Frequency-Domain Balanced Stochastic Truncation for Continuous and Discrete Time Systems

Hamid Reza Shaker

Abstract: A new method for relative error continuous and discrete time model order reduction is proposed. The reduction technique is based on two recently developed methods, namely frequency domain balanced truncation within a frequency bound and inner-outer factorization techniques. The proposed method is of interest for practical model order reduction because in this context it shows to keep the accuracy of the approximation as high as possible without sacrificing the computational efficiency. Numerical results show the accuracy and efficiency enhancement of the method.

Keywords: Frequency-domain reduction, model reduction, stochastic balanced structure.

1. INTRODUCTION

Over the past two decades, model reduction methods have become increasingly popular [1,12,21]. Such methods are designed to extract a reduced order state-space model that adequately describes the behavior of the system to study.

A low-order model for a large scale system brings us an easy implementation. As opposed to a high-order model that might require expensive or complicated hardware; the low-order model has less complicated and more easily available hardware. Further more, in the high order systems the analysis problems can not be solved within a reasonable time and cost. So in many cases, it is advisable to construct a reduced order model that approximates the physical behavior of the original system.

The reduction techniques are divided into two broad categories, namely singular value decomposition (SVD) based methods and moment matching based techniques. The first category consists of the methods like balanced truncation that preserves stability and has an upper bound for approximation error. Moment matching based methods like Krylov subspace method can be implemented iteratively, which leads to numerical efficient algorithms, but these do not automatically preserve stability and have no error bound [1,21]. Some of the proposed reduction methods are trying to reduce the absolute error and some others are trying to reduce the relative error as a measure for the approximation accuracy. The balanced stochastic truncation (BST) approach belongs to the family of relative error methods [17]. In contrast to absolute error methods like the balanced truncation (BT) or the singular perturbation approximation (SPA) method, the BST method has the main advantage in provision of a uniform approximation of the frequency response of the original system over the whole frequency domain, and particularly, in preservation of phase information [2]. For example, for a minimum-phase original system, the BST-approximation is also minimum-phase. However this is not generally true for the absolute error methods. In practical control engineering viewpoint usually a system is operating within a frequency bound and the system can be shut down outside of this frequency bound. Since we do not have to keep the approximation in permissible outside range of the operational bandwidth of the system, the accuracy can be increased if we confine the approximation within a frequency bound.

Bases on this idea the frequency domain balanced truncation within a frequency bound (FDBT) is proposed [3-8,13-16,20]. In this paper we proposed a new method for relative error model reduction which is based on BST and FDBT approaches and we call the proposed method by frequency domain balanced stochastic truncation (FBST). The proposed method is more accurate and more efficient than previous methods in the context of relative error model reduction like BST. The paper is organized as follows. In Section 2, we introduce some definitions, notations and concepts for BST. Section 3 consists of presenting FDBT algorithm and its properties. In Section 4, the FBST method based on some of the numerical recent algorithms like inner-outer factorization is presented.

Manuscript received July 29, 2006; revised July 24, 2007 and November 9, 2007; accepted January 22, 2008. Recommended by Editorial Board member Dong Hwan Kim under the direction of Editor Jae Weon Choi.

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In Section 5 the method is developed to discrete time systems. The method is applied to a practical CD player benchmark example in Section 6 and the results are shown. Finally in the last section some conclusions are given.

2. BALANCED STOCHASTIC TRUNCATION MODEL REDUCTION

Consider $G(s)$ a MIMO square transfer matrix with a minimal sate space realization $G:=(A,B,C,D)$ and of order n. If D is nonsingular it is possible to compute the left spectral factor $\psi(s)$ of $G(s)G^T(-s)$ satisfying:

$$\psi^T(-s)\psi(s)=G(s)G^T(-s). \tag{1}$$

The state space realization of G is called a balanced stochastic realization if:

$$W_c^G=W_o^\psi=diag(\sigma_1,\dots,\sigma_n), \tag{2}$$

where W_c^G is the controllability Gramian of $G(s)$, the matrix W_o^ψ is the observability Gramian of $\psi(s)$ and σ_i is the i^{th} Hankel singular value of the stable part of the so-called "phase matrix" $F(s)=(\psi^T(-s))^{-1}G(s)$. The singular values in (2) are ordered decreasingly [2,18,22].

We assume now that G is already stochastically balanced by an appropriate similarity transformation. The reduced model is obtained by eliminating the states related to the lowest set of singular values. The reduced model is stable and satisfies the relative error bound:

$$\left\|G^{-1}(G-G_r)\right\|_\infty \le \prod_{i=r+1}^{n}\frac{1+\sigma_i}{1-\sigma_i}-1. \tag{3}$$

This model reduction approach is called balanced stochastic truncation (BST) [2,19].

3. FREQUENCY-DOMAIN BALANCED TRUNCATION WITHIN A FREQUENCY BOUND

Consider the following n^{th} order state-space model representation of an asymptotic stable LTI system:

$$\begin{pmatrix}\dot{x}\\ y\end{pmatrix}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x\\ u\end{pmatrix}. \tag{4}$$

The problem is how to approximate the system with r^{th} order state-space model:

$$(A_r,B_r,C_r,D_r) \qquad \text{where} \quad r<n. \tag{5}$$

The approach which is commonly used and globally more accurate is the so-called Balanced Model Reduction introduced by Moore, for the first time [9]. In this method, the system is transformed to a basis where the states which are difficult to reach are simultaneously difficult to observe. Then, the reduced model is obtained simply by truncating the states which have this property. Considering the practical operation of the system within frequency bound and outside of it where some amount of inaccurate approximation is allowed, the accuracy can be improved if the balanced model reduction method is employed in the frequency band [3-8,13-16,20].

Controllability and observability Gramians in terms of w over a frequency bound $[w_1,w_2]$ are defined as following [3-8,20]:

$$\begin{aligned} W_{cf} &:= \frac{1}{2\pi}\int_{w_2}^{w_1}(Ijw-A)^{-1}BB^*(-Ijw-A^*)^{-1}dw,\\ W_{of} &:= \frac{1}{2\pi}\int_{w_2}^{w_1}(-Ijw-A^*)^{-1}C^*C(Ijw-A)^{-1}dw. \end{aligned} \tag{6}$$

Those are the solutions for the following Lyapunov equations:

$$\begin{aligned} AW_{cf}+W_{cf}A^* &= -(BB^*F^*+FBB^*),\\ A^*W_{of}+W_{of}A &= -(C^*CF+F^*C^*C), \end{aligned} \tag{7}$$

where F is defined by:

$$F=\int_{w_1}^{w_2}(Ijw-A)^{-1}dw \tag{8}$$

with an appropriate similarity transformation T and change of the basis, system realization in (1) can be transformed to a new balanced realization, so that the Gramians are equal and diagonal (in decreasing diagonal elements):

$$\bar{W}_{cf}=\bar{W}_{of}=diag(\sigma_1,\sigma_2,\dots,\sigma_n). \tag{9}$$

Here we have two important theorems that give us a physical interpretation for the reduction procedure [3-8,20].

Theorem 1: The frequency-domain controllability Gramian represents the energy flow of the system through each state variable within the frequency range $[w_1,w_2]$.

It means that if the unit white Gaussian noise test input signal $u(t)$, and state vector $x(t)$ of the system defined as follows:

$$u(jw)[u(jw)]^*=|u(jw)|^2=\begin{cases}1 & w_1\prec|w|\prec w_2\\ 0 & elsewhere,\end{cases}$$

$$x(t)=e^{At}Bu(t)\ \overset{\mathbb{F}}{\Leftrightarrow}(jwI-A)^{-1}Bu(jw)=X(jw).$$

The energy of the system through controllability Gramian is as follows:

$$E_x = \int_0^{\infty} x(t)x^*(t)dt$$
$$= \frac{1}{2\pi}\int_{w_1}^{w_2}(jwI - A)^{-1}Bu(jw)u^*(jw)$$
$$B^*(-jwI - A^*)^{-1}dw$$
$$= W_{cf}[w_1, w_2].$$

Theorem 2: The frequency-domain observability Gramian represents the energy flow of the system through each state variable within the frequency range $[w_1, w_2]$.

Consider a unit injected test signal x_0, where

$$x_0 x_0^* = \begin{cases} 1 & w_1 < w < w_2 \\ 0 & elsewhere. \end{cases}$$

Define output

$$y(t) \triangleq Ce^{At}x_0 \overset{F}{\Leftrightarrow} C(jwI - A)^{-1}x_0 = Y(jw).$$

The energy E_y of the system through observability Gramian is obtained by:

$$E_y = \int_0^{\infty} y(t)y^*(t)dt$$
$$= x_0^*\left[\frac{1}{2\pi}\int_{w_1}^{w_2}(-jwI - A^*)^{-1}C^*C(jwI - A)^{-1}dw\right]x_0$$

Now, x_0 being a white noise test signal, the result follows:

$$E_y = W_{cf}[w_1, w_2].$$

From the above theorems and (9), it is understood that for having a good approximation we should only truncate the states that are related to the lowest singular values in (9). This model reduction technique is called frequency domain balanced truncation within a frequency bound (FDBT). This method also preserves the stability and provides an error bound for absolute error.

4. FREQUENCY-DOMAIN STOCHASTIC BALANCED TRUNCATION

This section is the main part of this paper in which a new method for large scale model reduction is proposed. FBST keeps the advantages of BST within frequency bound and increases the accuracy of the approximation within a desired bounded frequency. In this model reduction method we do not have to involve the difficulties for solving Lyapanuv equations, we can implement numerical algorithm easily, approximating the integral to summation for finding Gramians under the definition (6). Numerical results in the next section show the accuracy and efficiency enhancement of the proposed method.

In FBST algorithm like BST at first we should find the left spectral factor $\psi(s)$ of $G(s)G^T(-s)$ satisfying (1). In order to compute the left spectral function we apply inner-outer factorization of [2,11] to factorize the state space realization $N = (A, W_c^G C^T + BD^T, -B^T(W_c^G)^{-1}, D^T)$ in the form $N_i(s)\psi(s)$ where $N_i(s)$ is the inner factor and $\psi(s)$ is the outer factor and the left spectral factor [2].

After the computation of the left spectral factor we change the state space representation by an appropriate similarity transform into stochastically balanced structure which we call that "frequency domain stochastic balanced realization". In the frequency domain stochastic realization the frequency domain controllability Gramian of $G(s)$ and the frequency domain observablity Gramian of the left spectral factor should be equal and diagonal and the diagonal elements should be in decreasing order:

$$W_{fc}^G = W_{fo}^{\psi} = diag(\sigma_1, ..., \sigma_n). \quad (10)$$

The reduced model is obtained by eliminating the states related to the lowest set of singular values. The reduced model is also stable and satisfies the relative error bound similar to (3):

$$\left\|G^{-1}(jw)(G(jw) - G_r(jw))\right\|_{\infty} \le \prod_{i=r+1}^{n}\frac{1+\sigma_i}{1-\sigma_i} - 1. \quad (11)$$

Inputs : system matrices (A,B,C,D) and the frequency ranges $[w_1,w_2]$
Outputs : reduced system matrices (A_r,B_r,C_r,D_r)

1- Form:
$N = (A, W_c^G C^T + BD^T, -B^T(W_c^G)^{-1}, D^T)$
2- Apply inner-outer factorization and find the left spectral factor $\psi(s)$
3- Compute the frequency domain controllability Gramian of the (A,B,C,D) system within a frequency bound $[w_1,w_2]$
4- Compute the frequency domain observability Gramian of the left spectral factor $\psi(s)$ within a frequency bound $[w_1,w_2]$
5- Find the similarity transformation T for stochastically balancing and balance the system stochastically.
6- Eliminate the states related to the lowest set of the singular values and find (A_r,B_r,C_r,D_r)

Fig. 1. FBST model reduction algorithm.

Fig. 1 shows the overall algorithm of FBST method.

The relation between FBST and BST reduction methods is exactly the same as the relation between balanced truncation and frequency domain balanced truncation methods. If we extend the frequency bound to $[-\infty,+\infty]$ in FBST, we achieve the same results as BST. FBST can be applied to all asymptotically stable dynamical systems which are square and nonsingular. Frequency domain stochastic balanced truncation in $[w_1,w_2]$ yields a uniformly good approximant over this frequency range instead of small absolute errors. Also it is possible to show that for minimal phase systems, frequency domain stochastic balanced truncation is the same as self-weighted frequency domain balanced truncation where the output weighting is given by $G^{-1}(s)$.

5. DISCRETE TIME FREQUENCY- DOMIAN STOCHATIC BALANCED TRUNCATION

For discrete time systems with transfer matrix $G(z)$ and state-space representation:

$$\begin{pmatrix} x[k+1] \\ y[k] \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x[k] \\ u[k] \end{pmatrix}, \qquad (12)$$

the frequency domain controllability and observabilty Gramians within $[w_1,w_2]$ are defined by:

$$W_{cf} := \frac{1}{2\pi}\int_{w_2}^{w_1} (Ie^{jw} - A)^{-1} BB^* (Ie^{-jw} - A^*)^{-1} dw,$$
$$W_{of} := \frac{1}{2\pi}\int_{w_2}^{w_1} (Ie^{-jw} - A^*)^{-1} C^* C (Ie^{jw} - A)^{-1} dw \qquad (13)$$

Inputs : system matrices (A,B,C,D) and the frequency ranges $[w_1,w_2]$ **Outputs :** reduced system matrices (A_r,B_r,C_r,D_r)
1- Form: $N = (A, W_c^G C^T + BD^T, -B^T (W_c^G)^{-1}, D^T)$ 2- Apply inner-outer factorization and find the left spectral factor $\psi(z)$ 3- Compute the frequency domain controllability Gramian of the (A,B,C,D) system within a frequency bound $[w_1,w_2]$ 4- Compute the frequency domain observability Gramian of the left spectral factor $\psi(z)$ within a frequency bound $[w_1,w_2]$ 5- Find the similarity transformation T for stochastically balancing and balance the system stochastically. 6- Eliminate the states related to the lowest set of the singular values and find (A_r,B_r,C_r,D_r)

Fig. 2. FBST model reduction algorithm.

[8], and $\psi(z)$ is the left spectral factor of $G(z)$ if [2]:

$$G(z)G^T(z^{-1}) = \psi^T(z^{-1})\psi(z) \qquad (14)$$

Discrete time FBST algorithm is similar to FBST algorithm for continuous case. The only differences are computation of the Gramians based on (11) instead of (6) and applying inner-outer factorization based on the algorithm in [10] instead of [11]. Fig. 2 shows the discrete time FBST model reduction algorithm.

6. PRACTICAL CD PLAYER BENCHMARK EXAMPLE

In this section we applied the proposed method to a strictly proper SISO practical CD player model of order 120 and compare it with BST method.

6.1. CD player benchmark example

One of the well-known practical applications of model order reduction is the control of CD player systems. The scheme of CD player mechanism is shown in Fig. 3. The control task is to achieve track following, which basically amounts to pointing the laser spot to the track of pits on the CD that is rotating. This mechanism consists of a swing arm on which a lens is mounted by means of two horizontal leaf springs. The rotation of the arm in the horizontal plane enable to read the spiral shaped disc-tracks, and the suspended lens is used to focus the spot on the disc. Due to the fact that the disc is not perfectly flat, and due to irregularities in the spiral of pits on the disc, a feedback system is needed. The higher the disc rate goes, the stronger are the demands on feedback controller. It is also required that the feedback system can sustain some level of external shocks. The challenge is to find a low-cost controller that can make the servo system faster and less sensitive to external shocks. In addition to this, it is required that all CD-players of a production set can be equipped with the same type of controller.

In practical point of view a high order model is needed to describe the vibrational behavior of the

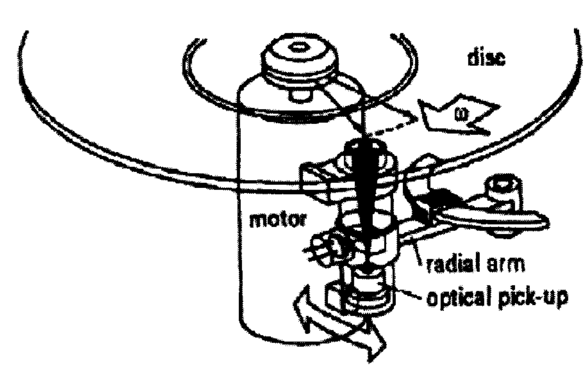

Fig. 3. CD Player.

electro-mechanical system over a large frequency range in order to anticipate the interaction with a controller of possible high-bandwidth. In many examples, the behavior of electrodynamics is predicted by means of finite element and substructuring method.

First the mechanism is divided in structural parts, and these components are modeled by means of finite-elements discretizations. The resulting model contains 60 vibration modes. In the other words the model that we want to design a low cost controller for that is of order 120 and we should reduce it.

6.2. Results and analysis

In this part the CD player model is reduced to 4th order model by applying both BST and FBST. Fig. 4 shows the Hankel singular values for FBST method related to the reduction frequency bound in Fig. 5. As we mentioned before, the Hankel singular values in balanced realization can give us useful information about the contribution of each state in input/output energy. In other words we can recognize the most important set of states to be preserved. Fig. 4 shows that the set of the first four states in frequency domain

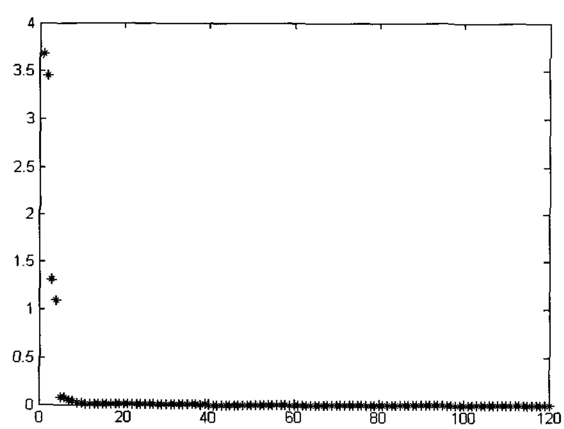

Fig. 4. FBST Hankel singular values.

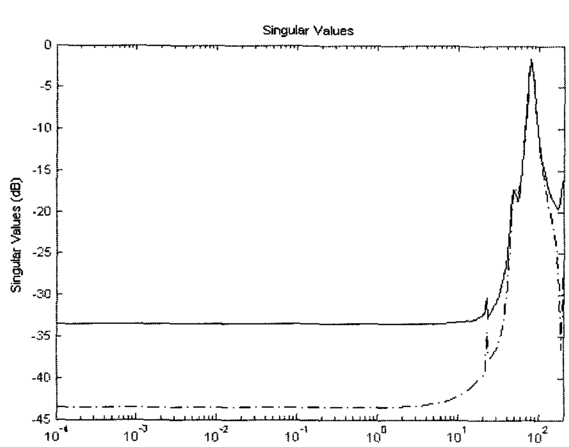

Fig. 5. FBST model reduction error (dash dotted) and BST model order reduction error (solid lines).

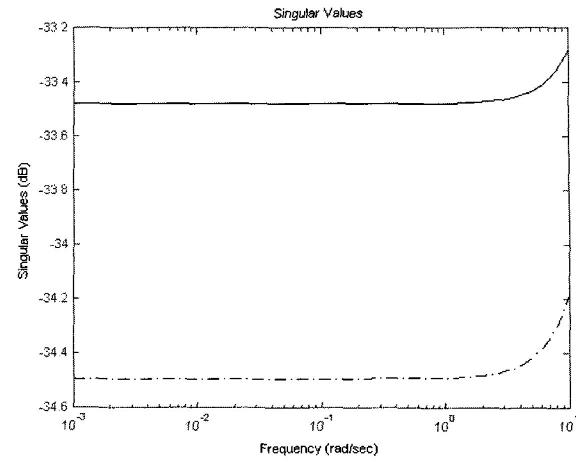

Fig. 6. FBST model reduction error (dash dotted) and BST model order reduction error (solid lines).

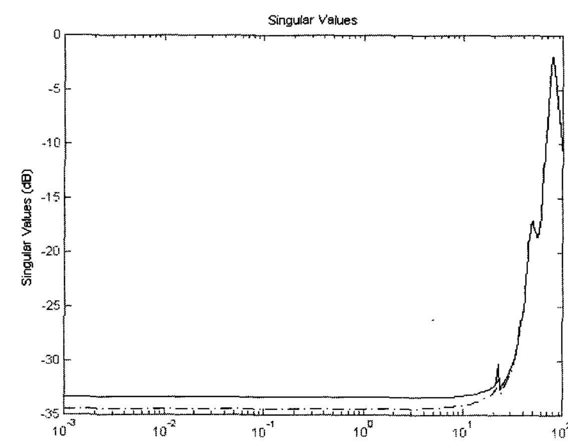

Fig. 7. FBST model reduction error (dash dotted) and BST model order reduction error (solid lines).

balanced realization has the most contribution in input/output energy. On the other hand from the gap between the first four singular values and the others, it is expected that the accuracy of approximation to 4th order model by truncation will be satisfactory. The infinity norm errors of BST and FBST are shown in Figs. 5-7 in different frequency bounds. As we can see in results, infinity norm approximation error in FBST method is less than approximation error by applying BST technique. The uniformity of the error in FBST is also satisfactory. The computational burden in FBST is less than BST because in FBST not only we do not have to solve Lyapunov equations for finding Gramians, but also we can easily compute frequency domain Gramian by approximating the integrals to summation in definitions.

7. CONCLUSIONS

In this paper, we have proposed a new model reduction method. The reduction method is based on stochastic balancing of a system within a frequency bound. Inner-Outer factorization is used in the numerical algorithm of the method as an accurate and efficient numerical approach.

The proposed method shows advantages in high accuracy, high efficiency, preservation of stability, and provision of error bounds which are suitable for the practical relative error model reduction.

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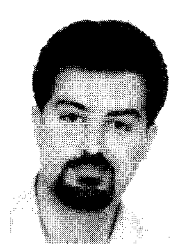

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