_{i,j}]$ in $\mathbb{M}_{m,n}(\mathbb{S})$, the matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the $(i, j)^{\text{th}}$ entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

We say that the matrix A *dominates* the matrix B if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$ in this case.

If A and B are matrices with $A \geq B$, then we let $A \setminus B$ denote the matrix C where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0; \\ a_{i,j} & \text{otherwise.} \end{cases}$$

Let $\mathbb{Z}(\mathbb{S})$ denote the center of the semiring \mathbb{S} . The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J , and O , respectively. The matrix $E_{i,j}$, called a *cell*, denotes

the matrix with 1 in (i, j) position and zero elsewhere. A *weighted cell* is any nonzero scalar multiple of a cell, i.e., $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathbb{S}$. Let R_i denote the matrix whose i^{th} row is all ones and all other rows are zero, and C_j denote the matrix whose j^{th} column is all ones and all other columns are zero. We let $|A|$ denote the number of nonzero entries in the matrix A . We denote by $A[i_1, \dots, i_k | j_1, \dots, j_l]$ the $k \times l$ -submatrix of A which lies in the intersection of the i_1, \dots, i_k rows and j_1, \dots, j_l columns.

Let $\Delta_{m,n} = \{(i, j) | i = 1, \dots, m; j = 1, \dots, n\}$. If $m = n$, we use the notation Δ_n instead of $\Delta_{n,n}$.

Let \mathbb{S} be a semiring, not necessary commutative. A map $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{m,n}(\mathbb{S})$ is called *linear operator* if T preserves matrix addition and scalar multiplication on both sides.

We say that a linear operator T *preserves* a set \mathbb{P} if $X \in \mathbb{P}$ implies that $T(X) \in \mathbb{P}$, or, if \mathbb{P} is a set of ordered pairs, that $(X, Y) \in \mathbb{P}$ implies $(T(X), T(Y)) \in \mathbb{P}$.

An operator T on $\mathbb{M}_{m,n}(\mathbb{S})$ is called a (P, Q, B) -operator if there exist permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$, and a matrix $B \in \mathbb{M}_{m,n}(\mathbb{S})$ with $B \geq J$ such that

$$(2.1) \quad T(X) = P(X \circ B)Q$$

for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$ or, $m = n$ and

$$(2.2) \quad T(X) = P(X \circ B)^t Q$$

for all $X \in \mathbb{M}_n(\mathbb{S})$, where X^t denotes the transpose of X . Operators of the form (2.1) are called *non-transposing (P, Q, B) -operators*; operators of the form (2.2) are *transposing (P, Q, B) -operators*.

An operator T is called a (U, V) -operator if there exist invertible matrices $U \in \mathbb{M}_m(\mathbb{S})$ and $V \in \mathbb{M}_n(\mathbb{S})$ such that

$$(2.3) \quad T(X) = UXV$$

for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$ or, $m = n$ and

$$(2.4) \quad T(X) = UX^tV$$

for all $X \in \mathbb{M}_n(\mathbb{S})$. Operators of the form (2.3) are called *non-transposing (U, V) -operators*; operators of the form (2.4) are *transposing (U, V) -operators*.

Unless otherwise specified, we will assume that \mathbb{S} is an antinegative semiring without zero divisors in the followings.

We recall some results proved in [2] for later use.

Theorem 2.1 ([2, Theorem 2.14]). *Let $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{m,n}(\mathbb{S})$ be a linear operator. Then the following are equivalent:*

- (1) T is bijective.
- (2) T is surjective.
- (3) There exists a permutation σ on $\Delta_{m,n}$ and units $b_{i,j} \in \mathbb{Z}(\mathbb{S})$ such that $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$ for all $(i, j) \in \Delta_{m,n}$.

Lemma 2.2 ([2, Lemma 2.16]). *Let $T : \mathbb{M}_{m,n}(\mathbb{S}) \rightarrow \mathbb{M}_{m,n}(\mathbb{S})$ be a linear operator which maps lines to lines and is defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, where σ is a permutation on $\Delta_{m,n}$, and $b_{i,j} \in \mathbb{Z}(\mathbb{S})$ are nonzero elements. Then T is a (P, Q, B) -operator.*

One can easily check that if $m = 1$ or $n = 1$ then all operators under consideration are (P, Q, B) -operators, if $m = n = 1$ then all operators under consideration are (P, P^t, B) -operators.

Henceforth we will always assume that $m, n \geq 2$. We say that $\mathbb{M}_{m,n}(\mathbb{S})$ has *full spanning column rank* if for each $k \leq \min\{m, n\}$, $\mathbb{M}_{m-k, n-k}(\mathbb{S})$ contains a matrix of spanning column rank $n - k$.

If $m \geq n$, then we can easily show that $\mathbb{M}_{m,n}(\mathbb{S})$ has full spanning column rank. But for $m < n$, $\mathbb{M}_{m,n}(\mathbb{S})$ may have or not have full spanning column rank according to a given semiring \mathbb{S} . For example, $\mathbb{M}_{3,4}(\mathbb{Z}^+)$ has full spanning column rank, while $\mathbb{M}_{3,4}(\mathbb{B})$ has not.

The spanning column ranks of matrix sums over semirings are restricted by the following list of inequalities established in [1] :

For $O \neq X, Y \in \mathbb{M}_{m,n}(\mathbb{S})$,

$$(2.5) \quad 1 \leq sc(X + Y) \leq n.$$

If \mathbb{S} is a subsemiring of \mathbb{R}^+ (the nonnegative reals), then

$$(2.6) \quad sc(X + Y) \geq |\rho(X) - \rho(Y)|.$$

If $O \neq X \in \mathbb{M}_{m,n}(\mathbb{S})$, $O \neq Y \in \mathbb{M}_{n,k}(\mathbb{S})$

$$(2.7) \quad sc(XY) \leq sc(Y).$$

As it was proved in [1] the above inequalities (2.5), (2.6) and (2.7) are sharp and the best possible.

The following example shows that an inequality, $\rho(A+B) \leq \rho(A) + \rho(B)$, for rank of sum of two matrices A and B over a field does not work for spanning column rank.

Example 2.3. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}_3(\mathbb{Z}^+), \quad B = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 0 & 0 \\ 2 & 4 & 8 \end{bmatrix} \in \mathbb{M}_3(\mathbb{Z}^+),$$

where \mathbb{Z}^+ is the semiring of nonnegative integers. Then $sc(A) + sc(B) = 2 < 3 = sc(A + B)$, and $sc(A + A) = 1 < 2 = sc(A) + sc(A)$ over \mathbb{Z}^+ .

Lemma 2.4. *Let B be a matrix in $\mathbb{M}_{m,n}(\mathbb{S})$ with $sc(B) = 1$. If all elements of B are units in $\mathbb{Z}(\mathbb{S})$, then $sc(X) = sc(P(X \circ B)Q)$ for all permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$.*

Proof. Let X be any matrix in $\mathbb{M}_{m,n}(\mathbb{S})$. If $Q \in \mathbb{M}_n(\mathbb{S})$ is a permutation matrix, it is clear that $sc(X) = sc(XQ)$. Thus, for all permutation matrices

$P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$, we have

$$sc(X) = sc(P^t P X Q) \leq sc(P X Q) \leq sc(X Q) = sc(X)$$

from (2.7), and hence $sc(X) = sc(P X Q)$. Thus, we suffice to claim that $sc(X) = sc(X \circ B)$.

Since $sc(B) = 1$, there exists a column $\mathbf{b}_i = (b_{1,i}, \dots, b_{m,i})^t \in \mathbb{S}^m$ such that $B = \mathbf{b}_i \mathbf{e} = [e_1 \mathbf{b}_i, e_2 \mathbf{b}_i, \dots, e_n \mathbf{b}_i]$ with $\mathbf{e} = (e_1, \dots, 1, \dots, e_n) \in \mathbb{S}^n$. Thus, for any matrix $X = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n] \in \mathbb{M}_{m,n}(\mathbb{S})$, we have $X \circ B = [(\mathbf{x}_1 \circ e_1 \mathbf{b}_i), (\mathbf{x}_2 \circ e_2 \mathbf{b}_i), \dots, (\mathbf{x}_n \circ e_n \mathbf{b}_i)]$. Let $\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_k}$ be any columns of X . Then we suffice to show that $\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_k}$ are spanning columns for all columns of X if and only if $(\mathbf{x}_{i_1} \circ e_{i_1} \mathbf{b}_i), \dots, (\mathbf{x}_{i_k} \circ e_{i_k} \mathbf{b}_i)$ are spanning columns for all columns of $X \circ B$. Let $sc(X) = k$ and $\mathbf{x}_1, \dots, \mathbf{x}_k$ be spanning columns for all columns of X without loss of generality. Then $\mathbf{x}_r = \sum_{j=1}^k c_j \mathbf{x}_j$ for all $r = 1, \dots, n$ with $c_j \in \mathbb{S}$. Then we have $(\mathbf{x}_r \circ e_r \mathbf{b}_i) = \sum_{j=1}^k c_j (\mathbf{x}_j \circ e_r \mathbf{b}_i)$, equivalently, $\mathbf{x}_1 \circ e_1 \mathbf{b}_i, \dots, \mathbf{x}_k \circ e_k \mathbf{b}_i$ are spanning columns for all columns of $X \circ B$.

Conversely, assume that $sc(X \circ B) = k$ and $(\mathbf{x}_1 \circ e_1 \mathbf{b}_i), \dots, (\mathbf{x}_k \circ e_k \mathbf{b}_i)$ are spanning columns for all columns of $X \circ B$ without loss of generality. Then for any column of $X \circ B$ we can write $(\mathbf{x}_r \circ e_r \mathbf{b}_i) = \sum_{j=1}^k f_j (\mathbf{x}_j \circ e_j \mathbf{b}_i)$ for $f_j \in \mathbb{S}$. Let $\mathbf{b}_i' = (b_{1,i}^{-1}, \dots, b_{m,i}^{-1})^t \in \mathbb{S}^m$. Then

$$(\mathbf{x}_r \circ e_r \mathbf{b}_i) \circ \mathbf{b}_i' = \left(\sum_{j=1}^k f_j (\mathbf{x}_j \circ e_j \mathbf{b}_i) \right) \circ \mathbf{b}_i',$$

equivalently

$$e_r \mathbf{b}_i \circ (\mathbf{b}_i \circ \mathbf{b}_i') \mathbf{x}_r = \left(\sum_{j=1}^k f_j e_j \mathbf{b}_i \circ \mathbf{b}_i' \right) \circ \mathbf{x}_j$$

since all entries of B are in $Z(\mathbb{S})$. Hence $e_r \mathbf{x}_r = \sum_{j=1}^k e_j f_j \mathbf{x}_j$ because $\mathbf{b}_i \circ \mathbf{b}_i' = (1, \dots, 1) \in \mathbb{S}^m$. That is, $\mathbf{x}_r = \sum_{j=1}^k e_r^{-1} f_j e_j \mathbf{x}_j$, equivalently, $\mathbf{x}_1, \dots, \mathbf{x}_k$ are spanning columns for all columns of X . \square

Lemma 2.5. *Let T be a (P, Q, B) -operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $sc(B) = 1$ and all elements of B are units. If \mathbb{S} is commutative, then T is a (U, V) -operator.*

Proof. Since T is a (P, Q, B) -operator, there exist permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$ such that $T(X) = P(X \circ B)Q$, or $m = n$ and $T(X) = P(X \circ B)^t Q$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. Since $sc(B) = 1$, there exists one column $\mathbf{b}_i = (b_{1,i}, \dots, b_{m,i})^t$ among the columns of B such that $B = \mathbf{b}_i \mathbf{e}$ with $\mathbf{e} = (e_1, \dots, e_{i-1}, 1, e_{i+1}, \dots, e_n)$. Since $b_{j,i}$ are units, e_j are invertible elements in \mathbb{S} for all $j = 1, \dots, n$. Let $D = \text{diag}(b_{1,i}, \dots, b_{m,i}) \in \mathbb{M}_m(\mathbb{S})$ and $E = \text{diag}(e_1, \dots, e_n) \in \mathbb{M}_n(\mathbb{S})$ be diagonal matrices. Since \mathbb{S} is commutative, it is straightforward to check that $X \circ B = D X E$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. For the case of $T(X) = P(X \circ B)Q$, if we let $U = P D$ and $V = E Q$, then $T(X) = U X V$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. If T is of the form $T(X) = P(X \circ B)^t Q$, then $U = P E$ and

$V = DQ$ shows that $T(X) = UX^tV$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. Thus the Lemma follows. \square

Let $X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be a matrix in $\mathbb{M}_{2,1}(\mathbb{Z}^+)$. Then we have that $sc(X) = 1$, but $sc(X^t) = 2$. Thus, in general, it is not true that for a matrix $X \in \mathbb{M}_{m,n}(\mathbb{S})$, $sc(X) = 1$ if and only if $sc(X^t) = 1$. But the following is obvious:

Lemma 2.6. *Let B be a matrix in $\mathbb{M}_{m,n}(\mathbb{S})$ and all elements of B are units in $\mathbb{Z}(\mathbb{S})$. Then $sc(B) = 1$ if and only if $sc(B^t) = 1$.*

Consider a matrix

$$(2.8) \quad \Xi = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in $\mathbb{M}_4(\mathbb{S})$. Then we can easily show that $sc(\Xi) = 4$ and $sc(\Xi^t) = 3$.

Below, we use the following notations in order to denote sets of matrices that arise as extremal cases in the inequalities (2.5) and (2.6):

$$\begin{aligned} \Gamma_1 &= \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid sc(X + Y) = 1\}; \\ \Gamma_n &= \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid sc(X + Y) = n\}; \\ \Gamma_R &= \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid sc(X + Y) = |\rho(X) - \rho(Y)|\}; \end{aligned}$$

In the following sections, we characterize linear operators that preserve the sets Γ_1 , Γ_n and Γ_R .

3. Linear preservers of Γ_1

In this section, we investigate the linear operators that preserve the set Γ_1 .

Theorem 3.1. *If T is a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$ that preserves Γ_1 , then T is a (P, Q, B) -operator, where $sc(B) = 1$ and all elements of B are units in $\mathbb{Z}(\mathbb{S})$. In particular, if \mathbb{S} is commutative, then T is a (U, V) -operator.*

Proof. Suppose T is a surjective linear operator that preserves Γ_1 . By Theorem 2.1, there exists a permutation σ on $\Delta_{m,n}$ and units $b_{i,j} \in \mathbb{Z}(\mathbb{S})$ such that $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i, j) \in \Delta_{m,n}$.

Assume that T does not preserve lines. Then, without loss of generality, we may assume that either $T(E_{1,1} + E_{1,2}) = b_{1,1}E_{1,1} + b_{1,2}E_{2,2}$ or $T(E_{1,1} + E_{2,1}) = b_{1,1}E_{1,1} + b_{2,1}E_{2,2}$. In either case, let $Y = O$ and X be either $E_{1,1} + E_{1,2}$ or $E_{1,1} + E_{2,1}$, so that $(X, Y) \in \Gamma_1$ while $(T(X), T(Y)) \notin \Gamma_1$, a contradiction. Thus T preserves lines. By Lemma 2.2, T is a (P, Q, B) -operator, where all elements of B are units in $\mathbb{Z}(\mathbb{S})$. Suppose $sc(B) \geq 2$. Then $(J, O) \in \Gamma_1$ and $sc(T(J)) = sc(B)$ or $sc(B^t)$ by Lemma 2.2, and hence by Lemma 2.6, $sc(T(J)) \geq 2$ so that $(T(J), O) \notin \Gamma_1$, a contradiction. Hence we have $sc(B) = 1$.

If \mathbb{S} is commutative, Lemma 2.5 shows that T is a (U, V) -operator. \square

In general, the converse of Theorem 3.1 may be true or not according to a given semiring \mathbb{S} . Obviously, by Lemma 2.4, all non-transposing (P, Q, B) -operators with $sc(B) = 1$ (all elements of B are units in $\mathbb{Z}(\mathbb{S})$) preserve Γ_1 . But the following example shows that transposing (P, Q, B) -operators may or not preserve Γ_1 according to given semirings

Example 3.2. (1) Consider the semiring \mathbb{Z}^+ of all nonnegative integers. Let $X = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \oplus O_{n-2} \in \mathbb{M}_n(\mathbb{Z}^+)$. Then we can easily show that $(X, O) \in \Gamma_1$, while $(X^t, O^t) \notin \Gamma_1$. So, the converse of Theorem 3.1 is not true in this case.

(2) Consider the binary Boolean semiring \mathbb{B} . Then it is straightforward that for a matrix $A \in \mathbb{M}_n(\mathbb{B})$, $sc(A) = 1$ if and only if all non-zero columns of A are the same. Thus all non-zero rows of A are the same and $sc(A^t) = 1$. That is, for any permutation matrices $P, Q \in \mathbb{M}_n(\mathbb{B})$, we have that $sc(A) = 1$ if and only if $sc(PA^tQ) = 1$. This shows that the converse of Theorem 3.1 is true in this case.

Over chain semiring, Theorem 3.1 can be generalized in the following way:

Theorem 3.3. *Let \mathbb{S} be a chain semiring. If T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$ that preserves Γ_1 , then followings are equivalent:*

- (1) T is surjective.
- (2) T strongly preserves Γ_1 .
- (3) T is a (P, Q) -operator.

Proof. Clearly, (3) implies that (1) and (2) are satisfied. Assume (1). By Theorem 3.1, T is a (P, Q, B) -operator, where $mc(B) = 1$ and all elements of B are units in $\mathbb{Z}(\mathbb{S})$. Since the only invertible element of a chain semiring is 1, it follows that $B = J$. Hence (3) is satisfied. Now, we suffice to show that (2) implies (3).

Assume (2). We want to show that there exists $\beta \in \mathbb{S}$ such that βT is surjective on $\mathbb{M}_{m,n}(\beta\mathbb{S})$. In order to show this, it suffice to check that for each pair of indices (i, j) there exist $Y \in \mathbb{M}_{m,n}(\mathbb{S})$ and $a \in \mathbb{S}$ such that $T(Y) = aE_{i,j}$. If this is not the case or if there is a cell whose image is not dominated by a cell, then there exists a $(0, 1)$ -matrix $N = [n_{i,j}]$ and a pair of indices (r, s) such that $n_{r,s} = 0$ and $T(N) \geq T(J)$. Let us show that there exists $\alpha \in \mathbb{M}_{m,n}(\mathbb{S})$ such that $T(\alpha J \setminus E_{r,s}) = T(\alpha J)$.

Let $T(N) = G = [g_{i,j}]$, $\alpha = \min\{g_{i,j} | g_{i,j} \neq 0\}$, and $T(J) = H = [h_{i,j}]$. Since N is a $(0, 1)$ -matrix, there exists a $(0, 1)$ -matrix M such that $J = N + M$. Thus $H = T(J) = T(N) + T(M) = G + T(M)$. Since \mathbb{S} is a chain semiring, $h_{i,j} \geq g_{i,j}$ for all (i, j) . By the choice of N , $T(N) \geq T(J)$. That is, $g_{i,j} = 0$ implies that $h_{i,j} = 0$. So, $\alpha h_{i,j} = \alpha = \alpha g_{i,j}$ by the definition of α . Thus, $T(\alpha N) = \alpha T(N) = \alpha T(J) = T(\alpha J)$. In the same way, it can be checked that if K is any $(0, 1)$ -matrix such that $N \leq K \leq J$, and $T(K) = R = [r_{i,j}]$, then $\alpha r_{i,j} = \alpha g_{i,j}$. Hence, $T(\alpha K) = T(\alpha J)$. Since $N \leq J \setminus E_{r,s} \leq J$, we have $T(\alpha J \setminus E_{r,s}) = T(\alpha J)$. Since $sc(\alpha J \setminus E_{r,s}) \neq 1$, we have $(\alpha J \setminus E_{r,s}, \alpha J \setminus$

$E_{r,s} \notin \Gamma_1$, while $(\alpha J, \alpha J) \in \Gamma_1$. Hence, $(T(\alpha J \setminus E_{r,s}), T(\alpha J \setminus E_{r,s})) \notin \Gamma_1$, while $(T(\alpha J), T(\alpha J)) \in \Gamma_1$, a contradiction with $T(\alpha J) = T(\alpha J \setminus E_{r,s})$. Thus there is no such a matrix N with a zero entry such that $T(N) \geq T(J)$. It follows that the image of a cell dominates only one cell and that for $\beta = \min\{h_{i,j} | T(J) = H = [h_{i,j}]\}$, βT is surjective on $\mathbb{M}_{m,n}(\beta\mathbb{S})$ and as above Theorem 3.1, is a (U, V) -operator since \mathbb{S} is a commutative chain semiring. That is, T is a (P, Q, B) -operator on $\mathbb{M}_{m,n}(\mathbb{S})$. Suppose that $B \neq J$. Then $b_{i,j} \neq 1$ for some (i, j) . Consider the matrix $X = E_{i,j} + b_{i,j}J$. Then $(X, X) \notin \Gamma_1$, while

$$\begin{aligned} T(X) &= T(E_{i,j}) + T(b_{i,j}J) = b_{i,j}T(E_{i,j}) + T(b_{i,j}J) \\ &= T(b_{i,j}E_{i,j} + b_{i,j}J) = T(b_{i,j}J), \end{aligned}$$

a contradiction to (2) since $(b_{i,j}J, b_{i,j}J) \in \Gamma_1$. Thus $B = J$ and T is a (P, Q) -operator. Hence (3) is satisfied. \square

4. Linear preservers of Γ_n

Recall that

$$\Gamma_n = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid sc(X + Y) = n\}.$$

Lemma 4.1. *Let $\mathbb{M}_{m,n}(\mathbb{S})$ have full spanning column rank, σ be a permutation of $\Delta_{m,n}$, and T be defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i, j) \in \Delta_{m,n}$, where all $b_{i,j}$ are units in $\mathbb{Z}(\mathbb{S})$. If T preserves Γ_n , then T preserves lines.*

Proof. Suppose T does not map lines to lines. Then there are two non-collinear cells E and F such that $T(E + F)$ is dominated by either R_i or C_j for some i and j .

If $T(E + F) \leq C_j$, we lose no generality in assuming that

$$T(E_{1,1} + E_{2,2}) = b_{1,1}E_{1,1} + b_{2,2}E_{2,1}.$$

If $n \leq m$, consider $A = E_{1,1} + E_{2,2} + \dots + E_{n,n}$. Then $T(A)$ has spanning column rank at most $n - 1$ since $b_{i,j}$ are invertible. That is, $(O, A) \in \Gamma_n$, while $(T(O), T(A)) \notin \Gamma_n$, a contradiction. Let us consider the case $m < n$. Since $\mathbb{M}_{m,n}(\mathbb{S})$ has full spanning column rank, there exists a matrix $X' \in \mathbb{M}_{m-2, n-2}(\mathbb{S})$ such that $sc(X') = n - 2$. Let us choose X' with the minimal number of non-zero entries. Let $X = O_2 \oplus X' \in \mathbb{M}_{m,n}(\mathbb{S})$. Thus $sc(X) = sc(X') = n - 2$. Hence $(E_{1,1} + E_{2,2}, X) \in \Gamma_n$. Since T preserves Γ_n , it follows that $(b_{1,1}E_{1,1} + b_{2,2}E_{2,1}, T(X)) \in \Gamma_n$, i.e., $sc(b_{1,1}E_{1,1} + b_{2,2}E_{2,1} + T(X)) = n$. Therefore $sc(T(X)[1, \dots, m | 3, \dots, n]) \geq n - 2$. Since the spanning column rank of any matrix cannot exceed the number of columns, $sc(T(X)[1, \dots, m | 3, \dots, n]) = n - 2$. Further, $|T(X)[1, \dots, m | 3, \dots, n]| < |X| = |X'|$ since T transforms cells to weighted cells and at least one cell has to be mapped into the 2nd column. Thus we can select an $(m - 2) \times (n - 2)$ submatrix of $T(X)[1, \dots, m | 3, \dots, n]$ whose spanning column rank is $n - 2$ and the number of whose nonzero entries are less than that of X' . This contradicts the choice of X' .

If $T(E + F) \leq R_i$, we may assume without loss of generality that $T(E_{1,1} + E_{2,2}) = b_{1,1}E_{1,1} + b_{2,2}E_{1,2}$. In this case, by considering the matrices $E_{1,1} + E_{2,2}$ and X chosen above, the result follows.

Thus, T preserves lines. □

Theorem 4.2. *Let T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $m \neq n$ or $m = n \geq 4$. If $\mathbb{M}_{m,n}(\mathbb{S})$ has full spanning column rank, then T preserves Γ_n if and only if T is a non-transposing (P, Q, B) -operator, where $sc(B) = 1$ and all elements of B are units in $\mathbb{Z}(\mathbb{S})$.*

Proof. Suppose T is a non-transposing (P, Q, B) -operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $sc(B) = 1$ and all elements $b_{i,j}$ of B are units in $\mathbb{Z}(\mathbb{S})$. Then it follows from Lemma 2.4 that T preserves Γ_n .

Conversely, assume that T preserves Γ_n . By applying Theorem 2.1 to Lemma 4.1, we have that T preserves lines. By Lemma 2.2, T is a (P, Q, B) -operator, where all elements of B are units in $\mathbb{Z}(\mathbb{S})$. Now, we will show that a transposing (P, Q, B) -operator does not preserve Γ_n . Suppose $m = n \geq 4$ and $T(X) = P(X \circ B)^t Q$ for some permutation matrices P and Q . Define an operator L on $\mathbb{M}_n(\mathbb{S})$ by $L(X) = P^t T(X) P = (X \circ B)^t Q P$. Since T preserves Γ_n if and only if L preserves Γ_n , we suffice to consider an operator L . Let $X = \Xi \oplus I_{n-4}$, where Ξ is the 4×4 matrix in (2.8). Then $sc(X) = n$ and $(X \circ B)^t$ has the 4th zero column. Thus, $(X, O) \in \Gamma_n$, while $(L(X), O) \notin \Gamma_n$ because $sc(L(X)) = sc((X \circ B)^t) \leq n - 1$. From this contradiction, we have established that T is a non-transposing (P, Q, B) -operator, where all elements of B are units in $\mathbb{Z}(\mathbb{S})$. That is, $T(X) = P(X \circ B)Q$.

It remains to show that $sc(B) = 1$. If not, we lose no generality in assuming that $B[1, 2|1, 2]$ has spanning column rank 2. Since $\mathbb{M}_{m,n}(\mathbb{S})$ has full spanning column rank, there exists a matrix $Y' \in \mathbb{M}_{m-2, n-2}(\mathbb{S})$ such that $sc(Y') = n - 2$. Consider matrices

$$X = \sum_{i=1}^2 (b_{i,1}^{-1} E_{i,1} + b_{i,2}^{-1} E_{i,2}) \quad \text{and} \quad Y = O_2 \oplus Y'$$

in $\mathbb{M}_{m,n}(\mathbb{S})$. Then $sc(X + Y) = n$ and hence $(X, Y) \in \Gamma_n$. But the first two columns of $(X + Y) \circ B$ are equal and hence it follows from Lemma 2.4 that

$$sc(T(X + Y)) = sc(P((X + Y) \circ B)Q) = sc((X + Y) \circ B) \leq n - 1,$$

that is, $(T(X), T(Y)) \notin \Gamma_n$, a contradiction. Thus, $sc(B) = 1$. □

Corollary 4.3. *Let T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $m \neq n$ or $m = n \geq 4$, and $\mathbb{M}_{m,n}(\mathbb{S})$ have full spanning column rank. If \mathbb{S} is commutative, and T preserves Γ_n then T is a non-transposing (U, V) -operator.*

Proof. Suppose T preserves Γ_n . By Theorem 4.2, T is a non-transposing (P, Q, B) -operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $sc(B) = 1$ and all elements $b_{i,j}$ of B are units in $\mathbb{Z}(\mathbb{S})$. Since \mathbb{S} is commutative, it follows from Lemma 2.5 that T is a non-transposing (U, V) -operator. □

5. Linear preservers of Γ_R

Recall that for $\mathbb{S} \subseteq \mathbb{R}^+$

$$\Gamma_R = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \mid sc(X + Y) = |\rho(X) - \rho(Y)|\}.$$

Lemma 5.1. *Let \mathbb{S} be any subsemiring of \mathbb{R}^+ , σ be a permutation of $\Delta_{m,n}$, and T be defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i, j) \in \Delta_{m,n}$, where all $b_{i,j}$ are units and $\min\{m, n\} \geq 3$. If T preserves Γ_R , then T preserves lines.*

Proof. Since the sum of three distinct weighted cells has spanning column rank at most 3, it follows that $\rho(T(E_{1,1} + E_{1,2} + E_{2,1})) \leq 3$. Now, for $X = E_{1,1} + E_{1,2} + E_{2,1}$ and $Y = E_{2,2}$, we have that $(X, Y) \in \Gamma_R$, and the image of Y is a single weighted cell, and hence $\rho(T(Y)) = 1$. Now, if $\rho(T(X)) = 3$, then $T(X + Y)$ must have spanning column rank 3 or 4, and hence $(T(X), T(Y)) \notin \Gamma_R$, a contradiction. If $\rho(T(X)) = 1$, clearly $(T(X), T(Y)) \notin \Gamma_R$ since $T(X + Y) \neq O$. Thus $\rho(T(X)) = 2$, and $sc(T(X + Y)) = 1$. However it is obvious that if a sum of four weighted cells has the spanning column rank 1, then they lie either in a line or in the intersection of two rows and two columns. The matrix $T(X + Y)$ is a sum of four weighted cells. These cells do not lie in a line since $\rho(T(X)) = 2$. Thus $T(X + Y)$ must be the sum of four weighted cells which lie in the intersection of two rows and two columns. Similarly, for any i, j, k, l , $T(E_{i,j} + E_{i,k} + E_{l,j} + E_{l,k})$ must lie in the intersection of two rows and two columns. It follows that any two rows must be mapped into two lines. By the bijectivity of T , if some pair of two rows is mapped into two rows (resp. columns), any pair of two rows is mapped into two rows (resp. columns). Similarly, if some pair of two columns is mapped into two rows (resp. columns), any pair of two columns is mapped into two rows (resp. columns).

Now, the image of three rows is contained in three lines, two of which are the image of two rows, thus, every row is mapped into a line. Similarly for columns. Thus, T preserves lines. □

Theorem 5.2. *Let \mathbb{S} be any subsemiring of \mathbb{R}^+ , $m \neq n$ or $m = n \geq 4$, and T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$. Then T preserves Γ_R if and only if T is a non-transposing (P, Q, B) -operator and $sc(B) = 1$.*

Proof. One can easily show that all non-transposing (P, Q, B) -operators preserve Γ_R .

Suppose T preserves Γ_R . By applying Theorem 2.1 to Lemma 5.1, T preserves lines. Lemma 2.2 implies that T is a (P, Q, B) -operator, where all elements of B are units. Now, we will show that if $m = n \geq 4$ and T is a transposing (P, Q, B) -operator, then T does not preserve Γ_R . Similar to the proof of Theorem 4.2, we suffice to consider an operator $X \rightarrow (X \circ B)^t Q P$, where $P, Q \in \mathbb{M}_n(\mathbb{S})$ are permutation matrices. Let $X = \Xi^t \oplus O_{n-4}$, where Ξ is the matrix in (2.8). Then we can easily show that

$$sc(X) = \rho(X) = \rho(X^t) = \rho((X \circ B)^t Q P) = 3$$

and

$$sc(X^t) = sc((X \circ B)^t QP) = 4.$$

Thus, $(X, O) \in \Gamma_R$, while $((X \circ B)^t QP, O) \notin \Gamma_R$. Therefore, T is a non-transposing (P, Q, B) -operator and hence $T(X) = P(X \circ B)Q$, where all elements of B are units.

Let us check that $sc(B) = 1$. Without loss of generality we assume that $n \leq m$. Let $Z = \sum_{k=n+1}^m \sum_{l=1}^n E_{k,l}$. Consider matrices

$$X = Z + \sum_{1 \leq j \leq i \leq n} E_{i,j} \quad \text{and} \quad Y = \sum_{1 \leq i < j \leq n} E_{i,j}.$$

Then we can easily show that

$$\rho(X) = n = \rho(T(X)) \quad \text{and} \quad \rho(Y) = n - 1 = \rho(T(Y)).$$

Thus, $sc(X + Y) = sc(J) = 1 = \rho(X) - \rho(Y)$, and hence $(X, Y) \in \Gamma_R$. Since T preserves Γ_R , it follows from Lemma 2.4 that

$$\begin{aligned} sc(B) &= sc(P(J \circ B)Q) = sc(P((X \circ B) + (Y \circ B))Q) \\ &= sc(T(X) + T(Y)) = \rho(T(X)) - \rho(T(Y)) = 1. \end{aligned}$$

Thus the theorem holds. □

References

- [1] L. B. Beasley and A. E. Guterman, *Rank inequalities over semirings*, J. Korean Math. Soc. **42** (2005), no. 2, 223–241.
- [2] ———, *Linear preservers of extremes of rank inequalities over semirings: Factor rank*, J. Math. Sci. (N. Y.) **131** (2005), 5919–5938.
- [3] L. B. Beasley, S. G. Lee, and S. Z. Song, *Linear operators that preserve pairs of matrices which satisfy extreme rank properties*, Linear Algebra Appl. **350** (2002), 263–272.
- [4] L. B. Beasley and N. J. Pullman, *Semiring rank versus column rank*, Linear Algebra Appl. **101** (1988), 33–48.
- [5] G. Marsaglia and George P. H. Styan, *Equalities and inequalities for ranks of matrices*, Linear and Multilinear Algebra **2** (1974/75), 269–292.
- [6] S. Pierce and others, *A Survey of Linear Preserver Problems*, Linear and Multilinear Algebra **33** (1992), no. 1-2, 1–129.
- [7] S. Z. Song and S. G. Hwang, *Spanning column ranks and their preservers of nonnegative matrices*, Linear Algebra Appl. **254** (1997), 485–495.
- [8] S. Z. Song and K. T. Kang, *Linear maps that preserve commuting pairs of matrices over general Boolean algebra*, J. Korean Math. Soc. **43** (2006), no. 1, 77–86.

DEPARTMENT OF MATHEMATICS
AND RESEARCH INSTITUTE FOR BASIC SCIENCES
CHEJU NATIONAL UNIVERSITY
JEJU 690-756, KOREA
E-mail address: szsong@cheju.ac.kr