Upper Bounds for the Performance of Turbo-Like Codes and Low Density Parity Check Codes

Kyuhyuk Chung and Jun Heo

Abstract: **Researchers have investigated many upper bound techniques applicable to error probabilities on the maximum likelihood (ML) decoding performance of turbo-like codes and low density parity check (LDPC) codes in recent years for a long codeword block size. This is because it is trivial for a short codeword block size. Previous research efforts, such as the simple bound technique [20] recently proposed, developed upper bounds for LDPC codes and turbo-like codes using ensemble codes or the uniformly interleaved assumption. This assumption bounds the performance averaged over all ensemble codes or all interleavers. Another previous research effort [21] obtained the upper bound of turbo-like code with a particular interleaver using a truncated union bound which requires information of the minimum Hamming distance and the number of codewords with the minimum Hamming distance. However, it gives the reliable bound only in the region of the error floor where the minimum Hamming distance is dominant, i.e., in the region of high signal-to-noise ratios. Therefore, currently an upper bound on ML decoding performance for turbo-like code with a particular interleaver and LDPC code with a particular parity check matrix cannot be calculated because of heavy complexity so that only average bounds for ensemble codes can be obtained using a uniform interleaver assumption. In this paper, we propose a new bound technique on ML decoding performance for turbo-like code with a particular interleaver and LDPC code with a particular parity check matrix using ML estimated weight distributions and we also show that the practical iterative decoding performance is approximately suboptimal in ML sense because the simulation performance of iterative decoding is worse than the proposed upper bound and no wonder, even worse than ML decoding performance. In order to show this point, we compare the simulation results with the proposed upper bound and previous bounds. The proposed bound technique is based on the simple bound with an approximate weight distribution including several exact smallest distance terms, not with the ensemble distribution or the uniform interleaver assumption. This technique also shows a tighter upper bound than any other previous bound techniques for turbo-like code with a particular interleaver and LDPC code with a particular parity check matrix.**

Index Terms: **Low-density parity-check (LDPC) codes, maximum likelihood (ML) decoding, turbo-like codes, weight distributions.**

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I. INTRODUCTION

Iterative decoding [1]–[6] for low-density parity-check (LDPC) codes and concatenated codes with interleavers represents a great advancement in communications theory because of their excellent performance. Parallel concatenated convolutional codes (PCCCs), also called turbo-like codes, and serial concatenated convolutional codes (SCCCs) with interleavers, introduced in [3], [4], [7]–[9], consist of simple binary convolutional codes connected through an interleaver in a parallel or in a serial manner. On the other hand, LDPC codes [1], [2] are very well liked because of their excellent performance and advantage for efficient parallel hardware implementation. Numerous simulations and bounds have demonstrated their remarkable performance [10]–[18].

In [19], the transfer function bounding techniques were applied to obtain the upper bounds on the bit-error probabilities and the word-error probabilities for maximum likelihood (ML) decoding of turbo codes. Since the transfer function bound is developed as a random coding bound with the uniformly interleaved assumption, it cannot be used to bound the performance of the turbo code with a particular interleaver and that of the LDPC code with a particular parity check matrix.

Moreover, the union bound cannot predict performance above the cutoff rate. There is a great demand to have bounds on performance that are useful for rates above the cutoff rate. Recently a proposed simple bound technique [20] showed a tight upper bound of repeat accumulate codes and LDPC codes by using an ensemble input-output weight distribution based on a uniform interleaver assumption above the cutoff rate. A truncated union bound [21] has been proposed in order to obtain the upper bound of the turbo-like code with a particular interleaver. This truncated union bound [21] uses information of the minimum Hamming distance and the number of codewords with the minimum Hamming distance from the particular code structures. However, it gives the reliable bound only in the region of the error floor where the minimum Hamming distance is dominant, i.e, in the region of high signal-to-noise ratios (SNRs).

In this paper, we propose a new bound technique for turbo-like code with a particular interleaver and LDPC code with a particular parity check matrix. The proposed bound technique is based on the simple bound with an approximate weight distribution including several exact smallest non-zero distance terms [21]–[23], not with the ensemble distribution or the uniform interleaver assumption. This bound can be used to predict ML decoding performance of turbo-like code with a particular interleaver and LDPC code with a particular parity check matrix without Monte-Carlo simulation suboptimal iterative decoding [1], [3], which is currently worse than the proposed upper bound and no wonder, even worse than ML decoding perfor-

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mance. The remainder of this paper is organized as follows. In Section II, we describe previous bounding techniques and compare transfer function bounds, simple bounds, and simulation results. In Section III, we present upper bounds for turbo-like code with a particular interleaver and LDPC code with a particular parity check matrix. In Section IV, we conclude the paper.

II. A REVIEW AND COMPARISON OF PREVIOUS BOUNDS

A. Transfer Function Bounds

For an ML decoder, a union bound on the probability of word error and bit error over an additive white Gaussian noise channel requires an input-output weight distribution. For the overall (N, K) code C , where N is the length of a codeword, K is the length of the information bits, and the code rate is $r \triangleq K/N$, $A_{w,d}$ denotes the number of codewords for an input sequence weight w and output codeword weight d. Then, the conditional probability of producing a codeword of weight d, given an input sequence of weight w, is

$$p(d|w) = \frac{A_{w,d}}{\sum_{d'} A_{w,d'}} = \frac{A_{w,d}}{\binom{K}{w}}. \quad (1)$$

where $(\cdot)$ is the number of combinations of size k from size n. The conditional probability distribution $p(d|w)$ for turbo codes is obtained by using the uniformly interleaved assumption [19].

The conditional probability that an ML decoder will choose a codeword of total weight d to the all-zero codeword is $Q(\sqrt{2dE_s/N_0})$. $Q(\cdot)$ is the complementary unit variance Gaussian distribution function. E_s is the energy per signal and N_0 is the one-sided noise spectral density and in turn E_s/N_0 is the channel symbol SNR. Then, the bit error probability P_b is upper bounded [19]:

$$P_b \le \sum_{w=1}^{K} \frac{w}{K}\binom{K}{w} \sum_{d} p(d|w) \left\{ Q\left(\sqrt{\frac{2dE_s}{N_0}}\right)\right\}. \quad (2)$$

The divergence properties of the transfer function bounds for turbo codes are observed above the cutoff rate [19] which is shown in Fig. 1.

B. A Simple Tight Bound

The performance of turbo-like codes is close to Shannon's channel capacity limit for moderate to large block sizes, so there is a need for bounds on performance that are useful for rates above the cutoff rate. In [20] such a simple bound on the probability of decoding error for block codes above the cutoff rate is derived in a closed form. This bound is simple because it does not require any integration or optimization in its final version. Consider a linear binary (N, K) block code C, where N is the codeword length and K the information frame length.

For a given code, d is the Hamming weight of a codeword. The upper bound [20] on the bit error rate (BER) with ML codeword decoding is given by

$$P_b \le \sum_{d=d_{min}}^{N-K+1} \min\left\{e^{-NE(c,d)}, e^{Ng(\delta)} Q\left(\sqrt{2cd}\right)\right\} \quad (3)$$

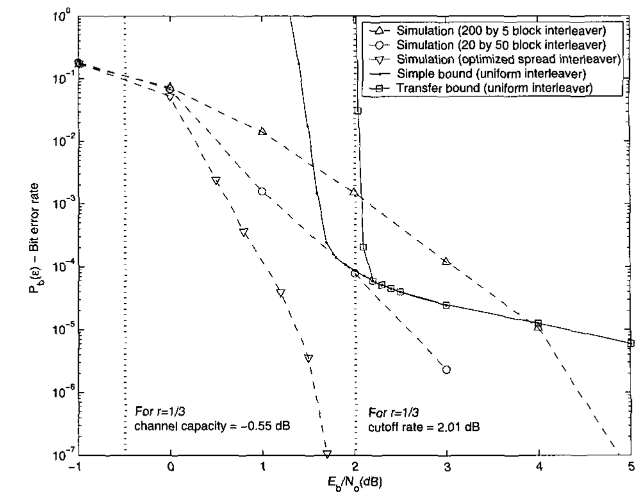

Fig. 1. Upper bounds with uniform interleavers and simulations with various interleavers for turbo codes ($r = 1/3, K = 1,000$).

where d_{min} is the minimum distance of the code and

$$E(c,d) \triangleq \begin{cases} \frac{1}{2}\ln[1 - 2c_0(\delta)f(c,\delta)] + \frac{c\,f(c,\delta)}{1+f(c,\delta)}, & \text{if } c_0(\delta) < c < \frac{e^{2g(\delta)}-1}{2\delta(2-\delta)}, \\ -g(\delta) + \delta c, & \text{otherwise} \end{cases} \quad (4)$$

where $\delta \triangleq d/N$, $c \triangleq r(E_b/N_0)$ with E_b being the energy per information bit, and

$$c_0(\delta) \triangleq \left(1 - e^{-2g(\delta)}\right)\frac{1-\delta}{2\delta}, \quad (5)$$

$$f(c,\delta) \triangleq \sqrt{\frac{c}{c_0(\delta)} + 2c + c^2} - c - 1, \quad (6)$$

$$g(\delta) \triangleq \frac{1}{N}\ln\left\{\sum_{w} \frac{w}{K} A_{w,d}\right\} \quad (7)$$

where $A_{w,d}$ is the input-output weight distribution defined in Section II-A. For the codeword error rate, (7) is changed differently as in [20]. In order to apply this simple bound to a particular code, the input-output weight distribution $A_{w,d}$ should be obtained for that particular code, which is usually very complicated.

C. Comparison of Transfer Function Bounds, Simple Bounds, and Simulation Results

Currently, an upper bound on ML decoding performance for turbo-like code with a particular interleaver and LDPC code with a particular parity check matrix cannot be calculated because of heavy complexity so that only average bounds for ensemble codes can be obtained using a uniform interleaver assumption. Fig. 1 shows comparisons of the average bounds (a transfer function bound and a simple bound) with the practical iterative decoding performance curves for turbo codes with various interleavers. The practical iterative decoding performance for turbo codes is known as suboptimal in ML sense. The curves for upper bounds and simulations on the waterfall region, not on the error floor region, approach the channel capacity and cutoff rate which are below the channel capacity as the performance of the code improves. It is meaningful to compare the upper bounds and simulations to the ultimate bound, i.e., the channel capacity because turbo codes are known to be a near channel-capacity achieving code. The turbo code uses constituent convolutional codes with the generator polynomial $(1+D^2)/(1+D+D^2)$. The rate of the turbo code is $1/3$. The three simulation results are obtained from three different interleavers with length $K = 1,000$. The first is an optimized spread interleaver [24], having the best performance among the three interleavers. The second is a block interleaver which reads bits in a 20 by 50 rectangular array row-wise and reads out column-wise. The third is a block interleaver with 200 by 5 rectangular array. The transfer function bound and the simple bound use the uniform interleaver with which only a single bound is obtained for the three different codes, i.e., one single transfer bound and one single simple bound, respectively. However, simulation results show that performance depends on the particular interleaver, that is, there are three different performance curves shown in Fig. 1.

Note that some performance curves of the iterative decoder can be worse than the upper bound on ML decoding performance because iterative detection is suboptimal.

Fig. 1 also shows a comparison of the simple bound with the transfer function bound. We observe that the simple bound is tighter than the transfer function bound at the low range of SNRs. At a BER of 10^{-1} the simple bound is about 0.7 dB tighter than the transfer function bound. It is well known that the transfer function bound diverges above the cutoff rate. The cutoff rate corresponds to $E_b/N_0 = 2.01$ dB for rate $1/3$ codes.

D. Truncated Union Bounds

A free distance d_{free} originally is a minimal Hamming distance between different encoded sequences of a convolutional code. Since a convolutional code does not use blocks, processing instead a continuous bitstream, the value of d_{free} applies to a quantity of errors located near to each other. However, practically, a block size is used for most systems using a convolutional code. Then, with a fixed block size and a proper trellis termination scheme d_{free} can be understood as the minimum distance, such as in PCCCs or turbo codes. For a linear binary code $C(N, K)$ with free distance d_{free}, we will denote by N_{free} its multiplicity (the number of codewords with weight d_{free}), and by w_{free} its information bit multiplicity (defined as the sum of the Hamming weights of the N_{free} information frames generating the codewords with weight d_{free}). For very high values of E_b/N_0, we can write the bit error rate P_b

$$P_b \simeq \frac{w_{free}}{K} Q\left(\sqrt{\frac{2E_b}{N_0}\frac{K}{N}d_{free}}\right). \quad (8)$$

For turbo-like codes, a better approximation can be obtained by including the other smallest terms of the distance spectrum [21].

By the symbol UB(j), we will denote the union bound expression, truncated to the contribute of the jth nonzero distance,

$$\text{UB}(j) = \sum_{i=1}^{j} \frac{w(i)}{K} Q\left(\sqrt{\frac{2E_b}{N_0}\frac{K}{N}d(i)}\right) \quad (9)$$

where $d(i)$ is the ith nonzero distance and $N(i)$ is the number of codewords with weight $d(i)$. The term $w(i)$ is defined as the sum of the Hamming weights of the $N(i)$ information frames generating the codewords with weight $d(i)$. In [21], branch and bound algorithms [25] for finding several smallest distances and their multiplicities were developed allowing performance of turbo codes and SCCCs to be approximated by truncated union bounds at high SNRs. Since these algorithms are based on the branch-and-bound method [25], complexity for finding the whole weight distribution is intractable.

III. UPPER BOUNDS FOR THE TURBO-LIKE CODE WITH A PARTICULAR INTERLEAVER AND THE LDPC CODE WITH A PARTICULAR PARITY CHECK MATRIX

The simple bound is the tightest closed-form upper bound on the decoding error rate [20]. We use the simple bound for the turbo-like code with a particular interleaver and the LDPC code with a particular parity check matrix. To use the simple bound, we need the conditional probability $p(d|w)$ that is defined in (1). But it is intractable to obtain $p(d|w)$ because of complexity. Thus, we want to obtain the ML estimator $\widehat{p}_{\text{MLE}}$ of $p(d|w)$. Since for a given input sequence weight w and output codeword weight d, we want to know $A_{w,d}$ in order to obtain the probability of picking one of the $A_{w,d}$ codewords among the $\binom{K}{w}$ codewords, this problem is the same as estimating the probability of picking one of white balls, when a ball is drawn with replacement from an urn that contains $A_{w,d}$ white balls and $[\binom{K}{w} - A_{w,d}]$ black balls. The indicator function $I_{w,d}(c)$ is defined by

$$I_{w,d}(c) = \begin{cases} 1, & \text{if } c \text{ is a codeword with } d \text{ given } w, \\ 0, & \text{otherwise} \end{cases} \quad (10)$$

where w is an input sequence weight and d is an output codeword weight. Then, $I_{w,d}(c)$ is the Bernoulli random variable with $p = p(d|w)$. If we let k be the number of codewords with the Hamming weight d for input sequences of the Hamming weight w among N_s generated sample codewords, k is the sum of the Bernoulli random variables associated with each of the N_s independent trials. Then, k is the binomial random variable with the following probability mass function

$$P(k|p) = \binom{N_s}{k} p^k (1-p)^{N_s-k} \quad (11)$$

for $k = 0, 1, \cdots, N_s$. In order to obtain the ML estimator $\widehat{p}_{\text{MLE}}$, we maximize the likelihood function $P(k|p)$

$$\widehat{p}_{\text{MLE}} = \max_p P(k|p) = \max_p \binom{N_s}{k} p^k (1-p)^{N_s-k}. \quad (12)$$

Differentiating the argument and setting the result equal to 0 give the solution

$$\widehat{p}_{\text{MLE}} = \frac{k}{N_s}. \quad (13)$$

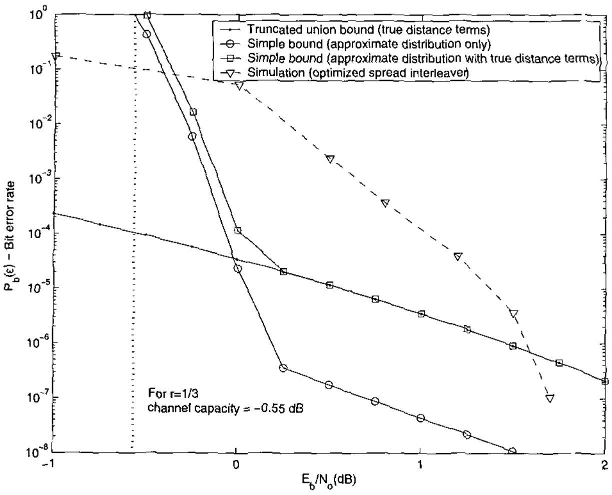

Fig. 2. Comparison of upper bounds of the several exact smallest distance terms, the approximate distribution only, and the approximate distribution with the several exact smallest distance terms (turbo code with optimized spread interleaver, ($r = 1/3,\ K = 1,000$)).

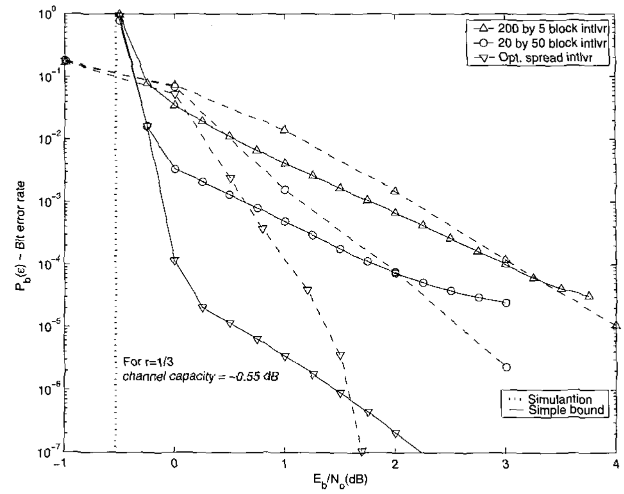

Fig. 3. Simple bounds using approximate distributions with the several exact smallest distance terms and iterative decoding simulation results for turbo codes with various interleavers ($r = 1/3,\ K = 1,000$).

It is also straightforward to verify that this solution is the global maximum. We approximate $p(d|w)$ by $\widehat{p}_{\text{MLE}}$. Then,

$$\widehat{A}_{w,d} = \binom{K}{w}\frac{k}{N_s} \tag{14}$$

where $\widehat{A}_{w,d}$ is the estimated weight distribution.

We choose $N_s = 10,000$ because the thresholds of the simple bounds for $N_s = 1,000,\ 10,000$, and $100,000$ were found in simulations to be similar for this code. In order to obtain $\widehat{A}_{w,d}$, we generate $N_s = 10,000$ codewords randomly for each input weight w, calculate k from the generated codewords, and obtain $\widehat{A}_{w,d}$ using (14). The several exact smallest distances $\{A_{w,d}\}_{d=d_{min}}^{d_{min}+10}$ are included in $\widehat{A}_{w,d}$ using the algorithm in [21] to approximate error floor region better at medium to high SNRs because for very high values of E_b/N_0 these several smallest distance terms are dominant, especially up to second or third terms. Thus, we included terms up to $d_{min} + 10$, conservatively, if we could calculate them.

In Figs. 2 and 3, the optimized spread interleaver of a turbo code is obtained by maximizing the interleaver bit locations [24]. In Fig. 2, we compare three cases, i.e.,

- when the several exact smallest distance terms are only considered,
- the approximate distributions only,
- the approximate distributions with the several exact smallest distance terms.

The practical iterative simulation decoding performances are approximately suboptimal in ML sense at the medium range of SNRs because the simulation performance of iterative decoding is worse in the medium SNR region than the proposed bound and no wonder, even worse than ML decoding performance. In order to show this point, we compare the simulation result with the proposed bound and other bounds.

In Fig. 3, simple bounds using approximate distributions with the several exact smallest distance terms are compared with iterative decoding simulation results for turbo codes with three different interleavers. We obtain three different accurate upper

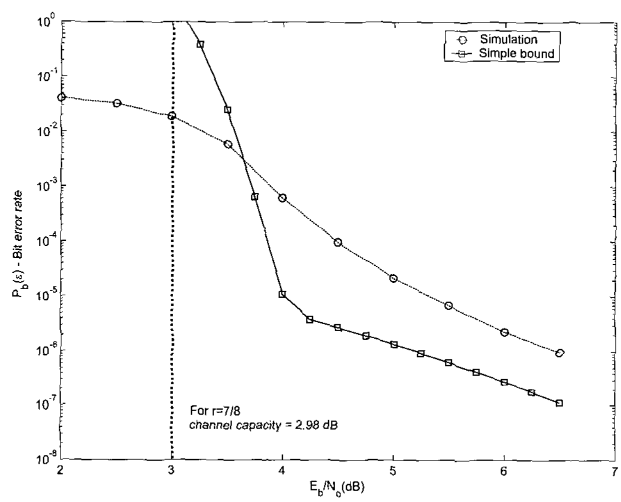

Fig. 4. Simple bound using an approximate distribution with the several exact distance term and iterative decoding simulation result for LDPC code with ($r = 7/8, (N, K) = (800, 700)$).

bounds for each interleaver. Again as in Fig. 2, the practical iterative simulation decoding performance is approximately suboptimal in ML sense at the medium range of SNRs because the simulation performance of the iterative decoding is worse in the medium SNR region than the proposed bound and even worse than the ML decoding performance. To clarify this point, we compare the simulation results with the proposed bounds.

As another application, we consider $(N, K) = (800, 700)$ regular LDPC codes. The rate for this LDPC code is $7/8$. In Fig. 4, the simple bound with both the exact minimum distance term of $d_{min} = 2$ and an approximate weight distribution is compared with the iterative decoding simulation performance at fixed iterations of 50 for rates $r = 7/8$. The minimum distance of the LDPC code is calculated using the algorithm in [23]. Note that a particular LDPC code is taken into account for both the codewords with the minimum distance and approximate weight distributions. In Fig. 4, the practical iterative simulation decoding performance is approximately suboptimal in ML sense at medium to high range of SNRs because the simulation performance of the iterative decoding is worse in the medium to high SNR region than the proposed bound and even worse than ML decoding performance. In order to emphasize this point, we compare the simulation result with the proposed bound.

We also observe in Figs. 3 and 4 that at the low range of SNRs the thresholds of the simple bound are approximately the channel capacity of rate $r = 1/3$, i.e., -0.55 dB and rate $r = 7/8$, i.e., 2.98 dB. Further research, however, will be required to support and prove such reasoning.

IV. CONCLUSION

We have presented a new bounding technique on ML decoding performance which is useful for turbo-like code with a particular interleaver and LDPC code with a particular parity check matrix. Unlike most of the previous bound techniques, this proposed bound does not use ensemble codes or the uniformly interleaved assumption, which bound the performance averaged over all ensemble codes or all interleavers. The proposed upper bound is based on the simple bound with estimated weight distributions including the several exact smallest distance terms. The proposed bound gives the good bounding information both on the water fall region and on the error floor region. This bound can be used to predict ML decoding performance of turbo-like code with a particular interleaver and LDPC code with a particular parity check matrix without Monte-Carlo iterative decoding simulation. This decoding is currently worse in some SNR region than the proposed upper bound and no wonder, it is even worse than ML decoding performance.

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