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NORM ESTIMATES AND UNIVALENCE CRITERIA FOR MEROMORPHIC FUNCTIONS

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Abstract. Norm estimates of the pre-Schwarzian derivatives are given for meromorphic functions in the outside of the unit circle. We deduce several univalence criteria for meromorphic functions from those estimates.

1. Introduction

Let $\mathscr{A}$ denote the set of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized so that $f(0) = 0$ and $f'(0) = 1$. The set $\mathscr{S}$ of univalent functions in $\mathscr{A}$ has been intensively studied by many authors. It is well recognized that the set Σ of univalent meromorphic functions F in the domain $\Delta = \{\zeta : |\zeta| > 1\}$ of the form

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n} \tag{1.1}$$

plays an indispensable role in the study of $\mathscr{S}$.

In parallel with the analytic case, we consider the set $\mathscr{M}$ of meromorphic functions in Δ with the expansion (1.1) around $\zeta = \infty$. For some technical reason, we also consider the set $\mathscr{M}_n$ of functions F in Σ of the form

$$F(\zeta) = \zeta + \frac{b_n}{\zeta^n} + \frac{b_{n+1}}{\zeta^{n+1}} + \cdots$$

for each nonnegative integer n. Note that $\mathscr{M}_0 = \mathscr{M}$.

Practically, it is an important problem to determine univalence of a given function in $\mathscr{A}$ or in $\mathscr{M}$. The best known conditions for univalence are probably those involving pre-Schwarzian or Schwarzian derivatives, which are defined by

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

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We define quantities for functions $f \in \mathscr{A}$ and $F \in \mathscr{M}$ by

$$\begin{aligned} \mathrm{B}(f) &= \sup_{|z|<1} (1-|z|^2)\, |zT_f(z)|\,, \\ \mathrm{B}^*(F) &= \sup_{|\zeta|>1} (|\zeta|^2-1)\, |\zeta T_F(\zeta)|\,, \\ \mathrm{N}(f) &= \sup_{|z|<1} (1-|z|^2)^2\, |S_f(z)|\,, \\ \mathrm{N}^*(F) &= \sup_{|\zeta|>1} (|\zeta|^2-1)^2\, |S_F(\zeta)|\,. \end{aligned}$$

Note that these quantities may take ∞ as their values. For example, if F has a pole at a finite point, then $\mathrm{B}^*(F) = \infty$. Those functions with finite norms constitute complex Banach spaces, which play a fundamental role in the universal Teichmüller space. See [19] for a survey on the universal Teichmüller space.

If $f \in \mathscr{A}$ and $F \in \mathscr{M}$ have the relation $f(z) = 1/F(1/z)$, then we can easily see that the relation

$$(1-|z|^2)^2 S_f(z) = (|\zeta|^2-1)^2 S_F(\zeta)$$

holds for $z = 1/\zeta$. In particular, we have $\mathrm{N}(f) = \mathrm{N}^*(F)$.

Nehari [18] proved the following univalence criteria except for the quasiconformal extension property, which is due to Ahlfors and Weill [1].

Theorem A. *Every $f \in \mathscr{S}$ satisfies $\mathrm{N}(f) \le 6$. Conversely, if $f \in \mathscr{A}$ satisfies $\mathrm{N}(f) \le 2$, then f must be univalent. Moreover, if $\mathrm{N}(f) \le 2k < 2$, then f extends to a k-quasiconformal mapping of the extended plane. The constants* 6 *and* 2 *are best possible. The same is true for meromorphic F.*

Here and hereafter, a quasiconformal mapping g is called k-quasiconformal if its Beltrami coefficient $\mu = g_{\bar z}/g_z$ satisfies $\|\mu\|_\infty \le k$. An extensive survey on those univalent functions in $\mathscr{S}$ or Σ which extend to quasiconformal mappings of the Riemann sphere was recently supplied by Krushkal [16].

Though $zf'(z)/f(z) = \zeta F'(\zeta)/F(\zeta)$, there is no such a simple relation between $zT_f(z)$ and $\zeta T_F(\zeta)$, and thus, between $\mathrm{B}(f)$ and $\mathrm{B}^*(F)$ for $f(z) = 1/F(\zeta),\ \zeta = 1/z$. Indeed, we have the formula

$$F'(\zeta) = \left(\frac{z}{f(z)}\right)^2 f'(z), \tag{1.2}$$

which leads to

$$-\frac{\zeta F''(\zeta)}{F'(\zeta)} = 2\left(1 - \frac{zf'(z)}{f(z)}\right) + \frac{zf''(z)}{f'(z)}.$$

Nevertheless, it is rather surprising that formally the same conclusion can be deduced for f and F. Compare Theorem B with Theorem C.

Theorem B. *Every* $f \in \mathscr{S}$ *satisfies* $\mathrm{B}(f) \le 6$. *Conversely, if* $f \in \mathscr{A}$ *satisfies* $\mathrm{B}(f) \le 1$, *then* $f \in \mathscr{S}$. *Moreover, if* $\mathrm{B}(f) \le k < 1$, *then* f *extends to a* k-*quasiconformal mapping of the extended plane. The constants* 6 *and* 1 *are best possible.*

The sufficiency of univalence and quasiconformal extendibility is due to Becker [7]. The sharpness of the constant 1 is due to Becker and Pommerenke [9]. The sharp inequality $\mathrm{B}(f) \le 6$ follows from a standard inequality appearing in coefficient estimation (see, e.g., [10, Theorem 2.4]).

Theorem C. *Every* $F \in \Sigma$ *satisfies* $\mathrm{B}^*(F) \le 6$. *Conversely, if* $F \in \mathscr{M}$ *satisfies* $\mathrm{B}^*(F) \le 1$, *then* $F \in \Sigma$. *Moreover, if* $\mathrm{B}^*(F) \le k < 1$, *then* F *extends to a* k-*quasiconformal mapping of the extended plane. The constants* 6 *and* 1 *are best possible.*

The sufficiency of univalence and quasiconformal extendibility is due to Becker [8]. The sharpness of the constant 1 is again due to Becker and Pommerenke [9]. On the other hand, the estimate $\mathrm{B}^*(F) \le 6$ lies deeper. Avhadiev [4] first showed the sharp inequality $\mathrm{B}^*(F) \le 6$ by appealing to Goluzin's inequality (see [11, p. 139]).

Note that many authors use a different norm for the pre-Schwarzian derivative of $f \in \mathscr{A}$, namely, $\|T_f\| = \sup_{|z|<1}(1-|z|^2)|T_f(z)|$, see [12], [13], [15] and [20]. By definition, we observe $\mathrm{B}(f) \le \|T_f\|$. The norm $\|T_f\|$ has some advantage such as invariance properties. For meromorphic functions, however, the corresponding norm is not suitable (see [19, §4.2]).

Recall that a plane domain $\Omega \subset \mathbb{C}$ is called *hyperbolic* if $\partial\Omega$ contains at least two points. The uniformization theorem ensures existence of the (complete) hyperbolic metric $\rho_\Omega(w)|dw|$ on Ω with constant Gaussian curvature -4. Let Ω be a hyperbolic plane domain such that $1 \in \Omega$ but $0 \notin \Omega$ and set

$$\Pi(\Omega) = \{F \in \mathscr{M} : F'(\zeta) \in \Omega \text{ for all } \zeta \in \Delta\}.$$

Set also $\Pi_n(\Omega) = \Pi(\Omega) \cap \mathscr{M}_n$ for $n = 0, 1, 2, \ldots$.

In [14], the quantity

$$W(\Omega) = \sup_{w \in \Omega} \frac{1}{|w|\rho_\Omega(w)}$$

is studied and called the *circular width* of Ω. Note that the circular width can also be expressed by $W(\Omega) = \sup_{z\in\mathbb{D}}(1-|z|^2)|p'(z)/p(z)|$, where $p : \mathbb{D} \to \Omega$ is any analytic universal covering projection of $\mathbb{D}$ onto Ω (We do not demand the condition $p(0) = 1$). For concrete values of circular widths of specific domains, see [14].

One of our main results in the present paper is an estimate of $\mathrm{B}^*(F)$ for $F \in \Pi_n(\Omega)$. The proof of the following theorem will be given in Section 2.

Theorem 1. *Let* Ω *be a hyperbolic domain such that* $1 \in \Omega$ *but* $0 \notin \Omega$. *For every* $F \in \Pi_n(\Omega)$, $n \geq 0$, *the inequality*

$$\mathrm{B}^*(F) \leq C_n W(\Omega)$$

holds, where C_n *are the constants given by* $C_0 = 2$ *and*

$$C_n = \sup_{0<r<1} \frac{(n+1)(1-r^2)r^{n-1}}{1-r^{2n+2}}, \quad n \geq 1. \tag{1.3}$$

As we shall show later (see Proposition 5), we have $C_1 = 2$ and $1 < C_n < (n+1)/n$ for $n \geq 2$. We note that an analytic counterpart of this theorem is known and it is much simpler to prove (see [13, Theorem 4.1]);

$$\mathrm{B}(f) \leq \|T_f\| \leq W(\Omega)$$

holds for $f \in \mathscr{A}$ *with* $f'(\mathbb{D}) \subset \Omega$.

The univalence criterion in the following is due to Aksent'ev [2] (see also [6, p. 11]). Later, Krzyż [17] gave quasiconformal extensions.

Theorem D (Aksent'ev, Krzyż)**.** *Let* $0 \leq k \leq 1$. *If* $F \in \mathscr{M}$ *satisfies the inequality*

$$|F'(\zeta) - 1| \leq k, \quad |\zeta| > 1, \tag{1.4}$$

then F *is univalent. Furthermore, if* $k < 1$, *then* F *extends to a* k*-quasiconformal mapping of the extended plane. The radii* 1 *and* k *are best possible.*

The above criterion implies univalence of $F \in \mathscr{M}$ when the range of F' is contained in the disk $|w - 1| < 1$. We remind the reader of the fact that the Noshiro-Warschawski theorem asserts that the condition $\mathrm{Re}\, f' > 0$ is sufficient for $f \in \mathscr{A}$ to be univalent (cf. [10, Theorem 2.16]). However, the meromorphic counterpart does not hold. Moreover, the range of F' cannot be enlarged to any disk of the form $|w - r| < r$, $r > 1$, to ensure univalence of F (Aksent'ev and Avhadiev [3], see also §4).

Applying Theorem 1 to specific domains Ω, we have several results similar to Theorem D. The following are a couple of examples. Note that the univalence criteria in Theorems 2 and 3 for the case $n = 0$ were first given by Avhadiev and Aksent'ev [5].

Let x_m be the unique solution to the equation

$${}_2F_1(1, -\tfrac{1}{m}; 1 - \tfrac{1}{m}; x) = \frac{1}{2}$$

in the interval $0 < x < 1$ for each integer $m \geq 2$ (see Section 4 for details). Put also $x_1 = x_2$.

Theorem 2. *Let* $n \geq 0$ *and* $0 \leq k \leq 1$. *Suppose that a function* $F \in \mathscr{M}_n$ *satisfies the condition*

$$|\arg F'(\zeta)| \leq \frac{k\pi}{4C_n}, \quad |\zeta| > 1,$$

then F must be univalent. Furthermore, if $k < 1$, then F extends to a k-quasiconformal mapping of the extended plane. As for univalence, the constant $\pi/(4C_n)$ cannot be replaced by any larger number than $(4/\pi)\arctan x_{n+1}$.

Note that $x_1 = x_2 \approx 0.4198$ and that

$$(4/\pi)\arctan x_1 \approx 0.506057 \approx 1.28866(\pi/8).$$

In the following univalence criterion, F' is even allowed to take values with negative real part. Let β_m be the unique solution to the equation

$$2\beta \int_0^{\pi/4} (\cot x)^{1/m} e^{2\beta(x-\pi/4)} dx = 1 \tag{1.5}$$

in $0 < \beta < \infty$ for each integer $m \geq 2$ (see Example 11 in Section 4). Set $\beta_1 = \beta_2$.

Theorem 3. *Let $n \geq 0$ and $0 \leq k \leq 1$. Suppose that a function $F \in \mathscr{M}_n$ satisfies the condition*

$$|\log |F'(\zeta)|| \leq \frac{k\pi}{4C_n}, \quad |\zeta| > 1,$$

then F must be univalent. Furthermore, if $k < 1$, then F extends to a k-quasiconformal mapping of the extended plane. As for univalence, the constant $\pi/(4C_n)$ cannot be replaced by any larger number than $\pi\beta_{n+1}/2$.

A numerical computation gives $\pi\beta_1/2 \approx 0.719122 \approx 1.83123(\pi/8)$. These results can also be translated into those for the functions $f \in \mathscr{A}$ by using the relation (1.2). The proofs of the above theorems and slightly more refined results will be presented in Section 5.

2. Proof of Theorem 1

Let Ω be a plane domain with $1 \in \Omega$ and $0, \infty \in \widehat{\mathbb{C}} \setminus \Omega$ and let p be an analytic universal covering map of $\mathbb{D}$ onto Ω with $p(0) = 1$. Let $F \in \Pi_n(\Omega)$ be given. When $n = 0$, the function F can be expressed in the form $F = F_0 + b_0$, where $F_0 \in \mathscr{M}_1$ and b_0 is a constant and hence $F_0 \in \Pi_1(\Omega)$. Recall that $C_0 = C_1 = 2$. Therefore, we may further assume that $n \geq 1$.

Let $\omega : \mathbb{D} \to \mathbb{D}$, $\omega(0) = 0$, be the lift of the mapping $z \mapsto F'(1/z)$ of $\mathbb{D}$ into Ω via the covering map $p : \mathbb{D} \to \Omega$, namely,

$$F'\left(\frac{1}{z}\right) = p(\omega(z)), \quad |z| < 1. \tag{2.1}$$

Since $F \in \mathscr{M}_n$, it can be expressed in the form

$$F(\zeta) = \zeta + \sum_{k=n}^{\infty} b_k \zeta^{-k}, \quad |\zeta| > 1,$$

we have

$$F'(1/z) = 1 - \sum_{k=n}^{\infty} kb_k z^{k+1} = 1 - \sum_{k=n+1}^{\infty} (k-1)b_{k-1}z^k, \quad |z| < 1.$$

In particular, ω has a zero of at least order $n+1$ at the origin. This implies that the function $\varphi(z) = \omega(z)/z^{n+1}$ is analytic and satisfies $|\varphi(z)| \le 1$ by the maximum modulus principle. We now apply the Schwarz-Pick lemma to the function φ to get

$$|\varphi'(z)| \le \frac{1 - |\varphi(z)|^2}{1 - |z|^2}, \quad |z| < 1,$$

and equivalently,

$$|z\omega'(z) - (n+2)\omega(z)| \le \frac{|z|^{2n+2} - |\omega(z)|^2}{|z|^n(1 - |z|^2)}, \quad |z| < 1. \tag{2.2}$$

In particular, we obtain

$$|z\omega'(z)| \le (n+2)|\omega(z)| + \frac{|z|^{2n+2} - |\omega(z)|^2}{|z|^n(1 - |z|^2)}, \quad |z| < 1. \tag{2.3}$$

The last inequality can be expressed by

$$(1 - |z|^2)|z|^{-1}|\omega'(z)| \le (1 - |\omega(z)|^2)F(|z|, |\omega(z)|), \quad |z| < 1, \tag{2.4}$$

where the function $F(r,s)$ is defined by

$$F(r,s) = \frac{(n+1)(1-r^2)r^n s + r^{2n+2} - s^2}{r^{n+2}(1-s^2)}.$$

Since $|\varphi(z)| \le 1$, we see that $|\omega(z)| \le |z|^{n+1}$ holds. We now show the following elementary result.

Lemma 4.

$$F(r,s) \le F(r, r^{n+1}) = \frac{(n+1)(1-r^2)r^{n-1}}{1 - r^{2n+2}}, \quad 0 \le s \le r^{n+1}.$$

Proof. We first see the inequality

$$\begin{aligned}
\frac{\partial F}{\partial s}(r,s) &= \frac{1+s^2}{r^{n+2}(1-s^2)^2}\left[(n+1)r^n(1-r^2) - 2(1-r^{2n+2})\frac{s}{1+s^2}\right] \\
&\ge \frac{1+s^2}{r^{n+2}(1-s^2)^2}\left[(n+1)r^n(1-r^2) - 2(1-r^{2n+2})\frac{r^{n+1}}{1+r^{2n+2}}\right] \\
&= \frac{(1+s^2)}{r^2(1-s^2)^2(1+r^{2n+2})}G(r), \quad 0 \le s \le r^{n+1},
\end{aligned}$$

because the function $s/(1+s^2)$ is increasing in $0 < s < 1$ and $s \le r^{n+1}$ is assumed. Here,

$$\begin{aligned} G(r) &= (n+1)(1-r^2)(1+r^{2n+2}) - 2r(1-r^{2n+2}) \\ &= (1-r^2)\left[(n+1)(1+r^{2n+2}) - 2r\sum_{j=0}^{n} r^{2j}\right] \\ &= (1-r^2)\left[(n+1)(1+r^{2n+2}) - r\sum_{j=0}^{n}(r^{2j}+r^{2n-2j})\right] \\ &= (1-r^2)\sum_{j=0}^{n}\left[(1+r^{2n+2}) - r(r^{2j}+r^{2n-2j})\right] \\ &= (1-r^2)\sum_{j=0}^{n}(1-r^{2j+1})(1-r^{2n+1-2j}) > 0. \end{aligned}$$

Therefore, we conclude that $(\partial F/\partial s)(r,s) > 0$ in $0 < s < r^{n+1}$, which implies the monotonicity of the function $F(r,s)$ in s. Thus the inequality $F(r,s) \le F(r, r^{n+1})$ holds in $0 \le s \le r^{n+1}$. □

We now complete the proof of Theorem 1. By taking the logarithmic derivative of the both sides of (2.1), we have the relation

$$\frac{-F''(1/z)}{z^2F'(1/z)} = \frac{p'(\omega(z))}{p(\omega(z))}\omega'(z), \quad |z| < 1.$$

Letting $\zeta = 1/z$, we thus obtain

$$(|\zeta|^2 - 1)\left|\frac{\zeta F''(\zeta)}{F'(\zeta)}\right| = (1-|z|^2)|z|^{-1}\left|\frac{p'(\omega(z))}{p(\omega(z))}\right||\omega'(z)|.$$

Recall here that C_n is nothing but the supremum of $F(r, r^{n+1})$ over $0 < r < 1$. We then make use of (2.4) and Lemma 4 to deduce the inequality

$$\begin{aligned} (|\zeta|^2 - 1)\left|\frac{\zeta F''(\zeta)}{F'(\zeta)}\right| &\le (1-|\omega(z)|^2)\left|\frac{p'(\omega(z))}{p(\omega(z))}\right| F(|z|, |z|^{n+1}) \\ &\le C_n(1-|\omega(z)|^2)\left|\frac{p'(\omega(z))}{p(\omega(z))}\right| \\ &\le C_n W(\Omega). \end{aligned}$$

The assertion of the theorem now follows.

Remark. The theorem is sharp if the relation $\rho_0 = r_0^{n+1}$ is satisfied by chance, where $r = r_0$ is the point where the maximum is attained in the definition of C_n and $r = \rho_0$ is the radius where the maximum is attained for $(1-|z|^2)|p'(z)/p(z)|$. Let w_0 be the maximum point of $(1-|z|^2)|p'(z)/p(z)|$ with $|w_0| = \rho_0$, and set $z_0 = r_0$. Then we choose ω_0 so that $\omega_0(z_0) = w_0$ and equalities hold in (2.2) and (2.3) at $z = z_0$ simultaneously (see the proof of Dieudonné's lemma in [10, p. 198]). Then, we actually have $\mathrm{B}^*(F) = C_n W(\Omega)$ in this case, where F is determined by $F'(1/z) = p(\omega_0(z))$ in $|z| < 1$.

As we mentioned in Section 1, we give some information about the constants C_n.

Proposition 5. *The constants* C_n *given by* (1.3) *satisfy the following*:

$$C_0 = C_1 = 2, \quad C_2 = \frac{3\sqrt{6(\sqrt{13}-1)}}{5+\sqrt{13}} \approx 1.37838, \tag{2.5}$$

$$1 < C_n < \frac{n+1}{n}, \quad n = 2, 3, \dots. \tag{2.6}$$

Proof. The relations in (2.5) can be checked in a straightforward way. We omit the details. We show only (2.6). Let $n \geq 2$ and set

$$g_n(x) = \frac{1 - x^{n+1}}{x^{(n-1)/2}(1-x)}, \quad x \in (0,1).$$

Then clearly, $C_n = (n+1)/\inf_{0<x<1} g_n(x)$. First note that

$$\lim_{x \to 1} g_n(x) = n + 1.$$

Therefore, we have $C_n \geq 1$. In order to show strictness, we set $x = 1 - \varepsilon$, $\varepsilon > 0$. Then

$$g_n(1-\varepsilon) = (n+1) - \frac{n+1}{2}\varepsilon + O(\varepsilon^2), \quad \varepsilon \to 0,$$

which implies that $g_n(x)$ is smaller than $n+1$ when $x < 1$ is close enough to 1. Therefore, $C_n > 1$.

We next show the reverse inequality. Since $g_n(x) \to +\infty$ as $x \to 0+$, the function g_n takes its minimum at a point in $(0,1)$. We now estimate $g_n(x)$ from below;

$$\begin{aligned}
g_n(x) &= x^{(1-n)/2} \sum_{j=0}^{n} x^j \\
&> x^{(1-n)/2} \sum_{j=0}^{n-1} x^j \\
&= x^{(1-n)/2} \sum_{j=0}^{n-1} \frac{x^j + x^{n-1-j}}{2} \\
&= \sum_{j=0}^{n-1} \frac{x^{j-(n-1)/2} + x^{(n-1)/2-j}}{2} \\
&\geq \sum_{j=0}^{n-1} 1 = n.
\end{aligned}$$

Thus we get the inequality $\min_{0<x\le 1} g_n(x) > n$, which in turn implies $C_n < (n+1)/n$. $\square$

3. A variant of Theorem 1

We give a variant of Theorem 1 in the present section. In the following theorem, the condition $p(0) = 1$ for the analytic universal covering map p of $\mathbb{D}$ onto Ω is required and the constant involved might not be computed easily, but the estimate is independent of n and better than Theorem 1 at least when $n = 0$.

Theorem 6. *Let Ω be a plane domain with $1 \in \Omega$ but $0, \infty \notin \Omega$ and let p be an analytic universal covering map of the unit disk $\mathbb{D}$ onto Ω with $p(0) = 1$. Then, for every $F \in \Pi(\Omega)$ the inequality*

$$\mathrm{B}^*(F) \le 2 \sup_{|z|<1} (1 - |z|) \left| \frac{p'(z)}{p(z)} \right|$$

holds.

Proof. The proof proceeds basically in the same line as in the previous section. In order to show that the constant is really independent of n for which $F \in \Pi_n(\Omega)$ holds, we prove the assertion under the additional assumption that $F \in \Pi_n(\Omega)$ for a fixed $n \ge 1$. We replace the inequality (2.4) by

$$(1 - |z|^2)|z|^{-1}|\omega'(z)| \le (1 - |\omega(z)|)H(|z|, |\omega(z)|), \quad |z| < 1, \tag{3.1}$$

where

$$H(r, s) = \frac{(n + 1)(1 - r^2)r^n s + r^{2n+2} - s^2}{r^{n+2}(1 - s)}.$$

Recall here that $|\omega(z)| \le |z|^{n+1}$ holds. Since the function $s^2 - 2s$ is decreasing in $0 < s < r^{n+1}$, we have

$$\begin{aligned} \frac{\partial H}{\partial s}(r, s) &= \frac{s^2 - 2s + (n + 1)(1 - r^2)r^n + r^{2n+2}}{r^{n+2}(1 - s)^2} \\ &\ge \frac{r^{2n+2} - 2r^{n+1} + (n + 1)(1 - r^2)r^n + r^{2n+2}}{r^{n+2}(1 - s)^2}. \end{aligned}$$

The numerator of the last term can be written in the form

$$\begin{aligned} & r^n \left[(n + 1)(1 - r^2) - 2r(1 - r^{n+1})\right] \\ &= r^n(1 - r) \left[(n + 1)(1 + r) - 2r(1 + r + r^2 + \cdots + r^n)\right] \\ &= r^n(1 - r) \sum_{j=0}^{n} \left(1 + r - 2r^{j+1}\right). \end{aligned}$$

It is now clear that $(\partial H/\partial s)(r, s) > 0$ in $0 < s \le r^{n+1}$. Thus $H(r, s)$ is increasing in s and therefore

$$H(r, s) \le H(r, r^{n+1}) = \frac{(n + 1)(1 - r^2)r^{n-1}}{1 - r^{n+1}} = g(r).$$

Since

$$g'(r) = \frac{(n+1)r^{n-2}\big((n-1)(1-r^2) - 2r^2(1-r^{n-1})\big)}{(1-r^{n+1})^2}$$
$$= \frac{(n+1)r^{n-2}(1-r)}{(1-r^{n+1})^2} \sum_{j=0}^{n-2} \Big[1 - r^{j+2} + r(1-r^{j+1})\Big] > 0,$$

the function $g(r)$ is increasing and thus $g(r) < g(1-) = 2$ for $0 \le r < 1$. Therefore, we obtain

$$\sup_{0<s\le r^{n+1}<1} H(r,s) = \sup_{0<r<1} g(r) = 2,$$

which is, indeed, independent of n.

The rest is same as in the previous section. We omit the details. $\square$

Since $1 - r \le 1 - r^2 = (1+r)(1-r) \le 2(1-r)$, the inequalities

$$\sup_{|z|<1} (1-|z|) \left|\frac{p'(z)}{p(z)}\right| \le W(\Omega) = \sup_{|z|<1} (1-|z|^2) \left|\frac{p'(z)}{p(z)}\right| \le 2 \sup_{|z|<1} (1-|z|) \left|\frac{p'(z)}{p(z)}\right|$$

hold. Thus, when $n = 0$, the estimate in Theorem 6 is better than that in Theorem 1.

4. Examples of non-univalent functions

In this section, we present non-univalent meromorphic functions in the class $\mathscr{M}$ to examine our univalence criteria given in introduction. First, we introduce the example given by Aksent'ev and Avhadiev [3].

Example 7. Let $r > 1$ be given and set $\Omega = \{w \in \mathbb{C} : |w - r| < r\}$. For a number $c \in (0, 1/2]$, we set $\Phi = G \circ F$, where $F(\zeta) = \zeta + c/\zeta$ and $G(\zeta) = \zeta + (1+c)^2/\zeta$. Then

$$\Phi'(\zeta) = 1 - \zeta^{-2} + c\psi(\zeta^{-2}),$$

where

$$\psi(z) = \psi_c(z) = -\frac{(c+3) - (c^2+3)z + (c^2-c)z^2}{(1+cz)^2}.$$

Note that the functions $1 - \zeta^{-2}$ and $\psi(\zeta^{-2})$ take the value 0 at $\zeta = \pm 1$. Since ψ_c is uniformly bounded in $\mathbb{D}$ and $\psi'(1) > 0$, in order to see that $F'(\mathbb{D}) \subset \Omega$ for sufficiently small c, it is enough to check that the (signed) curvature of the curve $\theta \mapsto \psi(e^{i\theta})$ is positive at $\theta = 0$, in other words, $\mathrm{Re}\,(1 + z\psi''(z)/\psi'(z))/|\psi'(z)|$ is positive at $z = 1$. A direct computation gives

$$1 + \frac{z\psi''(z)}{\psi'(z)} = \frac{3 - 10c + 2(c^2+c)z - c^2z^2}{(3-cz)(1+cz)},$$

which shows $\mathrm{Re}\,(1 + \psi''(1)/\psi'(1))/|\psi'(1)| > 0$ for a small enough $c > 0$ as required.

We see now that Φ is not univalent in Δ by observing that the two points $\pm i(1 + c + \sqrt{1 + 6c + c^2})/2$ in Δ are zeros of Φ.

The above example is qualitatively very nice but somewhat implicit because it is not simple to give a right value of c for a given $r > 1$. The next two examples are more concrete.

Example 8. We consider the function $F_m \in \mathscr{M}$ given by

$$\begin{aligned} F_m(\zeta) &= \zeta - 2\sum_{j=1}^{\infty} \frac{\zeta^{1-mj}}{mj-1} \\ &= \zeta\Big(2\,{}_2F_1(1, -\tfrac{1}{m}; 1-\tfrac{1}{m}; \zeta^{-m}) - 1\Big), \quad |\zeta| > 1, \end{aligned}$$

for each integer $m \geq 2$, where ${}_2F_1(a, b; c; x)$ stands for the hypergeometric function. Note that F_m has the m-fold symmetry

$$F_m(e^{2\pi i/m}\zeta) = e^{2\pi i/m} F_m(\zeta)$$

and belongs to the class $\mathscr{M}_{m-1}$. Since the function h_m defined by

$$h_m(x) = 2\,{}_2F_1(1, -\tfrac{1}{m}; 1-\tfrac{1}{m}; x) - 1 \quad (x \in (0,1))$$

has the properties that h_m is monotone decreasing, $h_m(0) = 1$ and $\lim_{x\to 1^-} h_m(x) = -\infty$, there is the unique point x_m such that $h(x_m) = 0$ in the interval $0 < x < 1$. Hence, the function F_m has the m zeros $e^{2\pi ij/m} x_m^{-1/m}$, $j = 0, 1, \ldots, m-1$, in Δ and, in particular, is not univalent in Δ. On the other hand, we have

$$F_m'(\zeta) = 1 + 2\sum_{j=1}^{\infty} \zeta^{-mj} = p(\zeta^{-m}),$$

where $p(z) = (1+z)/(1-z)$. It is a standard fact that p maps the unit disk onto the right half-plane $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$. Therefore, F_m' maps Δ onto $\mathbb{H}$ in an m-to-1 way and $\operatorname{Re} F_m' > 0$ holds.

In particular, we have shown the following.

Proposition 9. *For each integer $n \geq 0$, there is a non-univalent function F in the class $\mathscr{M}_n$ such that $\operatorname{Re} F'(\zeta) > 0$ in $|\zeta| > 1$.*

Note that the function F_2 in the above example can be expressed also by

$$F_2(\zeta) = \zeta - \log\frac{\zeta+1}{\zeta-1}, \quad |\zeta| > 1.$$

A numerical computation yields, for instance,

$$\begin{aligned} x_2 &\approx 0.419798, \\ x_3 &\approx 0.667508, \\ x_4 &\approx 0.808289. \end{aligned}$$

The above functions can be used to examine univalence criteria. Note that, for a function $F \in \mathscr{M}$, the new function

$$F^t(\zeta) = tF\Big(\frac{\zeta}{t}\Big), \quad |\zeta| > 1,$$

for $t \in (0,1)$ satisfies the relation $(F^t)'(\zeta) = F'(\zeta/t)$. For instance, for $m \geq 2$, the function $F_m^t(\zeta) = tF_m(\zeta/t)$ is not univalent as far as $t^m > x_m$, because $(\zeta/t)^{-m} = x_m$ has m roots in $|\zeta| > 1$ in this case. On the other hand, $(F_m^t)'$ has the range $\{w \in \mathbb{C} : w = (1+t^m z)/(1-t^m z) \text{ for some } z \in \mathbb{D}\} = \{w \in \mathbb{C} : |w - (1+t^{2m})/(1-t^{2m})| < 2t^m/(1-t^{2m})\}$. In this way, we have shown the following.

Proposition 10. *Let $\Omega_s = \{w \in \mathbb{C} : |w-(1+s^2)/(1-s^2)| < 2s/(1-s^2)\}$ and $n \geq 1$. If $s > x_{n+1}$, then the class $\Pi_n(\Omega_s)$ contains non-univalent functions.*

Example 11. The construction is similar to that of Example 8. First note that the analytic function $((1+z)/(1-z))^{i\beta}$ gives a universal covering projection of the unit disk onto the annulus $A = \{w \in \mathbb{C} : e^{-\pi\beta/2} < |w| < e^{\pi\beta/2}\}$ for a positive constant β. Let $G \in \mathscr{M}_{m-1}$ be the function determined by the relation $G'(\zeta) = ((\zeta^m+1)/(\zeta^m-1))^{i\beta}$ for an integer $m \geq 2$. Then G also has the m-fold symmetry. Let $h_\beta(z) = 1/z - \int_0^z t^{m-2} q_\beta(t^m)dt$, where $((1+z)/(1-z))^{i\beta} = 1 + zq_\beta(z)$, so that $G(\zeta) = h_\beta(1/\zeta)$. Now take any root ω of the polynomial $z^m + i$ and set $\varphi(\beta) = \omega h_\beta(\omega)$. Since $1 + ixq_\beta(ix) = ((1+ix)/(1-ix))^{i\beta} = \exp(2i\beta \operatorname{arctanh}(ix)) = \exp(-2\beta \arctan x)$, we have for $0 < r \leq 1$

$$\begin{aligned}\omega h_\beta(\omega r) &= \frac{1}{r} + \int_0^r it^{n-2} q_\beta(-it^m)dt \\ &= \frac{1}{r} - \int_0^r \left(\exp(2\beta \arctan(t^m)) - 1\right) t^{-2} dt.\end{aligned}$$

Thus,

$$\varphi(\beta) = 1 - \int_0^1 \left(\exp(2\beta \arctan(t^m)) - 1\right) t^{-2} dt.$$

Since $\varphi(0) = 1, \varphi(+\infty) = -\infty$ and

$$\varphi'(\beta) = -\int_0^1 t^{-2} \arctan(t^m) \exp(2\beta \arctan(t^m)) dt < 0,$$

there exists a unique β_m such that $\varphi(\beta_m) = 0$. We now simplify the equation $\varphi(\beta) = 0$. Performing integration by parts and then setting $x = \arctan(t^m)$, we have

$$\begin{aligned}\varphi(\beta) &= e^{\pi\beta/2} - 2\beta \int_0^{\pi/4} e^{2\beta x} (\tan x)^{-1/m} dx \\ &= e^{\pi\beta/2} \left(1 - 2\beta \int_0^{\pi/4} e^{2\beta(x-\pi/4)} (\cot x)^{1/m} dx\right).\end{aligned}$$

Thus we have arrived at the form in (1.5).

We now fix any $\beta > \beta_m$. Then $\omega h_\beta(\omega r) > 0$ for a small enough $r > 0$ whereas $\varphi(\beta) = \omega h_\beta(\omega) < 0$. Therefore, there exists an $\rho \in (0,1)$ such that $G(1/(\omega\rho)) = h_\beta(\omega\rho) = 0$. In particular, G has at least m zeros in Δ and thus is not univalent. By the above observations, we have the following proposition.

Proposition 12. *Let n be an integer with $n \geq 1$ and let $\beta > \beta_{n+1}$. Then there exists a non-univalent function $G \in \mathscr{M}_n$ such that $e^{-\pi\beta/2} < |G'(\zeta)| < e^{\pi\beta/2}$ for $|\zeta| > 1$.*

By a numerical computation, one has

$$\begin{aligned}\beta_2 &\approx 0.457807,\\ \beta_3 &\approx 0.786518,\\ \beta_4 &\approx 1.03144.\end{aligned}$$

5. Applications to univalence criteria

We combine Theorem 1 or Theorem 6 with Theorem C to deduce several univalence criteria for functions in $\mathscr{M}$. The same method can be applied also to $\mathscr{M}_n$ for $n \geq 1$, but we do not go into details here. In order to make statements concise, we introduce the notation $\Sigma(k)$ to designate the set of those functions in Σ which can be extended to k-quasiconformal mappings of the extended plane. For $k = 1$, simply we define $\Sigma(1) = \Sigma$ for convenience.

To examine Theorems 1 and 6, we assume Ω to be a disk containing 1 but not containing 0. Then we can express Ω as $\mathbb{D}(a, \rho) = \{w : |w - a| < \rho\}$, where $0 < \rho \leq |a|$ and $|1 - a| < \rho$. If we put $p(z) = a + \rho z$, then we compute

$$\begin{aligned} W(\mathbb{D}(a,\rho)) &= \sup_{z\in\mathbb{D}} (1-|z|^2)\frac{\rho}{|a+\rho z|} \\ &= \sup_{0<r<1} (1-r^2)\frac{\rho}{|a|-\rho r} \\ &= \frac{\rho}{|a|}\sup_{0<r<1} \frac{1-r^2}{1-(\rho/|a|)r} \\ &= \frac{2\rho/|a|}{1+\sqrt{1-(\rho/|a|)^2}}, \end{aligned}$$

where we have made a standard but tedious computation at the final step (see, for instance, [15, Lemma 4.2]). Therefore, by Theorem 1, we conclude that

$$\mathrm{B}^*(F) \leq \frac{2C_n\rho/|a|}{1+\sqrt{1-(\rho/|a|)^2}} \tag{5.1}$$

for $F \in \Pi_n(\mathbb{D}(a, \rho))$. It is easy to see that the right-hand side of the last inequality is less than or equal to k if and only if $\rho/|a| \leq 4C_n k/(4C_n^2 + k^2)$. Thus we can show the following by appealing to Theorem C.

Theorem 13. *Let n be an integer with $n \geq 0$ and $a \in \mathbb{C}$, $\rho > 0$ satisfy $\rho \leq |a|$ and $|a - 1| < \rho$. Suppose that*

$$\frac{\rho}{|a|} \leq \frac{4C_n k}{4C_n^2 + k^2}$$

for a constant k with $0 \leq k \leq 1$. Then $\Pi_n(\mathbb{D}(a, \rho)) \subset \Sigma(k)$.

We recall that Theorem D gives the stronger assertion $\Pi(\mathbb{D}(1,k)) \subset \Sigma(k)$ when $a = 1$ and $\rho = k$.

We next consider to apply Theorem 6. It is not simple to treat the case when a is not real. Therefore, we further assume that $a > 0$ for simplicity. Then the conformal map p of $\mathbb{D}$ onto $\mathbb{D}(a,\rho)$ with $p(0) = 1$ can be taken in the form $p(z) = (1+Az)/(1+Bz)$, where $-1 < B < A \le 1$. A simple computation gives us the relations $A = (\rho^2 - a^2 + a)/\rho$ and $B = (1-a)/\rho$.

First observe the expression (see [15, Lemma 4.1])

$$W = \sup_{z\in\mathbb{D}}(1-|z|)\left|\frac{p'(z)}{p(z)}\right| = \begin{cases} (A-B)\displaystyle\sup_{0<r<1}\frac{1-r}{(1-Ar)(1-Br)} & \text{if } A+B \le 0,\\ (A-B)\displaystyle\sup_{0<r<1}\frac{1-r}{(1+Ar)(1+Br)} & \text{if } A+B \ge 0.\end{cases}$$

At any event, we can easily see that $W = A - B$. Therefore, by Theorem 6, we obtain the estimate

$$\mathrm{B}^*(F) \le 2(A-B) = \frac{2(\rho^2-(a-1)^2)}{\rho} \tag{5.2}$$

for $F \in \Pi(\mathbb{D}(a,\rho))$. In the same way as above, we have the following.

Theorem 14. *Let $a > 0,\ \rho > 0$ satisfy $\rho \le a$ and $|a-1| < \rho$. Suppose that*

$$\rho^2 - (a-1)^2 \le \frac{k\rho}{2}$$

for a constant k with $0 \le k \le 1$. Then $\Pi(\mathbb{D}(a,\rho)) \subset \Sigma(k)$.

As an example, let us consider the disk $\Omega_s = \{w \in \mathbb{C} : |w-(1+s^2)/(1-s^2)| < 2s/(1-s^2)\}$. In this case, $A = s, B = -s$, and therefore,

$$\frac{4\rho/|a|}{1+\sqrt{1-(\rho/|a|)^2}} = 4s = 2(A-B),$$

which means that the estimates (5.1) with $n = 0$ and (5.2) are identical in this case. In particular, Theorems 13 and 14 both imply that $\Pi(\Omega_s) \subset \Sigma$ if $s \le 1/4$. This is, however, weaker than Theorem D because $\Omega_s \subset \mathbb{D}(1,1)$ for $s \le 1/3$. On the other hand, Proposition 10 implies that $\Pi(\Omega_s)$ is not contained in Σ for $s > x_2 \approx 0.4198$.

However, Theorems 13 and 14 may imply the inclusion $\Pi(\mathbb{D}(a,\rho)) \subset \Sigma$ for a disk $\mathbb{D}(a,\rho)$ which is not contained in $\mathbb{D}(1,1)$. For instance, by Theorem 14, we see that $\Pi(\mathbb{D}(3/2,4/5)) \subset \Sigma$ but $\mathbb{D}(3/2,4/5)$ is not a subset of $\mathbb{D}(1,1)$. By the way, this is not implied by Theorem 13.

We next recall basic results for the values of $W(\Omega)$ for special domains Ω. We set

$$S(\alpha,\gamma) = \{w \in \mathbb{C} : |\arg w - \gamma| < \pi\alpha/2\}$$
$$A(r_1,r_2) = \{w \in \mathbb{C} : r_1 < |w| < r_2\},$$

where $0 < \alpha \le 2,\ \gamma \in \mathbb{R}$ and $0 < r_1 < r_2 < \infty$. The domain $S(\alpha,\gamma)$ is called a sector with opening $\pi\alpha$ and vertex at 0. The domain $A = A(r_1,r_2)$ is called a round annulus centered at 0 with modulus $m = \log(r_2/r_1)$. We write $m = \operatorname{mod} A$. Then we have the following.

Lemma 15 ([14]).

$$W(S(\alpha,\gamma)) = 2\alpha, \quad 0 < \alpha \le 2,$$
$$W(A(r_1,r_2)) = \frac{2}{\pi}\log\frac{r_2}{r_1} = \frac{2}{\pi}\operatorname{mod} A(r_1,r_2), \quad 0 < r_1 < r_2 < \infty.$$

Combining this lemma with Theorems 1 and C, we can prove the following two results. Theorems 2 and 3 are just special cases of them up to non-univalent examples, which were supplied in the previous section.

Theorem 16. *Let $0 \le k \le 1$. If Ω is a sector with opening $k\pi/4$ and vertex at 0 such that $1 \in \Omega$, then $\Pi(\Omega) \subset \Sigma(k)$.*

Theorem 17. *Let $0 \le k \le 1$. If Ω is a round annulus centered at 0 with modulus $k\pi/4$ such that $1 \in \Omega$, then $\Pi(\Omega) \subset \Sigma(k)$.*

References

[1] L. V. Ahlfors and G. Weill, *A uniqueness theorem for Beltrami equations*, Proc. Amer. Math. Soc. **13** (1962), 975–978.

[2] L. A. Aksent'ev, *Sufficient conditions for univalence of regular functions* (Russian), Izv. Vysš. Učebn. Zaved. Matematika **1958** (1958) no. 3 (4), 3–7.

[3] L. A. Aksent'ev and F. G. Avhadiev, *A certain class of univalent functions* (Russian), Izv. Vysš. Učebn. Zaved. Matematika **1970** (1970) no. 10, 12–20.

[4] F. G. Avhadiev, *Conditions for the univalence of analytic functions* (Russian), Izv. Vysš. Učebn. Zaved. Matematika **1970** (1970) no. 11 (102), 3–13.

[5] F. G. Avhadiev and L. A. Aksent'ev, *Sufficient conditions for the univalence of analytic functions* (Russian), Dokl. Akad. Nauk SSSR **198** (1971), 743–746, English translation in Soviet Math. Dokl. **12** (1971), 859–863.

[6] ———, *Fundamental results on sufficient conditions for the univalence of analytic functions* (Russian), Uspehi Mat. Nauk **30** (1975), no. 4(184), 3–60, English translation in Russian Math. Surveys **30** (1975), 1–64.

[7] J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. **255** (1972), 23–43.

[8] ———, *Löwnersche Differentialgleichung und Schlichtheitskriterien*, Math. Ann. **202** (1973), 321–335.

[9] J. Becker and Ch. Pommerenke, *Schlichtheitskriterien und Jordangebiete*, J. Reine Angew. Math. **354** (1984), 74–94.

[10] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 259. Springer-Verlag, New York, 1983.

[11] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Translations of Mathematical Monographs, Vol. 26 American Mathematical Society, Providence, R.I. 1969.

[12] Y. C. Kim, S. Ponnusamy, and T. Sugawa, *Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives*, J. Math. Anal. Appl. **299** (2004), no. 2, 433–447.

[13] Y. C. Kim and T. Sugawa, *Growth and coefficient estimates for uniformly locally univalent functions on the unit disk*, Rocky Mountain J. Math. **32** (2002), no. 1, 179–200.

[14] ______, *A conformal invariant for nonvanishing analytic functions and its applications*, Michigan Math. J. **54** (2006), no. 2, 393–410.

[15] ______, *Norm estimates of the pre-Schwarzian derivatives for certain classes of univalent functions*, Proc. Edinb. Math. Soc. (2) **49** (2006), no. 1, 131–143.

[16] S. L. Krushkal, *Quasiconformal extensions and reflections*, Handbook of Complex Analysis: Geometric Function Theory. Vol. 2, 507–553, Elsevier, Amsterdam, 2005.

[17] J. G. Krzyż, *Convolution and quasiconformal extension*, Comment. Math. Helv. **51** (1976), no. 1, 99–104.

[18] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.

[19] T. Sugawa, *The universal Teichmüller space and related topics*, Proceedings of the International Workshop on Quasiconformal Mappings and their Applications (India) (S. Ponnusamy, T. Sugawa, and M. Vuorinen, eds.), Narosa Publishing House, 2007, pp. 261–289.

[20] S. Yamashita, *Norm estimates for function starlike or convex of order alpha*, Hokkaido Math. J. **28** (1999), no. 1, 217–230.

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