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CONJUGATE LOCI OF 2-STEP NILPOTENT LIE GROUPS SATISFYING $J_z^2 = \langle Sz, z\rangle A$

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Abstract. Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra which has an inner product $\langle\ ,\ \rangle$ and has an orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ for its center $\mathfrak{z}$ and the orthogonal complement $\mathfrak{v}$ of $\mathfrak{z}$. Then Each element z of $\mathfrak{z}$ defines a skew symmetric linear map $J_z : \mathfrak{v} \longrightarrow \mathfrak{v}$ given by $\langle J_z x, y\rangle = \langle z, [x, y]\rangle$ for all $x, y \in \mathfrak{v}$. In this paper we characterize Jacobi fields and calculate all conjugate points of a simply connected 2-step nilpotent Lie group N with its Lie algebra $\mathfrak{n}$ satisfying $J_z^2 = \langle Sz, z\rangle A$ for all $z \in \mathfrak{z}$, where S is a positive definite symmetric operator on $\mathfrak{z}$ and A is a negative definite symmetric operator on $\mathfrak{v}$.

1. Introduction

Let $\mathfrak{n}$ denote a finite dimensional Lie algebra over the real numbers. The Lie algebra $\mathfrak{n}$ is called 2-step nilpotent Lie algebra if $[x, [y, z]] = 0$ for any $x, y, z \in \mathfrak{n}$. A Lie group N is said to be 2-step nilpotent if its Lie algebra $\mathfrak{n}$ is 2-step nilpotent. Throughout, N will denote a simply connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ having center $\mathfrak{z}$. We shall use $\langle\ ,\ \rangle$ to denote either an inner product on $\mathfrak{n}$ or the induced left-invariant Riemannian metric tensor on N. Let $\mathfrak{v}$ denote the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$.

Each element z of $\mathfrak{z}$ defines a skew symmetric linear map $J_z : \mathfrak{v} \longrightarrow \mathfrak{v}$ given by $J_z(x) = (\mathrm{ad}x)^*(z)$ for all $x \in \mathfrak{v}$, where $(\mathrm{ad}x)^*(z)$ is the adjoint of $\mathrm{ad}x$ relative to the inner product $\langle\ ,\ \rangle$. More usefully J_z is defined by the equation

$$\langle J_z(x), y\rangle = \langle [x, y], z\rangle \tag{1.1}$$

for all $x, y \in \mathfrak{v}$. A 2-step nilpotent Lie group N with its Lie algebra $\mathfrak{n}$ is called *H-type* if it satisfies

$$J_z^2 = -\langle z, z\rangle I \text{ for all } z \in \mathfrak{z}.$$

The J-map was firstly introduced by A. Kaplan and used to study geometries of *H-type* groups [9, 10]. Also various aspects of *H-type* groups were investigated by Berndt, Tricerri, and Vanhecke [1]. The first general studies for 2-step nilpotent Lie groups were done by P. Eberlein [2, 3] and some related works

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followed. Especially, in 1997, Walschap [14] showed that for a nonsingular 2-step nilpotent Lie group with one dimensional center, the cut locus and the conjugate locus coincide, and he made an explicit determination of all first conjugate points in such a group. Gornet and Mast [4] showed that the first cut point of the starting point $\gamma(0)$ along a unit speed geodesic γ with initial velocity $\gamma'(0) = x_0 + z_0$ for $x_0 \in \mathfrak{v}$ and $z_0 \in \mathfrak{z}$ in a simply connected 2-step nilpotent Lie group N does not occur before length $\frac{2\pi}{\theta(z)}$, where $\theta(z)$ is the biggest of the norms of the eigenvalues of the skew-symmetric map J_z. Jang and Park later gave explicit formulas for all conjugate points along geodesics in any 2-step nilpotent Lie groups with one dimensional center [6]. And J. Kim [11] calculated all conjugate points of *H-type* groups. These last two works are generalized in a pseudo-Riemannian version by Jang, Parker, and Park [7, 8]. J. Lauret [13] introduced the notion of modified *H-type* by weakening the *H-type* condition.

Definition 1.1. A 2-step nilpotent Lie group $(N, \langle\ ,\ \rangle)$ is said to be a modified *H-type* group if for any nonzero $z \in \mathfrak{z}$

$$J_z^2 = \lambda(z)I \text{ for some } \lambda(z) < 0$$

or equivalently,

$$J_z^2 = -\langle Sz, z\rangle I \text{ for some positive definite symmetric operator } S \text{ on } \mathfrak{z}.$$

In [12], Y. Kim calculated all conjugate points along geodesics in a modified *H-type* group with two dimensional center.

More generally we can consider a class of 2-step nilpotent Lie groups $(N, \langle\ ,\ \rangle)$ satisfying the following condition

$$J_z^2 = \langle Sz, z\rangle A \text{ for all } z \in \mathfrak{z}, \tag{1.2}$$

where S is a positive definite symmetric operator on $\mathfrak{z}$ and A is a negative definite symmetric operator on $\mathfrak{v}$. Note that this class of 2-step nilpotent Lie groups contains all 2-step nilpotent groups with one dimensional center and all *H-type* groups, even all modified *H-type* groups. The definiteness of two operators A and S in (1.2) implies that we are in the nonsingular case, i.e., the map J_z has never zero eigenvalues. The main purpose of this paper is to characterize Jacobi fields and calculate all conjugate points and their multiplicities in a simply connected 2-step nilpotent Lie group N satisfying (1.2). Since N is endowed with a left invariant metric, we will only consider Jacobi fields and conjugate points along geodesics emanating from the identity element of N. In the remaining of this section we recall some facts about conjugate points. Also we will investigate some properties of simply connected 2-step nilpotent Lie groups satisfying (1.2) and state main results of this paper. In Section 2, we will give proofs of main results.

To study conjugate points, we use the Jacobi operator.

Definition 1.2. Along the geodesic γ, the *Jacobi operator* is given by

$$R_{\dot\gamma}\bullet = R(\bullet, \dot\gamma)\dot\gamma\,,$$

where R denotes the Riemannian curvature tensor.

For the reader's convenience, we recall that a *Jacobi field* along γ is a vector field along γ which is a solution of the *Jacobi equation*

$$\nabla_{\dot\gamma}^2 Y(t) + R_{\dot\gamma} Y(t) = 0$$

along γ, where ∇ denotes the Riemannian connection. The point $\gamma(t_0)$ is *conjugate* to the point $\gamma(0)$ if and only if there exists a nontrivial Jacobi field Y along γ such that $Y(0) = Y(t_0) = 0$. The multiplicity of $\gamma(t_0)$ is equal to the number of linearly independent of Jacobi fields $Y(t)$ with $Y(0) = Y(t_0) = 0$ and will be denoted by $\mathrm{mult}_{cp}(t_0)$. We will identify an element of $\mathfrak{n}$ with a left invariant vector field on N since T_eN may be identified with $\mathfrak{n}$, where e denotes the identity element of N.

For the reader's convenience, we provide the statement of Proposition 2.1 from [7].

Proposition 1.3. *Let γ be a geodesic in a simply connected 2-step nilpotent group N with $\gamma(0) = e$ and $\dot\gamma(0) = z_0 + x_0 \in \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{n}$. A vector field $Y(t) = z(t) + e^{tJ}u(t)$ along γ, where $z(t) \in \mathfrak{z}$ and $u(t) \in \mathfrak{v}$ for each t, is a Jacobi field if and only if*

$$\begin{aligned} \dot z(t) - [e^{tJ}u(t), e^{tJ}x_0] &= \zeta\,, \\ e^{tJ}\ddot u(t) + e^{tJ}J\dot u(t) - J_\zeta e^{tJ}x_0 &= 0\,, \end{aligned}$$

where $J = J_{z_0}$ and $\zeta \in \mathfrak{z}$ is a constant and $e^{tJ} = \sum_{n=0}^{\infty} \frac{t^n J^n}{n!}$.

The following example shows one way to construct 2-step nilpotent Lie groups satisfying (1.2) from a finite collection of *H-type* Lie algebras with same dimensional centers.

Example 1.4. Let $\{\mathfrak{n}_i | i = 1, \ldots, m\}$ be a finite collection of *H-type* Lie algebras with metrics $\langle\ ,\ \rangle_i$ and bracket operations $[\ ,\ ]_i$ and let $\mathfrak{n}_i = \mathfrak{z}_i \oplus \mathfrak{v}_i$ be their orthogonal decompositions, where $\mathfrak{z}_i$ and $\mathfrak{v}_i$ are centers and orthogonal complements, respectively for $i = 1, \ldots, m$. Assume that all dimensions of the centers $\mathfrak{z}_i$ are equal. Then without loss of generality we may assume that all $\mathfrak{z}_i$ are same space with the metric $\langle\ ,\ \rangle_1$ and denote it by $\mathfrak{z}$. Subsequently we have a new *H-type* algebra $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_m$ with center $\mathfrak{z}$ and its orthogonal complements $\mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_m$ by giving a new metric $\langle\ ,\ \rangle$

$$\langle z_1 + \sum_{i=1}^m x_i, z_2 + \sum_{i=1}^m y_i \rangle = \langle z_1, z_2\rangle_1 + \sum_{i=1}^m \langle x_i, y_i\rangle_i$$

for $z_1, z_2 \in \mathfrak{z}$ and $x_i, y_i \in \mathfrak{v}_i$, $i = 1, \ldots, m$ and a new bracket operation $[\ ,\ ]$

$$[\sum_{i=1}^m x_i, \sum_{i=1}^m y_i] = \sum_{i=1}^m [x_i, y_i]_i$$

for $x_i,\ y_i \in v_i,\ i=1,\ldots,m$. For a positive definite symmetric operator S on $\mathfrak{z}$ and positive distinct reals $\lambda_1,\ldots,\lambda_m$, we now give a new metric $\langle\langle\ ,\ \rangle\rangle$ on $\mathfrak{n}$ by

$$\langle\langle z_1+\sum_{i=1}^m x_i, z_2+\sum_{i=1}^m y_i\rangle\rangle = \langle Sz_1, z_2\rangle + \sum_{i=1}^m \langle \frac{1}{\lambda_i}x_i, y_i\rangle,$$

where $x_i, y_i \in v_i,\ z_1, z_2 \in \mathfrak{z}$ and $\langle\ ,\ \rangle$ denotes the *H-type* metric on $\mathfrak{n}$. Let J_z^* and J_z be as in (1.1) for $\langle\langle\ ,\ \rangle\rangle$ and $\langle\ ,\ \rangle$ respectively. Then we have

$$\begin{aligned}
\langle\langle J_z^*\sum_{i=1}^m x_i, \sum_{i=1}^m y_i\rangle\rangle &= \langle\langle z, [\sum_{i=1}^m x_i, \sum_{i=1}^m y_i]\rangle\rangle = \langle\langle z, \sum_{i=1}^m [x_i, y_i]\rangle\rangle \\
&= \sum_{i=1}^m \langle\langle z, [x_i, y_i]\rangle\rangle = \sum_{i=1}^m \langle Sz, [x_i, y_i]\rangle \\
&= \sum_{i=1}^m \langle J_{Sz}x_i, y_i\rangle = \sum_{i=1}^m \langle\langle \lambda_i J_{Sz}x_i, y_i\rangle\rangle \\
&= \langle\langle J_{Sz}\sum_{i=1}^m \lambda_i x_i, \sum_{i=1}^m y_i\rangle\rangle \\
&= \langle\langle J_{Sz}\sum_{i=1}^m \sqrt{-A}x_i, \sum_{i=1}^m y_i\rangle\rangle \\
&= \langle\langle J_{Sz}\sqrt{-A}\sum_{i=1}^m x_i, \sum_{i=1}^m y_i\rangle\rangle
\end{aligned}$$

for $z \in \mathfrak{z},\ x_i, y_i \in \mathfrak{v}_i$ and the operators $A = -\lambda_i^2 I,\ \sqrt{-A} = \lambda_i I$ on each subspaces $\mathfrak{v}_i,\ i=1,\ldots,m$. Thus we get $J_z^* = J_{Sz}\sqrt{-A}$ for any $z \in \mathfrak{z}$. This and commutativity between $\sqrt{-A}$ and J_{Sz} imply that

$$(J_z^*)^2 = J_{Sz}^2(-A) = (-\langle Sz, Sz\rangle I)(-A) = \langle\langle Sz, z\rangle\rangle A.$$

Thus the simply connected 2-step nilpotent Lie group N with its Lie algebra $\mathfrak{n}$ and a left invariant metric $\langle\langle\ ,\ \rangle\rangle$ is a group satisfying (1.2).

The following proposition shows that the above examples exhaust all possibilities for 2-step nilpotent Lie groups satisfying (1.2) and shows some elementary properties of such groups.

Proposition 1.5. *Let N be a 2-step nilpotent group with a left invariant metric $\langle\langle\ ,\ \rangle\rangle$ satisfying* (1.2) *and let $-\lambda_1^2,\ldots,-\lambda_m^2$ be all distinct eigenvalues of A. Then for $\mathfrak{z}$ the center of the Lie algebra $\mathfrak{n}$ of N and $\mathfrak{v}_i$ the eigenspace of A corresponding to $-\lambda_i^2$ the followings hold.*

(1) *Every subspace $\mathfrak{v}_i$ of the orthogonal complement $\mathfrak{v}$ of $\mathfrak{z}$ is J_z-invariant and $AJ_z = J_zA$ for all $z \in \mathfrak{z}$.*
(2) *$\mathfrak{v}_i \perp \mathfrak{v}_j$ for $i \neq j,\ i,j=1,2,\ldots,m$.*
(3) *$[\mathfrak{v}_i, \mathfrak{v}_j] = \{0\}$ for $i \neq j,\ i,j=1,2,\ldots,m$ and $J_z\mathfrak{v}_i \subset \mathfrak{v}_i$ for every $z \in \mathfrak{z}$, $i=1,2,\ldots,m$.*

(4) *Each subspace* $\mathfrak{z}\oplus\mathfrak{v}_i$ *of* $\mathfrak{n}$, $i = 1, \dots, m$ *is a modified* H*-type Lie algebra.*
(5) *The metric* $\langle\ ,\ \rangle$ *on* $\mathfrak{n}$ *defined by*

$$\langle x + z, y + w\rangle = \langle\langle(\sqrt{-A})x, y\rangle\rangle + \langle\langle S^{-1}z, w\rangle\rangle \tag{1.3}$$

for $x, y \in \mathfrak{v} = \sum_{i=1}^{m}\mathfrak{v}_i$ *and* $z, w \in \mathfrak{z}$ *is an* H*-type metric.*

Proof. Since properties 1, 2, and 3 directly follows from (1.1) and (1.2), we omit proofs of them. It is clear that $\mathfrak{z}\oplus\mathfrak{v}_i$ is a subalgebra of $\mathfrak{n}$ for every $i \in \{1, 2, \dots, m\}$. Also we can see that $J_z^2 x = -\lambda_i^2\langle\langle Sz, z\rangle\rangle x$ for every $x \in \mathfrak{v}_i$ and every $z \in \mathfrak{z}$. This means that $J_z^2 = -\langle\langle S'z, z\rangle\rangle I$ on $\mathfrak{v}_i$ for every $z \in \mathfrak{z}$ and $S' = \lambda_i^2 S$. So $\mathfrak{z}\oplus\mathfrak{v}_i$ is a modified *H-type* Lie algebra.

For a $z \in \mathfrak{z}$ let J_z^* be as in (1.1) for the metric $\langle\ ,\ \rangle$ defined by (1.3). Then we find

$$\begin{aligned}\langle J_z^* x, y\rangle &= \langle z, [x, y]\rangle = \langle\langle S^{-1}z, [x, y]\rangle\rangle \\ &= \langle\langle J_{S^{-1}z}x, y\rangle\rangle = \langle \sqrt{-A}^{-1} J_{S^{-1}z}x, y\rangle\end{aligned}$$

for every $x, y \in \mathfrak{v} = \sum_{i=1}^{m}\mathfrak{v}_i$. This implies that $J_z^* = \sqrt{-A}^{-1} J_{S^{-1}z}$. Thus we have $(J_z^*)^2 = (-A)^{-1}\langle\langle SS^{-1}z, S^{-1}z\rangle\rangle A = \langle\langle z, S^{-1}z\rangle\rangle(-I) = -\langle z, z\rangle I$. □

Here are some characterizations on 2-step nilpotent groups satisfying $J_z^2 = \langle Sz, z\rangle A$, which will be useful for computations.

Lemma 1.6. *Let* N *be a simply connected nonsingular 2-step nilpotent group endowed with a left invariant metric* $\langle\ ,\ \rangle$*. Then for a positive definite symmetric operator* S *on the center* $\mathfrak{z}$ *of its Lie algebra* n *and a negative definite symmetric operator* A *on the orthogonal complement* $\mathfrak{v}$ *of* $\mathfrak{z}$ *the following statements for* N *are all equivalent.*

(1) *The equality* $J_z^2 = \langle Sz, z\rangle A$ *holds for all* $z \in \mathfrak{z}$.
(2) *The equality* $J_z J_{z'} + J_{z'} J_z = 2\langle Sz, z'\rangle A$ *holds for all* $z, z' \in \mathfrak{z}$.
(3) *The equality* $\langle J_z x, J_{z'} x\rangle = -\langle Sz, z'\rangle\langle Ax, x\rangle$ *holds for all* $z, z' \in \mathfrak{z}$ *and for all* $x \in \mathfrak{v}$.
(4) *The equality* $\langle J_z x, J_z y\rangle = -\langle Sz, z\rangle\langle Ax, y\rangle$ *holds for all* $z \in \mathfrak{z}$ *and for all* $x, y \in \mathfrak{v}$.
(5) *The equality* $[x, J_z x] = -\langle Ax, x\rangle Sz$ *holds for all* $x \in \mathfrak{v}$ *and* $z \in \mathfrak{z}$.

Proof. To prove all equivalences we can proceed in the cyclic order (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (1). Since all steps can be verified in standard ways by polarization or by the fact that $\langle J_z x, y\rangle = -\langle x, J_z y\rangle$ for all $x, y \in \mathfrak{v}$ and $z \in \mathfrak{z}$, here we only show the step (5) ⇒ (1). By hypothesis (5), we get

$$[x + y, J_z(x + y)] = -\langle A(x + y), x + y\rangle Sz$$

for all $x, y \in \mathfrak{v}$ and for all $z \in \mathfrak{z}$. This and hypothesis (5) implies that

$$[x, J_z y] + [y, J_z x] = -2\langle Ax, y\rangle Sz$$

for all $x, y \in \mathfrak{v}$ and for all $z \in \mathfrak{z}$. Thus we have

$$\langle z, [x, J_z y] + [y, J_z x]\rangle = -2\langle Ax, y\rangle\langle Sz, z\rangle$$

for all $x, y \in \mathfrak{v}$ and for all $z \in \mathfrak{z}$. From this it follows that

$$\langle J_z^2 x, y\rangle = \langle Sz, z\rangle\langle Ax, y\rangle$$

for all $x, y \in \mathfrak{v}$ and for all $z \in \mathfrak{z}$. This imply that $J_z^2 = \langle Sz, z\rangle A$ for all $z \in \mathfrak{z}$. □

Corollary 1.7. *Let N be a simply connected 2-step nilpotent Lie group with its Lie algebra $n = \mathfrak{z} \oplus \mathfrak{v}$ satisfying* (1.2). *Then the following equalities holds*

$$[J_{z_1}J_{z_2}x, J_{z_1}x] = -\langle Sz_1, z_1\rangle\lambda^2[J_{z_2}x, x],$$

$$[J_{z_2}x, J_{z_1}x] = [J_{z_1}J_{z_2}x, x],$$

for all $z_1, z_2 \in \mathfrak{z}$ with $\langle Sz_1, z_2\rangle = 0$ and eigenvector x of A with an eigenvalue $-\lambda^2$.

Proof. By items (2) and (5) in Lemma 1.6 we have

$$\begin{aligned}[J_{z_1}J_{z_2}x, J_{z_1}x] &= -\langle AJ_{z_1}x, J_{z_1}x\rangle Sz_2 = \langle J_{z_1}^2 Ax, x\rangle Sz_2\\ &= -\langle Sz_1, z_1\rangle\lambda^2\langle Ax, x\rangle Sz_2 = \langle Sz_1, z_1\rangle\lambda^2[x, J_{z_2}x],\end{aligned}$$

which proves the first equality. Note that

$$\langle z_1, [J_{z_2}x, J_{z_1}x]\rangle = \langle -J_{z_1}^2 J_{z_2}x, x\rangle = \lambda^2\langle Sz_1, z_1\rangle\langle J_{z_2}x, x\rangle = 0$$

and

$$\langle z_1, [J_{z_1}J_{z_2}x, x]\rangle = \langle J_{z_1}^2 J_{z_2}x, x\rangle = -\lambda^2\langle Sz_1, z_1\rangle\langle J_{z_2}x, x\rangle = 0.$$

Also we have for every $\zeta \in \mathfrak{z}$ with $\langle S\zeta, z_1\rangle = 0$

$$\langle \zeta, [J_{z_2}x, J_{z_1}x]\rangle = \langle J_\zeta J_{z_1}J_{z_2}x, x\rangle = \langle \zeta, [J_{z_1}J_{z_2}x, x]\rangle.$$

These three equalities imply that

$$\langle z, [J_{z_2}x, J_{z_1}x]\rangle = \langle z, [J_{z_1}J_{z_2}x, x]\rangle$$

for all $z \in \mathfrak{z}$. So we can conclude that the second equality holds. □

From now on, N will denote a simply connected 2-step nilpotent Lie group with a left invariant metric $\langle\ ,\ \rangle$ satisfying (1.2) for a fixed negative definite symmetric operator A on $\mathfrak{v}$ and a fixed positive definite symmetric operator S on $\mathfrak{z}$. Assume that $\mathfrak{z}$ and $\mathfrak{v}$ are decomposed as direct sums $\oplus_{k=1}^{l}\mathfrak{z}_k$ and $\oplus_{i=1}^{m}\mathfrak{v}_i$, respectively where $\mathfrak{z}_k$ and $\mathfrak{v}_i$ are eigenspaces of S and A corresponding to eigenvalues α_k and $-\lambda_i^2$, respectively. For simplicity we will use the notation

$$\mu_i = \sqrt{\langle Sz_0, z_0\rangle}\lambda_i$$

for $i = 1, 2, \ldots, m$.

Let γ be a geodesic in N with $\gamma(0) = e$ and $\dot\gamma(0) = z_0 + x_0 \in \mathfrak{z} \oplus \mathfrak{v}$, respectively, and let $J = J_{z_0}$. We may assume γ is normalized so that $\langle\dot\gamma, \dot\gamma\rangle = 1$. As usual, $\mathbb{Z}^*$ denotes the set of all integers with 0 removed.

Remark 1.8. We adapt the usual notation from number theory and write $a|b$ to denote that b is a nonzero integral multiple of a for real a, b. Otherwise we write $a \nmid b$. It seems necessary to explain the meanings of some summations and direct sums subscripted by these division notations, which will be found in the following results. In the statements and proofs of Propositions 1.11-12, Theorem 1.13 and Corollary 1.14, $\sum_{\frac{\lambda_i}{n}|\lambda_h}$ and $\sum_{\frac{\lambda_i}{n}\nmid\lambda_h}$ mean a summation over all h in the set $\{1, 2, \dots, m\}$ that have the property that λ_h is an integral multiple of λ_i/n and a summation over all h in the set $\{1, 2, \dots, m\}$ that do not have the property, respectively (Please note that the integers i and n are fixed in both cases). The direct sums are similarly explained.

From (1.2) and $e^{tJ} = \sum_{n=0}^{\infty} \frac{(tJ)^n}{n!}$ we have

$$e^{tJ} = \cos(\mu_i t)I + \frac{1}{\mu_i}\sin(\mu_i t)J, \text{ on } \mathfrak{v}_i \tag{1.4}$$

for $i = 1, 2, \dots, m$. The following lemma is useful to understand multiplicity equations in the statements of our results and proofs of main results.

Lemma 1.9. *The operator* $(e^{\pm tJ} - I) : \mathfrak{v}_i \longrightarrow \mathfrak{v}_i$, $i = 1, 2, \dots, m$ *is either the zero map or an invertible map depending on whether* t *is an integral multiple of* $\frac{2\pi}{\mu_i}$ *or not, respectively. For* $t = \frac{2\pi}{\mu_i}n$, *where* $n \in \mathbb{Z}^*$, *the orthogonal complement* $\mathfrak{v}$ *of* $\mathfrak{z}$ *in* $\mathfrak{n}$ *is orthogonally decomposed into*

$$\mathfrak{v} = \mathrm{Im}(e^{\pm tJ} - I) \bigoplus \ker(e^{\pm tJ} - I),$$

where $\mathrm{Im}(e^{\pm tJ} - I) = \bigoplus_{\frac{\lambda_i}{n}\nmid\lambda_h} \mathfrak{v}_h$ *and* $\ker(e^{\pm tJ} - I) = \bigoplus_{\frac{\lambda_i}{n}|\lambda_h} \mathfrak{v}_h$.

For completeness we will state characterizations of Jacobi fields and calculations of conjugate points in simple cases which can be derived by direct calculation using Proposition 1.3.

Proposition 1.10. *Under these assumptions, the following hold.*

(1) *if* $z_0 = 0$ *and* $x_0 \neq 0$, *then a vector field* $Y(t) = z(t) + u(t)$ *along* γ *with* $z(t) \in \mathfrak{z}$, $u(t) \in \mathfrak{v}$ *for every* t *is a Jacobi field with* $Y(0) = 0$ *if and only if* $z(t) = \frac{t^3}{6}\langle Ax_0, x_0\rangle S\zeta + \frac{t^2}{2}[v, x_0] + t\zeta$ *and* $u(t) = \frac{t^2}{2}J_\zeta x_0 + tv$ *for a vector* $v \in \mathfrak{v}$ *and a vector* $\zeta \in \mathfrak{z}$.

(2) *if* $z_0 \neq 0$ *and* $x_0 = 0$, *then a vector field* $Y(t) = z(t) + e^{tJ}u(t)$ *along* γ *with* $z(t) \in \mathfrak{z}$, $u(t) \in \mathfrak{v}$ *and* $J = J_{z_0}$ *is a Jacobi field with* $Y(0) = 0$ *if and only if* $z(t) = t\zeta$ *and* $u(t) = (e^{-tJ} - I)v$ *for a vector* $\zeta \in \mathfrak{z}$ *and a vector* $v \in \mathfrak{v}$.

Proposition 1.11. *Under these assumptions, the following hold.*

(1) *if* $z_0 = 0$ *and* $x_0 \neq 0$, *then there is no conjugate point to* $\gamma(0)$ *along* γ;

(2) *if* $z_0 \neq 0$ *and* $x_0 = 0$, *then* $\gamma(t)$ *is conjugate to* $\gamma(0)$ *along* γ *if and only if*

$$t \in \cup_{i=1}^{m} \frac{2\pi}{\mu_i}\mathbb{Z}^*,$$

and $\mathrm{mult}_{cp}(\frac{2\pi}{\mu_i}n) = \sum_{\frac{\lambda_i}{n}|\lambda_h} \dim \mathfrak{v}_h$.

Now we will state the main results of this paper which will be proved in Section 2 of this paper. We will use properties 1-3 in Proposition 1.5 without comments.

Proposition 1.12. *Let γ be a geodesic of N with $\gamma(0) = e$ and $\gamma'(0) = z_0 + x_0$, $z_0 \neq 0 \neq x_0$. Also assume that x_0 is decomposed as $x_0 = \sum_{i=1}^m x_i$, where $x_i \in \mathfrak{v}_i$ for $i = 1, \ldots, m$. Then a vector field $Y(t) = z(t) + e^{tJ}u(t)$ along γ with $z(t) \in \mathfrak{z}$, $u(t) \in \mathfrak{v}$ for all t and $J = J_{z_0}$ is a Jacobi field with $Y(0) = 0$ if and only if $z(t)$ and $u(t)$ are given by (1.6) and (1.5) for a constant c , a constant vector $\zeta \in \mathfrak{z}$, $\langle Sz_0, \zeta\rangle = 0$ and a vector $v_0 = \sum_{i=1}^m v_i$, $v_i \in \mathfrak{v}_i$, $i = 1, 2, \ldots, m$.*

$$u(t) = ctx_0 + (e^{-tJ} - I)v_0 + \frac{1}{2\langle Sz_0, z_0\rangle}(e^{-2tJ} - e^{-tJ})A^{-1}J_\zeta x_0, \tag{1.5}$$

$$\begin{aligned} z(t) = & \sum_{i=1}^m (\frac{1}{\mu_i}\sin\mu_i t - \frac{1}{2}t - \frac{1}{4\mu_i}\sin 2\mu_i t)\big[v_i, x_i\big] \\ & + \sum_{i=1}^m (\frac{1}{2\mu_i^3}\sin\mu_i t - \frac{1}{2\mu_i^2}t)\big[J_\zeta x_i, x_i\big] \\ & + \sum_{i=1}^m \frac{1}{\mu_i^2}(\frac{3}{4} - \cos\mu_i t + \frac{1}{4}\cos 2\mu_i t)\big[v_i, Jx_i\big] \\ & + \sum_{i=1}^m \frac{1}{2\mu_i^4}(1 - \cos\mu_i t)\big[J_\zeta x_i, Jx_i\big] \\ & + \sum_{i=1}^m \frac{1}{2\mu_i^2}(\frac{1}{2\mu_i}\sin 2\mu_i t - t)\big[Jv_i, Jx_i\big] \\ & + \sum_{i=1}^m \frac{1}{4\mu_i^2}(\cos 2\mu_i t - 1)\big[Jv_i, x_i\big] + (cz_o + \zeta)t. \end{aligned} \tag{1.6}$$

Theorem 1.13. *Let γ be such a geodesic in N with $z_0 \neq 0 \neq x_0$. Also assume that z_0 and x_0 are decomposed as $z_0 = \sum_{k=1}^l z_k$ and $x_0 = \sum_{i=1}^m x_i$, where $z_k \in \mathfrak{z}_k$ for $k = 1, \ldots, l$ and $x_i \in \mathfrak{v}_i$ for $i = 1, \ldots, m$. Then $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ if and only if either*

$$t_0 \in \cup_{i=1}^m \frac{2\pi}{\mu_i}\mathbb{Z}^*,$$

or

$$t_0 \in B = \left\{ t \in \mathbb{R} \;\middle|\; \sum_{k=1}^l \frac{\alpha_k(1 + \alpha_k h_1(t))}{t + \alpha_k h_2(t)}\langle z_k, z_k\rangle = 0 \right\}, \tag{1.7}$$

where

$$h_1(t) = \sum_{i=1}^{m}(1 - \frac{\mu_i t}{2}\cot\frac{\mu_i t}{2})\frac{\langle x_i, x_i\rangle}{\langle Sz_0, z_0\rangle} \tag{1.8}$$

and

$$h_2(t) = \sum_{i=1}^{m}\frac{1}{2\mu_i^3}(\sin\mu_i t - \mu_i t)\langle Ax_i, x_i\rangle. \tag{1.9}$$

If $t_0 \in B$*, then* $\mathrm{mult}_{cp}(t_0) = 1$*. For* $t_0 = \frac{2\pi n}{\mu_i}$*, the multiplicity is as follows.*

If $x_0 \notin \mathrm{Im}(e^{-t_0 J} - I) = \bigoplus_{\frac{\lambda_i}{n}\nmid\lambda_h}\mathfrak{v}_h$*, then*

$$\mathrm{mult}_{cp}(t_0) = \dim\ker(e^{-t_0 J} - I) - 1.$$

If $x_0 \in \mathrm{Im}(e^{-t_0 J} - I)$*, then*

$$\mathrm{mult}_{cp}(t_0) = \begin{cases} \dim\ker(e^{t_0 J} - I) + 1 \ \textit{if} \ \displaystyle\sum_{k=1}^{l}\frac{\alpha_k(1+\alpha_k h_3(t_0))}{t_0 + \alpha_k h_4(t_0)}\langle z_k, z_k\rangle = 0 \\ \dim\ker(e^{t_0 J} - I) \ \textit{if} \ \displaystyle\sum_{k=1}^{l}\frac{\alpha_k(1+\alpha_k h_3(t_0))}{t_0 + \alpha_k h_4(t_0)}\langle z_k, z_k\rangle \neq 0, \end{cases}$$

where

$$h_3(t) = \sum_{\frac{\lambda_i}{n}\nmid\lambda_h}(1 - \frac{\mu_h t}{2}\cot\frac{\mu_h t}{2})\frac{\langle x_h, x_h\rangle}{\langle Sz_0, z_0\rangle} \tag{1.10}$$

and

$$h_4(t) = \sum_{\frac{\lambda_i}{n}\nmid\lambda_h}\frac{1}{2\mu_h^3}(\sin\mu_h t - \mu_h t)\langle Ax_h, x_h\rangle. \tag{1.11}$$

Corollary 1.14 ([6, Theorem 2])**.** *Let* γ *be such a geodesic in a 2-step nilpotent group* N *whose Lie algebra* $\mathfrak{n}$ *has one dimensional center* $\mathfrak{z}$ *with* $z_0 \neq 0 \neq x_0$*. And let* $-\lambda_1^2, -\lambda_2^2, \ldots, -\lambda_m^2$ *be all distinct eigenvalues of* J_z^2 *for a unit vector* z *in the center* $\mathfrak{z}$*. Then for the decomposition* $x_0 = \sum_{i=1}^m x_i$*, (*x_i *is contained the eigenspace of* J_z^2*,* v_i *with respect to the eigenvalue* $-\lambda_i^2$*,* $i = 1, \ldots, m$*) and* $\mu_i = \sqrt{\langle z_0, z_0\rangle}\lambda_i$*,* $i = 1, \ldots, m$*,* $\gamma(t_0)$ *is conjugate to* $\gamma(0)$ *if and only if* $t_0 \in \cup_{i=1}^m \frac{2\pi}{\mu_i}\mathbb{Z}^* \cup B$*, where*

$$B = \left\{ t \in \mathbb{R} \;\middle|\; \sum_{i=1}^{m}\langle x_i, x_i\rangle\frac{\mu_i t}{2}\cot\frac{\mu_i t}{2} = 1 \right\}. \tag{1.12}$$

If $t_0 \in B$*, then* $\mathrm{mult}_{cp}(t_0) = 1$*. For* $t_0 = \frac{2\pi n}{\mu_i}$*, the multiplicity are as follows.*

If $x_0 \nmid \mathrm{Im}(e^{-t_0 J} - I) = \bigoplus_{\frac{\lambda_i}{n}\nmid\lambda_h}\mathfrak{v}_h$*, then*

$$\mathrm{mult}_{cp}(t_0) = \dim\ker(e^{-t_0 J} - I) - 1.$$

If $x_0 \in \mathrm{Im}(e^{-t_0 J} - I)$, *then*

$$\mathrm{mult}_{cp}(t_0) = \begin{cases} \dim\ker(e^{t_0 J} - I) + 1 \ \textit{if} \displaystyle\sum_{\frac{\lambda_i}{n}\nmid\lambda_h} \frac{\mu_h t_0 \langle x_h, x_h\rangle}{2}\cot\frac{\mu_h t}{2} = 1 \\ \dim\ker(e^{t_0 J} - I) \ \textit{if} \displaystyle\sum_{\frac{\lambda_i}{n}\nmid\lambda_h} \frac{\mu_h t_0 \langle x_h, x_h\rangle}{2}\cot\frac{\mu_h t}{2} \neq 1. \end{cases}$$

Proof. Let z be a unit vector in the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{n}$. Since $\mathfrak{z}$ is one dimensional, there exists a constant c such that $z_0 = cz$. then we have $J_{z_0}^2 = c^2 J_z^2 = \langle z_0, z_0\rangle J_z^2$. Thus, the given group N satisfies (1.2) with $S = I$ and $A = J_z^2$.

So, we can apply results of Theorem 1.13 to this group N. Since $S = I$, we may assume that $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$ in (1.7).

This imply that the condition

$$\sum_{k=1}^{l} \frac{\alpha_k(1+\alpha_k h_1(t))}{t + \alpha_k h_2(t)}\langle z_k, z_k\rangle = 0 \text{ in } (1.7)$$

can be simplified as

$$1 + h_1(t) = 0, \tag{1.13}$$

where

$$h_1(t) = \sum_{i=1}^{m}\left(1 - \frac{\mu_i t}{2}\cot\frac{\mu_i t}{2}\right)\frac{\langle x_i, x_i\rangle}{\langle z_0, z_0\rangle}, \ (\mu_i = \lambda_i |z_0|).$$

Multiplying the value $\langle z_0, z_0\rangle$ at both sides of (1.13), we have

$$\langle z_0, z_0\rangle + \sum_{i=1}^{m}\left(1 - \frac{\mu_i t}{2}\cot\frac{\mu_i t}{2}\right)\langle x_i, x_i\rangle = 0$$

or

$$\langle z_0, z_0\rangle + \sum_{i=1}^{m}\langle x_i, x_i\rangle - \sum_{i=1}^{m}\langle x_i, x_i\rangle\left(\frac{\mu_i t}{2}\cot\frac{\mu_i t}{2}\right) = 0.$$

Since $\sum_{i=1}^{m}\langle x_i, x_i\rangle = \langle x_0, x_0\rangle$, the above equation becomes

$$\sum_{i=1}^{m}\langle x_i, x_i\rangle\left(\frac{\mu_i t}{2}\cot\frac{\mu_i t}{2}\right) = 1.$$

Therefore $\gamma(t_0)$ is conjugate to $\gamma(0)$ if and only if $t_0 \in \cup_{i=1}^{m}\frac{2\pi}{\mu_i}\mathbb{Z}^* \cup B$, where B is the set defined by (1.12).

When $t_0 = \frac{2\pi n}{\mu_i} = \frac{2\pi n}{|z_0|\lambda_i}$, the condition

$$\sum_{k=1}^{l} \frac{\alpha_k(1+\alpha_k h_3(t_0))}{t_0 + \alpha_k h_4(t_0)}\langle z_k, z_k\rangle = 0,$$

where $h_3(t)$ and $h_4(t)$ are given by (1.10) and (1.11) with $S = I$ becomes

$$1 + h_3(t_0) = 0$$

because of $\alpha_1 = \cdots = \alpha_l = 1$.

If $x_0 \in \mathrm{Im}(e^{-t_0 J} - I)$, then x_0 is decomposed as $x_0 = \sum_{\frac{\lambda_i}{n} \nmid \lambda_h} x_h$, where $x_h \in \mathfrak{v}_h$.

As before

$$1 + h_3(t_0) = 0$$

is equivalent to

$$\sum_{\frac{\lambda_i}{n} \nmid \lambda_h} \frac{\mu_h t_0 \langle x_h, x_h \rangle}{2} \cot \frac{\mu_h t_0}{2} = 1.$$

So we have the desired multiplicity formulas. □

To derive the following corollary from Theorem 1.13, we need to note that if N is *H-type*, then $e^{-t_0 J} = I$ for every $t_0 \in \frac{2\pi}{|z_0|}\mathbb{Z}^*$.

Corollary 1.15 ([1, 8])**.** *Let γ be such a geodesic in an H-type group N, $z_0 \neq 0 \neq x_0$. Then $\gamma(t_0)$ is conjugate to $\gamma(0)$ if and only if $t_0 \in \frac{2\pi}{|z_0|}\mathbb{Z}^* \cup B$, where*

$$B = \left\{ t \in \mathbb{R} \;\middle|\; \langle x_0, x_0 \rangle \frac{|z_0| t}{2} \cot \frac{|z_0| t}{2} = 1 \right\}.$$

If $t_0 \in B$, then $\mathrm{mult}_{cp}(t_0) = 1$. *If $t_0 = \frac{2\pi n}{|z_0|}$,* $\mathrm{mult}_{cp}(t_0) = \dim \mathfrak{v} - 1$.

2. Proofs of main results

Proof of Proposition 1.12. Assume that $Y(t) = z(t) + e^{tJ}u(t)$ is a nontrivial Jacobi field along γ with $Y(0) = 0$. Then by Proposition 1.3 and the fact that the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{n}$ can be decomposed into a direct sum $\mathfrak{z} = [[z_0]] \oplus [[Sz_0]]^\perp$ of the subspace $[[z_0]]$ generated by the vector z_0 and the orthogonal complement of $[[Sz_0]]$, $[[Sz_0]]^\perp$, which is not an orthogonal decomposition in general, we may assume that

$$\dot{z}(t) - [e^{tJ}u(t), e^{tJ}x_0] = cz_0 + \zeta, \tag{2.1}$$

$$e^{tJ}\ddot{u}(t) + e^{tJ}J\dot{u}(t) - J_{cz_0+\zeta}e^{tJ}x_0 = 0 \tag{2.2}$$

for a constant c and a constant vector $\zeta \in \mathfrak{z}$ with

$$\langle Sz_0, \zeta \rangle = 0. \tag{2.3}$$

By direct computations we can show that the general solution of equation (2.2) satisfying $u(0) = 0$ is given by (1.5). To show this, we used the fact that $e^{-tJ}J_\zeta = J_\zeta e^{tJ}$; this follows from item (2) in Lemma 1.6 and (2.3). Since v_0 in (1.5) is decomposed as $v_0 = \sum_{i=1}^m v_i$, where v_i is contained in the eigensubspace $\mathfrak{v}_i$ of A, substituting (1.5) for $u(t)$ in (2.1) gives us

$$\dot{z}(t) - \sum_{i=1}^{m} \left[v_i - e^{tJ}v_i + \frac{1}{2\langle Sz_0, z_0 \rangle}(e^{-tJ} - I)A^{-1}J_\zeta x_i, e^{tJ}x_i \right] = cz_0 + \zeta\,. \tag{2.4}$$

Using (1.4), from (2.4) we find

$$\begin{aligned}&\dot{z}(t)-\sum_{i=1}^{m}\big[(1-\cos\mu_i t)v_i-\frac{1}{\mu_i}\sin\mu_i tJv_i-\frac{1}{2\mu_i^2}\{(\cos\mu_i t-1)I\\&-\frac{1}{\mu_i}\sin\mu_i tJ\}J_\zeta x_i,\cos\mu_i tx_i+\frac{1}{\mu_i}\sin\mu_i tJx_i\big]\\=\;&cz_0+\zeta\end{aligned}$$

or

$$\begin{aligned}&\dot{z}(t)-\sum_{i=1}^{m}\big[v_i+\frac{1}{2\mu_i^2}J_\zeta x_i,\cos\mu_i tx_i+\frac{1}{\mu_i}\sin\mu_i tJx_i\big]\\&+\sum_{i=1}^{m}\cos^2\mu_i t\big[v_i+\frac{1}{2\mu_i^2}J_\zeta x_i,x_i\big]+\sum_{i=1}^{m}\frac{1}{\mu_i^2}\sin^2\mu_i t\big[Jv_i-\frac{1}{2\mu_i^2}JJ_\zeta x_i,Jx_i\big]\\&+\sum_{i=1}^{m}\frac{1}{\mu_i}\cos\mu_i t\sin\mu_i t\{\big[v_i+\frac{1}{2\mu_i^2}J_\zeta x_i,Jx_i\big]+\big[Jv_i-\frac{1}{2\mu_i^2}JJ_\zeta x_i,x_i\big]\}\\=\;&cz_o+\zeta.\end{aligned}$$

Integrating this under the condition $z(0)=0$, we have

$$\begin{aligned}z(t)=\;&\sum_{i=1}^{m}\big[v_i+\frac{1}{2\mu_i^2}J_\zeta x_i,\frac{1}{\mu_i}\sin\mu_i tx_i+\frac{1}{\mu_i^2}(1-\cos\mu_i t)Jx_i\big]\\&-\sum_{i=1}^{m}\frac{1}{2}(t+\frac{1}{2\mu_i}\sin 2\mu_i t)\big[v_i+\frac{1}{2\mu_i^2}J_\zeta x_i,x_i\big]\\&-\sum_{i=1}^{m}\frac{1}{2\mu_i^2}(t-\frac{1}{2\mu_i}\sin 2\mu_i t)\big[Jv_i-\frac{1}{2\mu_i^2}JJ_\zeta x_i,Jx_i\big]\\&-\sum_{i=1}^{m}\frac{1}{4\mu_i^2}(1-\cos 2\mu_i t)\{\big[v_i+\frac{1}{2\mu_i^2}J_\zeta x_i,Jx_i\big]+\big[Jv_i-\frac{1}{2\mu_i^2}JJ_\zeta x_i,x_i\big]\}\\&+(cz_o+\zeta)t\end{aligned}$$

or

$$\begin{aligned}z(t)=\;&\sum_{i=1}^{m}(\frac{1}{\mu_i}\sin\mu_i t-\frac{1}{2}t-\frac{1}{4\mu_i}\sin 2\mu_i t)\big[v_i,x_i\big]\\&+\sum_{i=1}^{m}(\frac{1}{2\mu_i^3}\sin\mu_i t-\frac{1}{4\mu_i^2}t-\frac{1}{8\mu_i^3}\sin 2\mu_i t)\big[J_\zeta x_i,x_i\big]\\&+\sum_{i=1}^{m}\frac{1}{\mu_i^2}(\frac{3}{4}-\cos\mu_i t+\frac{1}{4}\cos 2\mu_i t)\big[v_i,Jx_i\big]\\&+\sum_{i=1}^{m}\frac{1}{2\mu_i^4}(\frac{3}{4}-\cos\mu_i t+\frac{1}{4}\cos 2\mu_i t)\big[J_\zeta x_i,Jx_i\big]\end{aligned}$$

$$+\sum_{i=1}^{m}\frac{1}{2\mu_i^2}\Big(\frac{1}{2\mu_i}\sin 2\mu_i t - t\Big)\big[Jv_i, Jx_i\big]$$
$$+\sum_{i=1}^{m}\frac{1}{4\mu_i^4}\Big(t - \frac{1}{2\mu_i}\sin 2\mu_i t\Big)\big[JJ_\zeta x_i, Jx_i\big]$$
$$+\sum_{i=1}^{m}\frac{1}{4\mu_i^2}(\cos 2\mu_i t - 1)\big[Jv_i, x_i\big]$$
$$+\sum_{i=1}^{m}\frac{1}{8\mu_i^4}(1-\cos 2\mu_i t)\big[JJ_\zeta x_i, x_i\big]$$
$$+(cz_o+\zeta)t.$$

Since $[JJ_\zeta x_i, Jx_i] = -\mu_i^2[J_\zeta x_i, x_i]$ and $[J_\zeta x_i, Jx_i] = [JJ_\zeta x_i, x_i]$ by Corollary 1.7, the above equation becomes (1.6). We showed that if a vector field $Y(t) = z(t) + e^{tJ}u(t)$ along γ is a Jacobi field with $Y(0) = 0$, then $u(t)$ and $z(t)$ must be of the forms (1.5) and (1.6) respectively for a constant c, a vector $\zeta \in \mathfrak{z}$ which is orthogonal to the vector Sz_0 and a vector $v_0 = \sum_{i=1}^m v_i \in \oplus_{i=1}^m \mathfrak{v}_i$. Conversely it is easy to see that such $Y(t)$ is a Jacobi field along γ with $Y(0) = 0$. □

Proof of Theorem 1.13. First Assume that $\gamma(t_0)$ is a conjugate point along γ. Then there exists a nontrivial Jacobi field $Y(t) = z(t) + e^{tJ}u(t)$ along γ for $z(t) \in \mathfrak{z}$ and $u(t) \in \mathfrak{v}$ satisfying $Y(0) = Y(t_0) = 0$. By Proposition 1.11 we may assume $u(t)$ and $z(t)$ are of the forms (1.5) and (1.6) respectively for a constant c, a vector $\zeta \in \mathfrak{z}$ with (2.3) and a vector $v_0 = \sum_{i=1}^m v_i \in \oplus_{i=1}^m \mathfrak{v}_i = \mathfrak{v}$. Now assume that $t_0 \notin \cup_{i=1}^m \frac{2\pi}{\mu_i}\mathbb{Z}^*$, which implies that $e^{-t_0 J} - I$ is invertible on $\mathfrak{v}$ by Lemma 1.9. Then since $u(t_0) = 0$, we have

$$v_i = -(e^{-t_0J} - I)^{-1} c t_0 x_i + \frac{1}{2\mu_i^2} e^{-t_0 J} J_\zeta x_i,\ i = 1, 2, \ldots, m. \tag{2.5}$$

Using (1.4) and the following identity

$$\big(e^{-t_0J} - I\big)^{-1} = -\frac{1}{2}I + \frac{1}{2\mu_i}\cot\frac{\mu_i t_0}{2} J \text{ on } \mathfrak{v}_i$$

for $i = 1, 2, \ldots, m$, from (2.5) we have

$$v_i = \frac{1}{2}ct_0x_i - \frac{ct_0}{2\mu_i}\cot\frac{\mu_i t_0}{2}Jx_i + \frac{1}{2\mu_i^2}\cos\mu_i t_0 J_\zeta x_i - \frac{1}{2\mu_i^3}\sin\mu_i t_0 JJ_\zeta x_i$$

for $i = 1, 2, \ldots, m$. This implies

$$[v_i, x_i] = -\frac{ct_0}{2\mu_i}\cot\frac{\mu_i t_0}{2}[Jx_i, x_i] + \frac{1}{2\mu_i^2}\cos\mu_i t_0[J_\zeta x_i, x_i]$$
$$-\frac{1}{2\mu_i^3}\sin\mu_i t_0[JJ_\zeta x_i, x_i],$$

$$= \frac{1}{2}ct_0[x_i, Jx_i] + \frac{1}{2\mu_i^2}\cos\mu_i t_0[J_\zeta x_i, Jx_i] - \frac{1}{2\mu_i^3}\sin\mu_i t_0[JJ_\zeta x_i, Jx_i],$$

$$[Jv_i, Jx_i] = \frac{ct_0\mu_i}{2}\cot\frac{\mu_i t_0}{2}[x_i, Jx_i] + \frac{1}{2\mu_i^2}\cos\mu_i t_0[JJ_\zeta x_i, Jx_i] + \frac{1}{2\mu_i}\sin\mu_i t_0[J_\zeta x_i, Jx_i],$$

$$[Jv_i, x_i] = \frac{1}{2}ct_0[Jx_i, x_i] + \frac{1}{2\mu_i^2}\cos\mu_i t_0[JJ_\zeta x_i, x_i] + \frac{1}{2\mu_i}\sin\mu_i t_0[J_\zeta x_i, x_i],\ i = 1, 2, \ldots, m.$$

Replacing these into (1.6) and after some computations we find

$$\begin{aligned} z(t) &= ct_0\sum_{i=1}^{m}(\frac{1}{2} + \frac{1}{2}\sin\mu_i t\cot\frac{\mu_i t_0}{2} - \frac{1}{2}\cos\mu_i t - \frac{\mu_i t}{2}\cot\frac{\mu_i t_0}{2})\frac{\langle x_i, x_i\rangle}{\langle Sz_0, z_0\rangle}Sz_0 \\ &+ \sum_{i=1}^{m}\frac{1}{2\mu_i}(\sin\mu_i(t_0 - t) + \frac{1}{2}\sin\mu_i(2t - t_0) \\ &\qquad + \mu_i t - \sin\mu_i t - \frac{1}{2}\sin\mu_i t_0)\frac{\langle x_i, x_i\rangle}{\langle Sz_0, z_0\rangle}S\zeta \\ &+ \sum_{i=1}^{m}\frac{1}{4\mu_i^4}(2 + \cos\mu_i(2t - t_0) + \cos\mu_i t_0 \\ &\qquad - 2\cos\mu_i(t - t_0) - 2\cos\mu_i t)[J_\zeta x_i, Jx_i] \\ &+ (cz_o + \zeta)t. \end{aligned} \tag{2.6}$$

From (2.6) we find

$$\begin{aligned} z(t_0) &= ct_0\sum_{i=1}^{m}(1 - \frac{\mu_i t_0}{2}\cot\frac{\mu_i t_0}{2})\frac{\langle x_i, x_i\rangle}{\langle Sz_0, z_0\rangle}Sz_0 \\ &\quad + \sum_{i=1}^{m}\frac{1}{2\mu_i}(\mu_i t_0 - \sin\mu_i t_0)\frac{\langle x_i, x_i\rangle}{\langle Sz_0, z_0\rangle}S\zeta \\ &\quad + (cz_o + \zeta)t_0. \end{aligned}$$

Then we have

$$z(t_0) = ct_0 h_1(t_0)Sz_0 + h_2(t_0)S\zeta + ct_0 z_0 + t_0\zeta \tag{2.7}$$

for $h_1(t)$ and $h_2(t)$ given by (1.8) and (1.9). Let $z_0 = \sum_{k=1}^{l} z_k,\ \zeta = \sum_{k=1}^{l}\zeta_k \in \oplus_{k=1}^{l}\mathfrak{z}_k$ be decompositions of z_0 and ζ, where the $\mathfrak{z}_k$ are the eigenspaces of the operator S with the corresponding eigenvalues α_k. Then $z(t_0) = 0$ and (2.7) imply

$$\zeta_k = -\frac{ct_0(1 + \alpha_k h_1(t_0))}{t_0 + \alpha_k h_2(t_0)}z_k \tag{2.8}$$

for every $k \in \{1, 2, \dots, l\}$. This and (2.3) imply that

$$c \sum_{k=1}^{l} \frac{\alpha_k(1+\alpha_k h_1(t_0))}{t_0+\alpha_k h_2(t_0)} \langle z_k, z_k \rangle = 0.$$

Since $c \neq 0$ (otherwise, $Y(t) \equiv 0$) we can see that $t_0 \in B$. The multiplicity follows the fact that v_i and ζ are uniquely determined by c. Conversely if $t_0 \in B$, consider an arbitrary constant c and a vector $\zeta = \sum_{k=1}^{l} \zeta_k$ for ζ_k given by (2.8). Then ζ satisfies (2.3) since $t_0 \in B$. Also consider $v_i,\ i = 1, \dots, m$ given by (2.5) for such c and ζ. Then a Jacobi field $Y(t) = z(t) + e^{tJ}u(t)$ along γ, where $z(t)$ and $u(t)$ are given by (1.6) and (1.5) for such c, ζ, v_i and $v_0 = \sum_{i=1}^{m} v_i$ satisfies $Y(0) = Y(t_0) = 0$. Thus $\gamma(t_0)$ is conjugate to $\gamma(0)$.

We now assume that

$$t_0 = \frac{2\pi}{\mu_i} n \text{ for some } i \in \{1, 2, \dots, m\}, \tag{2.9}$$

where $n \in \mathbb{Z}^*$.

Lemma 1.9 implies that if $\frac{\lambda_i}{n} | \lambda_h$, then $e^{-t_0 J} - I = 0$ on $\mathfrak{v}_h$ and if $\frac{\lambda_i}{n} \nmid \lambda_h$, then $e^{-t_0 J} - I$ is invertible on $\mathfrak{v}_h$. We proceed with two cases separately. If $x_0 \notin \mathrm{Im}(e^{-t_0 J} - I)$, then $u(t_0) = 0$ and (1.5) imply that $c = 0$. Thus we have

$$u(t) = (e^{-tJ} - I)v_0 + \frac{1}{2\langle Sz_0, z_0 \rangle}(e^{-2tJ} - e^{-tJ})A^{-1}J_\zeta x_0\,. \tag{2.10}$$

From (2.9), (2.10), $u(t_0) = 0$ and Lemma 1.9 we have

$$v_0 = \sum_{\frac{\lambda_i}{n}|\lambda_h} v_h + \sum_{\frac{\lambda_i}{n}\nmid\lambda_h} \frac{1}{2\mu_h^2} e^{-t_0 J} J_\zeta x_h,$$

where each v_h in the first summation is arbitrary in $\mathfrak{v}_h$. Replacing these into (1.6) and after some computations we have

$$\begin{aligned} z(t) = & \sum_{\frac{\lambda_i}{n}|\lambda_h} (\frac{1}{\mu_h}\sin\mu_h t - \frac{1}{2}t - \frac{1}{4\mu_h}\sin 2\mu_h t)\big[v_h, x_h\big] \\ & + \sum_{\frac{\lambda_i}{n}|\lambda_h} (\frac{1}{2\mu_h^3}\sin\lambda_h t - \frac{1}{2\mu_h^2}t)\big[J_\zeta x_h, x_h\big] \\ & + \sum_{\frac{\lambda_i}{n}|\lambda_h} \frac{1}{\mu_h^2}(\frac{3}{4} - \cos\mu_h t + \frac{1}{4}\cos 2\mu_h t)\big[v_h, Jx_h\big] \\ & + \sum_{\frac{\lambda_i}{n}|\lambda_h} \frac{1}{2\mu_h^4}(1 - \cos\mu_h t)\big[J_\zeta x_h, Jx_h\big] \\ & + \sum_{\frac{\lambda_i}{n}|\lambda_h} \frac{1}{2\mu_h^2}(\frac{1}{2\mu_h}\sin 2\mu_h t - t)\big[Jv_h, Jx_h\big] \end{aligned}$$

$$+\sum_{\frac{\lambda_i}{n}|\lambda_h}\frac{1}{4\mu_h^2}(\cos 2\mu_h t-1)\big[Jv_h,x_h\big]$$
$$+\sum_{\frac{\lambda_i}{n}\nmid\lambda_h}\frac{1}{2\mu_h}(\sin\mu_h(t_0-t)+\frac{1}{2}\sin\mu_h(2t-t_0)+\mu_h t$$
$$-\sin\mu_h t-\frac{1}{2}\sin\mu_h t_0)\frac{\langle x_h,x_h\rangle}{\langle Sz_0,z_0\rangle}S\zeta$$
$$+\sum_{\frac{\lambda_i}{n}\nmid\lambda_h}\frac{1}{4\mu_h^4}(2+\cos\mu_h(2t-t_0)+\cos\mu_h t_0-2\cos\mu_h(t-t_0)$$
$$-2\cos\mu_h t)[J_\zeta x_h,Jx_h]+t\zeta.$$

Since $\langle[Jv_h,Jx_h],\zeta\rangle=-\mu_h^2\langle[v_h,x_h],\zeta\rangle$, it follows from the above equation and (2.9) that

$$\begin{aligned}\langle z(t_0),\zeta\rangle &= \frac{t_0\langle S\zeta,\zeta\rangle}{2\langle Sz_0,z_0\rangle}\sum_{\frac{\lambda_i}{n}|\lambda_h}\langle x_h,x_h\rangle\\ &\quad+\frac{\langle S\zeta,\zeta\rangle}{2\langle Sz_0,z_0\rangle}\sum_{\frac{\lambda_i}{n}\nmid\lambda_h}\frac{1}{\mu_h}(\mu_h t_0-\sin\mu_h t_0)\langle x_h,x_h\rangle+t_0\langle\zeta,\zeta\rangle.\end{aligned}$$

If $\zeta\neq 0$, the sign of this formula coincides with the sign of t_0. Thus the condition $z(t_0)=0$ implies that $\zeta=0$ and we have

$$u(t)=(e^{-tJ}-I)v_0$$

for some $v_0\in\ker(e^{-t_0J}-I)$, and

$$\begin{aligned}z(t)=&\sum_{\frac{\lambda_i}{n}|\lambda_h}(\frac{1}{\mu_h}\sin\mu_h t-\frac{1}{2}t-\frac{1}{4\mu_h}\sin 2\mu_i h)\big[v_h,x_h\big]\\ &+\sum_{\frac{\lambda_i}{n}|\lambda_h}\frac{1}{\mu_h^2}(\frac{3}{4}-\cos\mu_h t+\frac{1}{4}\cos 2\mu_h t)\big[v_h,Jx_h\big]\\ &+\sum_{\frac{\lambda_i}{n}|\lambda_h}\frac{1}{2\mu_h^2}(\frac{1}{2\mu_h}\sin 2\mu_h t-t)\big[Jv_h,Jx_h\big]\\ &+\sum_{\frac{\lambda_i}{n}|\lambda_h}\frac{1}{4\mu_h^2}(\cos 2\mu_h t-1)\big[Jv_h,x_h\big]\end{aligned}\tag{2.11}$$

for the decomposition $v_0=\sum_{\frac{\lambda_i}{n}|\lambda_h}v_h$, $v_h\in\mathfrak{v}_h$. (2.9) and (2.11) imply that

$$z(t_0)=-\frac{t_0}{2}\sum_{\frac{\lambda_i}{n}|\lambda_h}([v_h,x_h]+\frac{1}{\mu_h^2}[Jv_h,Jx_h]).\tag{2.12}$$

Since $\langle [Jv_h, Jx_h], \zeta\rangle = -\mu_h^2 \langle [v_h, x_h], \zeta\rangle$ holds for every vector $\zeta \in \mathfrak{z}$ with $\langle S\zeta, z_0\rangle = 0$, it follows from (2.12) that

$$\langle z(t_0), \zeta\rangle = 0$$

for every vector $\zeta \in \mathfrak{z}$ with $\langle S\zeta, z_0\rangle = 0$. Thus the condition $z(t_0) = 0$ is equivalent to

$$\langle z(t_0), z_0\rangle = t_0 \sum_{\frac{\lambda_i}{n}|\lambda_h} \langle v_h, Jx_h\rangle = t_0\langle v_0, Jx_0\rangle = 0.$$

Therefore $z(t_0) = 0$ is equivalent to $\langle v_0, Jx_0\rangle = 0$. Since Jx_0 is not perpendicular to the subspace $\ker(e^{-t_0J} - I)$ by the condition $x_0 \notin \mathrm{Im}(e^{t_0J} - I)$ and Lemma 1.9, the multiplicity is $\dim \ker(e^{-t_0J} - I) - 1$.

Now assume that $x_0 \in \mathrm{Im}(e^{-t_0J} - I)$ and $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ for $t_0 = \frac{2\pi}{\mu_i} n$. Then $x_0 = \sum_{\frac{\lambda_i}{n}\nmid\lambda_h} x_h$ by Lemma 1.9. This and $u(t_0) = 0$ imply that

$$v_h = -(e^{-t_0J} - I)^{-1} ct_0 x_h + \frac{1}{2\mu_h^2} e^{-t_0J} J_\zeta x_h \tag{2.13}$$

for h with $\frac{\lambda_i}{n} \nmid \lambda_h$, and v_h are arbitrary for h with $\frac{\lambda_i}{n}|\lambda_h$. Replacing these into (1.6) and after some computations we get

$$\begin{aligned} z(t_0) \quad = \quad & \left(\sum_{\frac{\lambda_i}{n}\nmid\lambda_h} (ct_0 - \frac{ct_0^2\mu_h}{2} \cot\frac{\mu_h t_0}{2}) \frac{\langle x_h, x_h\rangle}{\langle Sz_0, z_0\rangle}\right) Sz_0 \\ & + \left(\sum_{\frac{\lambda_i}{n}\nmid\lambda_h} \frac{1}{2\mu_h^3} (\sin\mu_h t_0 - \lambda_h t_0)\langle Ax_h, x_h\rangle\right) S\zeta + (cz_0 + \zeta)t_0. \end{aligned}$$

Then we have

$$z(t_0) = ct_0 h_3(t_0) Sz_0 + h_4(t_0) S\zeta + ct_0 z_0 + t_0\zeta$$

for $h_3(t)$ and $h_4(t)$ given by (1.10) and (1.11). Let $z_0 = \sum_{k=1}^l z_k$, $\zeta = \sum_{k=1}^l \zeta_k \in \oplus_{k=1}^l \mathfrak{z}_k$ be decompositions of z_0 and ζ. Since $z(t_0) = 0$, from the above equation we have

$$\zeta_k = -\frac{ct_0(1 + \alpha_k h_3(t_0))}{t_0 + \alpha_k h_4(t_0)} z_k \tag{2.14}$$

for $k = 1, 2, \ldots, l$. This and (2.3) imply that

$$\sum_{k=1}^{l} c\frac{\alpha_k(1 + \alpha_k h_3(t_0))}{t_0 + \alpha_k h_4(t_0)} \langle z_k, z_k\rangle = 0.$$

If $\sum_{k=1}^l \frac{\alpha_k(1+\alpha_k h_3(t_0))}{t_0+\alpha_k h_4(t_0)} \langle z_k, z_k\rangle = 0$, then a Jacobi field $Y(t)$ satisfying $Y(0) = Y(t_0) = 0$ is determined by a constant c and a vector $v_0 \in \ker(e^{-t_0J} - I)$. Thus the multiplicity is $\dim \ker(e^{-t_0J} - I) + 1$. If $\sum_{k=1}^l \frac{\alpha_k(1+\alpha_k h_3(t_0))}{t_0+\alpha_k h_4(t_0)} \langle z_k, z_k\rangle \neq 0$, then $c = 0$ and a Jacobi field $Y(t)$ with $Y(0) = Y(t_0) = 0$ is determined by a vector $v_0 \in \ker(e^{-t_0 J} - I)$. Thus the multiplicity is $\dim \ker(e^{-t_0 J} - I)$. Conversely if $t_0 = \frac{2\pi}{\mu_i} n$ and $x_0 \notin \operatorname{Im}(e^{-t_0 J} - I)$, then a Jacobi field $Y(t) = z(t) + e^{tJ} u(t)$, where $u(t) = (e^{-tJ} - I)v_0$, $v_0 \in \ker(e^{-t_0 J} - I)$ with $\langle v_0, Jx_0 \rangle = 0$ and $z(t)$ is given by (2.11) for $v_h \in \mathfrak{v}_h$, components of the decomposition $v_0 = \sum_{\frac{\lambda_i}{n} | \lambda_h} v_h$. Then we can see that $Y(0) = Y(t_0) = 0$, which implies that $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ. If $t_0 = \frac{2\pi}{\mu_i} n$ and $x_0 \in \operatorname{Im}(e^{-t_0 J} - I)$, then consider a constant c (when $\sum_{k=1}^{l} \frac{\alpha_k(1+\alpha_k h_3(t_0))}{t_0 + \alpha_k h_4(t_0)} \langle z_k, z_k \rangle \neq 0$ for $h_3(t)$, $h_4(t)$ given by (1.10) and (1.11), $c = 0$) and a vector $\zeta = \sum_{k=1}^{l} \zeta_k$ for ζ_k given by (2.14). Also consider arbitrary v_h in $\mathfrak{v}_h$ for $\frac{\lambda_i}{n} | \lambda_h$, v_h given by (2.13) for $\frac{\lambda_i}{n} \nmid \lambda_h$ and $v_0 = \sum_{\frac{\lambda_i}{n} | \lambda_h} v_h + \sum_{\frac{\lambda_i}{n} \nmid \lambda_h} v_h$. Then a Jacobi field $Y(t) = z(t) + e^{tJ} u(t)$, where $u(t)$ and $z(t)$ are given by (1.5) and (1.6) for such c, ζ, v_h and v_0 satisfies $Y(0) = Y(t_0) = 0$, which implies $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ. □

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