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DECOMPOSITIONS OF COMPLETE MULTIPARTITE GRAPHS INTO GREGARIOUS 6-CYCLES USING COMPLETE DIFFERENCES

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Abstract. The complete multipartite graph $K_{n(2t)}$ having n partite sets of size $2t$, with $n \geq 6$ and $t \geq 1$, is shown to have a decomposition into *gregarious* 6-cycles, that is, the cycles which have at most one vertex from any particular partite set. Complete sets of differences of numbers in $\mathbb{Z}_n$ are used to produce starter cycles and obtain other cycles by rotating the cycles around the n-gon of the partite sets.

1. Introduction

Edge-disjoint decompositions of graphs into cycles of a fixed length have been considered in a number of different ways. After a series of developments, necessary and sufficient conditions for a complete graph of odd order, or a complete graph of even order minus an 1-factor, to have a decomposition into cycles of some fixed length have recently been obtained (see [1], [7] and [8] as well as their references). The key factor for all this work was the decomposition of complete bipartite graphs obtained by Sotteau ([9]). Many authors began to consider cycle decompositions with special properties such as resolvable cycle decompositions ([3], [4], [6]). Billington and Hoffman ([2]) introduced the notion of a *gregarious* cycle in a tripartite graph, and the notion of gregarious cycles has been modified in following papers ([2], [3], [5]).

A few years ago, Šajna ([8]) showed that the complete multipartite graph $K(2, 2, \ldots, 2)$ has a decomposition into m-cycles if and only if m divides the number of edges. However, the decomposition was by arbitrary cycles, not by gregarious ones. It seems that the requirement of gregariousness makes the problem more complicated. Recently, Billington and Hoffman ([2]) and Cho et el. ([5]) independently produced gregarious 4-cycle decompositions for certain complete multipartite graphs.

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In this paper, as a sequel to the earlier paper ([5]), we will consider complete multipartite graphs with partite sets of the same even cardinality and will show that these graphs have decompositions into gregarious 6-cycles if the numbers of edges is divisible by 6. Thus, the result in this article may be considered as a contribution to the decomposition problem in the direction of generalizing the results in [6] and [8]. When the size of partite sets is odd or the length of the cycle is odd, the problem seems to be more difficult to handle.

We first make our definition of gregarious cycles precise. We call a cycle in a multipartite graph *gregarious* if at most one vertex of the cycle comes from any particular partite set. For simplicity, we say that a graph is γ_6*-decomposable* if it is decomposable into γ_6*-cycles*, i.e., gregarious 6-cycles, and a decomposition into γ_6-cycles will be called a γ_6*-decomposition*.

Throughout the paper, $K_{n(2t)}$ will denote $K(2t, 2t, \ldots, 2t)$, the complete multipartite graph with n partite sets of $2t$ elements.

Now, we state the main theorem of the paper.

Theorem 1.1. *Let $n \geq 6$ and 6 divide $2n(n-1)$, the number of edges in $K_{n(2)}$. Then $K_{n(2t)}$ has a γ_6-decomposition for every positive integer t.*

We will prove the above theorem in the subsequent sections. In fact, we will prove the following special case of Theorem 1.1, and then will obtain Theorem 1.1 as a corollary.

Theorem 1.1′. *Let $n \geq 6$ and 6 divide $2n(n-1)$. Then $K_{n(2)}$ has a γ_6-decomposition.*

Proof of Theorem 1.1. By Theorem 1.1′, there is a γ_6-decomposition Φ of $K_{n(2)}$. We adopt the standard "expanding points method" used in [4] or [5]. Replace each vertex a of $K_{n(2)}$ by t new vertices labeled $a_1, a_2, \ldots, a_t$, and then join all vertices a_i to all vertices b_j if ab was an edge in $K_{n(2)}$. Then the resulting graph is $K_{n(2t)}$. If $\lambda = \langle a, b, c, d, e, f \rangle$ is a γ_6-cycle in Φ, then

$$\lambda_{ij} = \langle a_i, b_j, c_i, d_j, e_i, f_j \rangle \qquad (i = 1, 2, \ldots, t, \quad j = 1, 2, \ldots, t)$$

are t^2 mutually disjoint γ_6-cycles of $K_{n(2t)}$ (see Figure 1). The collection of all such γ_6-cycles of $K_{n(2t)}$ obtained from all cycles in Φ constitutes a γ_6-decomposition of $K_{n(2t)}$. □

From now on, we will concentrate on proving Theorem 1.1′. However, if n is odd then the conclusion can be easily obtained from the following known result.

Lemma 1.2. ([1], [8]) *Let n be an odd integer and m any positive integer. Then, K_n has a decomposition into m-cycles if and only if m divides $\frac{n(n-1)}{2}$.*

Theorem 1.3. *Let n be an odd integer and suppose 6 divides $\frac{n(n-1)}{2}$. Then $K_{n(2)}$ has a γ_6-decomposition.*

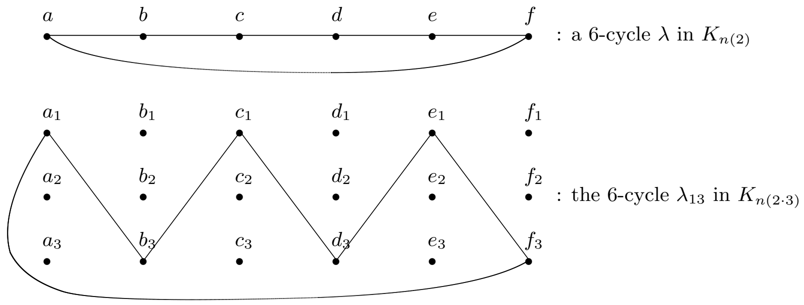

Figure 1

Proof. Let the vertices of K_n be $v_0, v_1, v_2, \ldots, v_{n-1}$, and let the partite sets of $K_{n(2)}$ be $\{0, \overline{0}\}$, $\{1, \overline{1}\}$, ..., $\{n-1, \overline{n-1}\}$. By the preceding lemma, K_n has a decomposition Φ of K_n into 6-cycles. If $\lambda = \langle v_{i_0}, v_{i_1}, \ldots, v_{i_5}\rangle$ is a 6-cycle in Φ, we can produce four cycles

$$\lambda_1 = \langle i_0, i_1, i_2, i_3, i_4, i_5\rangle, \qquad \lambda_2 = \langle i_0, \overline{i_1}, i_2, \overline{i_3}, i_4, \overline{i_5}\rangle,$$
$$\lambda_3 = \langle \overline{i_0}, \overline{i_1}, \overline{i_2}, \overline{i_3}, \overline{i_4}, \overline{i_5}\rangle, \qquad \lambda_4 = \langle \overline{i_0}, i_1, \overline{i_2}, i_3, \overline{i_4}, i_5\rangle$$

from λ. Clearly, they are mutually disjoint γ_6-cycles of $K_{n(2)}$, and the collection of all such γ_6-cycles obtained from each 6-cycle in Φ is a γ_6-decomposition of $K_{n(2)}$. □

However, if n is even, K_n does not have a cycle decomposition, and hence we can not apply Lemma 1.2. So, we need a different method. The method we are about to develop in the subsequent sections can be applied to all cases.

In Section 2, we introduce feasible sequences of differences of numbers in $\mathbb{Z}_n$ and explain the method for producing γ_6-cycles from feasible sequences. In Section 3, we prove Theorem 1.1′ by producing appropriate feasible sequences and generating γ_6-cycles.

2. Cycles from feasible sequences of differences

For $K_{n(2)}$, let the partite sets be $A_0 = \{0, \overline{0}\}$, $A_1 = \{1, \overline{1}\}$, ..., and $A_{n-1} = \{n-1, \overline{n-1}\}$. Thus, the elements in $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ are used as indices of the partite sets and as vertices of the graph as well. An edge between a vertex in A_i and another vertex in A_j is called an *edge of distance d* for some d with $0 < d \leq \frac{n}{2}$ if $|i-j| = d$, where the arithmetic is done in $\mathbb{Z}_n$. In particular, if $d = \frac{n}{2}$, then the edges of distance d are called the *diagonal edges*. For example, the edges $0\overline{4}$, 73, $\overline{7}\,\overline{2}$ and $\overline{8}3$ are all edges of distance 4 in $K_{9(2)}$, and the edges $4\,\overline{9}$ and $\overline{0}\,\overline{5}$ are diagonal edges of $K_{10(2)}$.

Put $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm\frac{n-1}{2}\}$ if n is odd and $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm\frac{n-2}{2}, \frac{n}{2}\}$ if n is even. Then, $\mathcal{D}_n$ is a complete set of differences of two distinct numbers in $\mathbb{Z}_n$. A sequence $\rho = (r_1, r_2, \ldots, r_6)$ of differences in $\mathcal{D}_n$ is called a *feasible sequence*, or an *f-sequence* for simplicity, if

(i) $\sum_{i=1}^{6} r_i = 0$, that is, the total sum of the terms of the sequence is zero, and

(ii) $\sum_{i=p}^{q} r_i \neq 0$ for all p, q with $1 < p$ or $q < 6$, that is, any proper partial sum of consecutive entries is nonzero,

where the arithmetic is done in $\mathbb{Z}_n$.

Let $\rho = (r_1, r_2, \ldots, r_6)$ be any sequence, which may not be feasible, of differences of $\mathcal{D}_n$. The *sequence of initial sums*, or the *s-sequence* for short, of ρ is the sequence $\sigma_\rho = (s_0, s_1, s_2, \ldots, s_5)$ of elements in $\mathbb{Z}_n$, where $s_0 = 0$ and $s_i = \sum_{j=1}^{i} r_j$ for $i = 1, 2, \ldots, 5$. Note that, $s_i = s_{i-1} + r_i$ for each $i = 1, 2, \ldots, 5$ and $s_5 + r_6 = s_0$.

With the above notation, the sequence σ_ρ represents the sequence of partite sets which a 6-cycle traverses, and the feasibility of ρ guarantees that the cycle is proper and gregarious. Now, the following lemma is trivial from the definitions.

Lemma 2.1. *Let $\sigma_\rho = (s_0, s_1, s_2, \ldots, s_5)$ be the s-sequence of a sequence $\rho = (r_1, r_2, \ldots, r_6)$ of differences in $\mathcal{D}_n$. Then ρ is an f-sequence if and only if $\sum_{i=1}^{6} r_i = 0$ and all entries of σ_ρ are mutually distinct.*

Let ϕ^+ and ϕ^- be mappings of $\mathbb{Z}_n$ into $\cup_{i=0}^{n-1} A_i$ defined by $\phi^+(i) = i$ and $\phi^-(i) = \bar{i}$ for all i in $\mathbb{Z}_n$. A *flag* is a sequence $\phi^* = (\phi_0, \phi_1, \ldots, \phi_5)$ where $\phi_i = \phi^+$ or ϕ^- for $i = 1, 2, 3, 4, 5$. Given such a flag ϕ^*, we also use the same notation ϕ^* to denote the mapping defined by $\phi^*(s_0, s_1, \ldots, s_5) = \langle \phi_0(s_0), \phi_1(s_1), \ldots, \phi_5(s_5) \rangle$ for every sequence $(s_0, s_1, \ldots, s_5)$ of distinct elements in $\mathbb{Z}_n$. Note that $\phi^*(s_0, s_1, \ldots, s_5)$ is a γ_6-cycle.

Let $\tau : \mathbb{Z}_n \to \mathbb{Z}_n$ be the mapping defined by $\tau(i) = i+1$ for all i in $\mathbb{Z}_n$. Then, $\tau^j(i) = i+j$ for all i, j in $\mathbb{Z}_n$ and τ^n is the identity mapping. We can extend each τ^j to a mapping $\tau_*^j : \mathbb{Z}_n^6 \to \mathbb{Z}_n^6$ by defining $\tau_*^j(s_0, s_1, \ldots, s_5) = (\tau^j(s_0), \tau^j(s_1), \ldots, \tau^j(s_5))$.

Now, if we are given a pair (ρ, ϕ^*) consisting of an f-sequence and a flag, we can produce a class $\{\phi^*(\tau_*^j(\sigma_\rho)) \mid \in \mathbb{Z}_n\}$ of γ_6-cycles. For example, if $\rho = (r_1, r_2, \ldots, r_6)$ and $\phi^* = (\phi^+, \phi^-, \phi^-, \phi^+, \phi^+, \phi^-)$, then $\sigma_\rho = (s_0, s_1, s_2, \ldots, s_5)$ and the γ_6-cycles in the class are:

$$
\begin{array}{lcllllll}
\phi^*(\tau_*^0(\sigma_\rho)) & = & \langle\, 0, & \overline{s_1}, & \overline{s_2}, & s_3, & s_4, & \overline{s_5}\,\rangle, \\
\phi^*(\tau_*^1(\sigma_\rho)) & = & \langle\, 1, & \overline{s_1+1}, & \overline{s_2+1}, & s_3+1, & s_4+1, & \overline{s_5+1}\,\rangle, \\
\phi^*(\tau_*^2(\sigma_\rho)) & = & \langle\, 2, & \overline{s_1+2}, & \overline{s_2+2}, & s_3+2, & s_4+2, & \overline{s_5+2}\,\rangle, \\
\vdots & \vdots & & & \vdots & & & \\
\phi^*(\tau_*^k(\sigma_\rho)) & = & \langle\, k, & \overline{s_1+k}, & \overline{s_2+k}, & s_3+k, & s_4+k, & \overline{s_5+k}\,\rangle, \\
\vdots & \vdots & & & \vdots & & & \\
\phi^*(\tau_*^{n-1}(\sigma_\rho)) & = & \langle\, n-1, & \overline{s_1-1}, & \overline{s_2-1}, & s_3-1, & s_4-1, & \overline{s_5-1}\,\rangle.
\end{array}
$$

Note that every column on the right-hand side has one vertex from every partite set. Thus, each edge of the form $p\,\overline{q}$ appears as the first edge of a γ_6-cycle above if $q-p=s_1=r_1$. Each edge of the form $\overline{p}\,\overline{q}$ appears as the second edge of a γ_6-cycle above if $q-p=s_2-s_1=r_2$. Similarly, each edge of the form $\overline{p}\,q$ with $q-p=r_3$, of the form $p\,q$ with $q-p=r_4$, of the form $p\,\overline{q}$ with $q-p=r_5$, and of the form $\overline{p}\,q$ with $q-p=r_6$, appears in the γ_6-cycles above.

This procedure is the method we will use to obtain a γ_6-decomposition of $K_{n(2)}$. The main problem then is how to choose pairs of f-sequences and flags so that, in the γ_6-cycles produced by these pairs, each of the edge $p\,q$, $\overline{p}\,q$, $p\,\overline{q}$ and $\overline{p}\,\overline{q}$ with $q-p=d$ appears exactly once for every distance d with $1\le d\le \frac{n}{2}$. Note that we sometimes need to produce a class with only $\frac{n}{2}$ γ_6-cycles when n is even.

For an integer k, a class containing k γ_6-cycles will be called a k-class. We will use n-classes and $\frac{n}{2}$-classes, each generated from a give γ_6-cycle using it as the *starter cycle*.

If $\lambda=(v_1,v_2,\ldots,v_6)$ is a 6-cycle, then edge v_iv_{i+1} will be called the ith edge of λ for $i=1,2,3,4,5$, and v_6v_1 will be called the last edge of λ.

3. Proof of Theorem 1.1′

The number of edges in $K_{n(2)}$ is $4\cdot\binom{n}{2}=2n(n-1)$. Thus, for $K_{n(2)}$ to be γ_6-decomposable, $2n(n-1)$ must be divisible by 6. That is, $n\equiv 0,1,3$ or 4 (mod 6). Of course, we always assume $n\ge 6$. We divide the proof into four cases depending on n modulo 6.

Case (1). Suppose $n\equiv 1 \pmod 6$ and put $n=6k+1$ with $k\ge 1$. The number of edges in $K_{n(2)}$ is $2(6k+1)\,6k=12kn$ and we will produce $2kn$ mutually disjoint γ_6-cycles in $2k$ n-classes. We have $\mathcal{D}_n=\{\pm1,\pm2,\ldots,\pm3k\}$ here. We partition $\mathcal{D}_n$ into k sets $T_i=\{\pm(3i+1),\pm(3i+2),\pm(3i+3)\}$ for $i=0,1,2,\ldots,k-1$. For each i, put

$$\rho_i=(3i{+}1,-(3i{+}2),3i{+}3,3i{+}2,-(3i{+}1),-(3i{+}3)),$$

and we have $\sigma_{\rho_i}=(0,3i{+}1,n{-}1,3i{+}2,6i{+}4,3i{+}3)$. Since $6k\ge 6(i+1)=6i+6$ for $i=0,1,2,\ldots,k-1$, all entries of σ_{ρ_i} are mutually distinct. Thus, ρ_i is an f-sequence by Lemma 2.1. Now, we choose two flags

$$\phi_1^*=(\phi^+,\phi^+,\phi^+,\phi^+,\phi^-,\phi^-)$$

and

$$\phi_2^*=(\phi^-,\phi^+,\phi^-,\phi^-,\phi^-,\phi^+).$$

Then, using the γ_6-sequences $\phi_1^*(\sigma_{\rho_i})$ and $\phi_2^*(\sigma_{\rho_i})$ as starter cycles, we generate two n-classes $C_i=\{\phi_1^*(\tau_*^j(\sigma_{\rho_i}))\mid j\in\mathbb{Z}_n\}$ and $D_i=\{\phi_2^*(\tau_*^j(\sigma_{\rho_i}))\mid j\in\mathbb{Z}_n\}$, respectively, as below:

$$
\begin{array}{ll}
(C_i) & \langle\ 0,\ 3i{+}1, n{-}1, 3i{+}2, \overline{6i{+}4}, \overline{3i{+}3}\ \rangle,\\
& \langle\ 1,\ 3i{+}2,\ 0,\ 3i{+}3, \overline{6i{+}5}, \overline{3i{+}4}\ \rangle,\\
& \langle\ 2,\ 3i{+}3,\ 1,\ 3i{+}4, \overline{6i{+}6}, \overline{3i{+}5}\ \rangle,\\
& \qquad\vdots\\
& \langle n{-}2, 3i{-}1, n{-}3, 3i{-}2, \overline{6i{+}2}, \overline{3i{+}1}\ \rangle,\\
& \langle n{-}1,\ 3i,\ n{-}2, 3i{-}1, \overline{6i{+}3}, \overline{3i{+}2}\ \rangle.
\end{array}
\qquad
\begin{array}{ll}
(D_i) & \langle\ \overline{0},\ 3i{+}1, \overline{n{-}1}, \overline{3i{+}2}, \overline{6i{+}4}, 3i{+}3\ \rangle,\\
& \langle\ \overline{1},\ 3i{+}2,\ \overline{0},\ \overline{3i{+}3}, \overline{6i{+}5}, 3i{+}4\ \rangle,\\
& \langle\ \overline{2},\ 3i{+}3,\ \overline{1},\ \overline{3i{+}4}, \overline{6i{+}6}, 3i{+}5\ \rangle,\\
& \qquad\vdots\\
& \langle \overline{n{-}2}, 3i{-}1, \overline{n{-}3}, \overline{3i{-}2}, \overline{6i{+}2}, 3i{+}1\ \rangle,\\
& \langle \overline{n{-}1},\ 3i,\ \overline{n{-}2}, \overline{3i{-}1}, \overline{6i{+}3}, 3i{+}2\ \rangle.
\end{array}
$$

For p, q in $\mathbb{Z}_n$ with $q-p = 3i+1$, we have the following observations.

(i) Each edge $p\,q$ appears as the first edge of a cycle in E_1.

(ii) Each edge $\overline{p}\,\overline{q}$ appears in the form $\overline{q}\,\overline{p}$ as the fifth edge of a cycle in E_1.

(iii) Each edge $\overline{p}\,q$ appears as the first edge of a cycle in E_2.

(iv) Each edge $p\,\overline{q}$ appears in the form $\overline{q}\,p$ as the fifth edge of a cycle in E_2.

Since no other edges of the above cycles have distance $3i+1$, each of the edges $p\,q$, $\overline{p}\,\overline{q}$, $\overline{p}\,q$ and $p\,\overline{q}$ with $p-q = 3i+1$ appears exactly once in the above cycles. Similarly, we see that this is true when $p-q = 3i+2$ or $p-q = 3i+3$, as well. Therefore, every edge of distance $3i+1$, $3i+2$ or $3i+3$ in $K_{n(2)}$ appears exactly once in γ_6-cycles of the two n-classes, and so the γ_6-cycles are mutually disjoint.

If we perform the preceding procedure for each T_i for $i = 0, 1, 2, \ldots, k{-}1$, we obtain $2k$ n-classes of γ_6-cycles, in which every edge of any nonzero distance appears exactly once. Consequently, the γ_6-cycles in $\bigcup_{i=0}^{k-1}(C_i \cup D_i)$ constitute a γ_6-decomposition of $K_{n(2)}$. Clearly, this decomposition is *circular* in the sense that it is invariant under τ^*.

Example 3.1. Let $n = 6 \cdot 2 + 1 = 13$. Then $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm 6\}$. By the procedure in Case (1), we have $\rho_0 = (1, -2, 3, 2, -1, -3)$ and $\rho_1 = (4, -5, 6, 5, -4, -6)$, and so

$$\sigma_{\rho_0} = (0, 1, 12, 2, 4, 3), \qquad \sigma_{\rho_1} = (0, 4, 12, 5, 10, 6).$$

The n-classes C_1, D_1, C_2, D_2 generated from the γ_6-cycles $\phi_1^*(\sigma_{\rho_0})$, $\phi_2^*(\sigma_{\rho_0})$, $\phi_1^*(\sigma_{\rho_1})$ and $\phi_2^*(\sigma_{\rho_1})$, respectively, are as below:

$\langle 0, 1, 12, 2, \overline{4}, \overline{3}\rangle$,	$\langle \overline{0}, 1, \overline{12}, \overline{2}, \overline{4}, 3\rangle$,	$\langle 0, 4, 12, 5, \overline{10}, \overline{6}\rangle$,	$\langle \overline{0}, 4, \overline{12}, \overline{5}, \overline{10}, 6\rangle$,
$\langle 1, 2, 0, 3, \overline{5}, \overline{4}\rangle$,	$\langle \overline{1}, 2, \overline{0}, \overline{3}, \overline{5}, 4\rangle$,	$\langle 1, 5, 0, 6, \overline{11}, \overline{7}\rangle$,	$\langle \overline{1}, 5, \overline{0}, \overline{6}, \overline{11}, 7\rangle$,
$\langle 2, 3, 1, 4, \overline{6}, \overline{5}\rangle$,	$\langle \overline{2}, 3, \overline{1}, \overline{4}, \overline{6}, 5\rangle$,	$\langle 2, 6, 1, 7, \overline{12}, \overline{8}\rangle$,	$\langle \overline{2}, 6, \overline{1}, \overline{7}, \overline{12}, 8\rangle$,
$\langle 3, 4, 2, 5, \overline{7}, \overline{6}\rangle$,	$\langle \overline{3}, 4, \overline{2}, \overline{5}, \overline{7}, 6\rangle$,	$\langle 3, 7, 2, 8, \overline{0}, \overline{9}\rangle$,	$\langle \overline{3}, 7, \overline{2}, \overline{8}, \overline{0}, 9\rangle$,
$\langle 4, 5, 3, 6, \overline{8}, \overline{7}\rangle$,	$\langle \overline{4}, 5, \overline{3}, \overline{6}, \overline{8}, 7\rangle$,	$\langle 4, 8, 3, 9, \overline{1}, \overline{10}\rangle$,	$\langle \overline{4}, 8, \overline{3}, \overline{9}, \overline{1}, 10\rangle$,
$\langle 5, 6, 4, 7, \overline{9}, \overline{8}\rangle$,	$\langle \overline{5}, 6, \overline{4}, \overline{7}, \overline{9}, 8\rangle$,	$\langle 5, 9, 4, 10, \overline{2}, \overline{11}\rangle$,	$\langle \overline{5}, 9, \overline{4}, \overline{10}, \overline{2}, 11\rangle$,
$\langle 6, 7, 5, 8, \overline{10}, \overline{9}\rangle$,	$\langle \overline{6}, 7, \overline{5}, \overline{8}, \overline{10}, 9\rangle$,	$\langle 6, 10, 5, 11, \overline{3}, \overline{12}\rangle$,	$\langle \overline{6}, 10, \overline{5}, \overline{11}, \overline{3}, 12\rangle$,
$\langle 7, 8, 6, 9, \overline{11}, \overline{10}\rangle$,	$\langle \overline{7}, 8, \overline{6}, \overline{9}, \overline{11}, 10\rangle$,	$\langle 7, 11, 6, 12, \overline{4}, \overline{0}\rangle$,	$\langle \overline{7}, 11, \overline{6}, \overline{12}, \overline{4}, 0\rangle$,
$\langle 8, 9, 7, 10, \overline{12}, \overline{11}\rangle$,	$\langle \overline{8}, 9, \overline{7}, \overline{10}, \overline{12}, 11\rangle$,	$\langle 8, 12, 7, 0, \overline{5}, \overline{1}\rangle$,	$\langle \overline{8}, 12, \overline{7}, \overline{0}, \overline{5}, 1\rangle$,
$\langle 9, 10, 8, 11, \overline{0}, \overline{12}\rangle$,	$\langle \overline{9}, 10, \overline{8}, \overline{11}, \overline{0}, 12\rangle$,	$\langle 9, 0, 8, 1, \overline{6}, \overline{2}\rangle$,	$\langle \overline{9}, 0, \overline{8}, \overline{1}, \overline{6}, 2\rangle$,
$\langle 10, 11, 9, 12, \overline{1}, \overline{0}\rangle$,	$\langle \overline{10}, 11, \overline{9}, \overline{12}, \overline{1}, 0\rangle$,	$\langle 10, 1, 9, 2, \overline{7}, \overline{3}\rangle$,	$\langle \overline{10}, 1, \overline{9}, \overline{2}, \overline{7}, 3\rangle$,
$\langle 11, 12, 10, 0, \overline{2}, \overline{1}\rangle$,	$\langle \overline{11}, 12, \overline{10}, \overline{0}, \overline{2}, 1\rangle$,	$\langle 11, 2, 10, 3, \overline{8}, \overline{4}\rangle$,	$\langle \overline{11}, 2, \overline{10}, \overline{3}, \overline{8}, 4\rangle$,
$\langle 12, 0, 11, 1, \overline{3}, \overline{2}\rangle$.	$\langle \overline{12}, 0, \overline{11}, \overline{1}, \overline{3}, 2\rangle$.	$\langle 12, 3, 11, 4, \overline{9}, \overline{5}\rangle$.	$\langle \overline{12}, 3, \overline{11}, \overline{4}, \overline{9}, 5\rangle$.

Case (2). Suppose $n \equiv 4 \pmod 6$ and put $n = 6k+4$ with $k \geq 1$. The number of edges is $2(6k+4)(6k+3) = 6(2k+1)n$, and we need to produce

$(2k+1)n$ mutually disjoint γ_6-cycles. We note that $\frac{n}{2} = 3k+2$ and we have $\mathcal{D}_n = \{\pm 1, \pm 2, \ldots, \pm(3k+1), 3k+2\}$.

Take the subset $\{\pm 1, \pm 2, \pm 3, \ldots, \pm(3k-5), \pm(3k-4), \pm(3k-3)\}$ of $\mathcal{D}_n$ and partition it into $k-1$ subsets $T_i = \{\pm(3i+1), \pm(3i+2), \pm(3i+3)\}$ for $i = 0, 1, \ldots, k-2$. With each T_i for $i = 0, 1, \ldots, k-2$, we proceed exactly the same way as in Case (1). That is, for each i, with the f-sequence $\rho_i = (3i+1, -(3i+2), 3i+3, 3i+2, -(3i+1), -(3i+3))$ and the flags $\phi_1^* = (\phi^+, \phi^+, \phi^+, \phi^+, \phi^-, \phi^-)$ and $\phi_2^* = (\phi^-, \phi^+, \phi^-, \phi^-, \phi^-, \phi^+)$, we generate two n-classes as in Case (1) from the γ_6-cycles $\phi_1^*(\sigma_{\rho_i})$ and $\phi_2^*(\sigma_{\rho_i})$, respectively. Then, we obtain $2(k-1)$ n-classes C_i and D_i of γ_6-cycles for $i = 0, 1, \ldots, k-2$, in which every edge of distance d appears exactly once for d with $1 \le d \le 3k-3$.

Now, we take care of edges of distance d with $3k-2 \le d \le 3k+2$. Here, we need a more complicated procedure to handle the diagonal edges. Put

$$\eta = (3k-2, -(3k-1), 3k, 3k+2, -3k, -(3k+1)),$$

and we have $\sigma_\eta = (0, 3k-2, 6k+3, 3k-1, 6k+1, 3k+1)$. Since $n \ge 10$, the components of σ_η are mutually distinct and so η is an f-sequences by Lemma 2.1. We choose four flags

$$\psi_1^* = (\phi^-, \phi^-, \phi^+, \phi^-, \phi^-, \phi^+), \qquad \psi_3^* = (\phi^-, \phi^+, \phi^-, \phi^+, \phi^+, \phi^-),$$
$$\psi_2^* = (\phi^-, \phi^-, \phi^+, \phi^+, \phi^-, \phi^-), \qquad \psi_4^* = (\phi^-, \phi^+, \phi^-, \phi^-, \phi^+, \phi^+).$$

Using the γ_6-cycles $\psi_i^*(\sigma_\eta)$ for $i = 1, 2, 3, 4$, we generate four $\frac{n}{2}$-classes:

$$F_1 = \{\psi_1^*(\tau_*^j(\sigma_\eta)) \mid 0 \le j \le 3k+1\}, \qquad F_3 = \{\psi_3^*(\tau_*^j(\sigma_\eta)) \mid 0 \le j \le 3k+1\},$$
$$F_2 = \{\psi_2^*(\tau_*^j(\sigma_\eta)) \mid 3k+2 \le j \le 6k+3\}, F_4 = \{\psi_4^*(\tau_*^j(\sigma_\eta)) \mid 3k+2 \le j \le 6k+3\}.$$

The $\frac{n}{2}$-classes are as below:

$$\begin{array}{ll}
(F_1) & \langle \overline{0}, \overline{3k-2}, 6k+3, \overline{3k-1}, \overline{6k+1}, 3k+1 \rangle, \\
 & \langle \overline{1}, \overline{3k-1}, 0, \overline{3k}, \overline{6k+2}, 3k+2 \rangle, \\
 & \vdots \\
 & \langle \overline{3k}, \overline{6k-2}, 3k-1, \overline{6k-1}, \overline{3k-3}, 6k+1 \rangle, \\
 & \langle \overline{3k+1}, \overline{6k-1}, 3k, \overline{6k}, \overline{3k-2}, 6k+2 \rangle. \\
(F_2) & \langle \overline{3k+2}, \overline{6k}, 3k+1, 6k+1, \overline{3k-1}, \overline{6k+3} \rangle, \\
 & \langle \overline{3k+3}, \overline{6k+1}, 3k+2, 6k+2, \overline{3k}, \overline{0} \rangle, \\
 & \vdots \\
 & \langle \overline{6k+2}, \overline{3k-4}, 6k+1, 3k-3, \overline{6k-1}, \overline{3k-1} \rangle, \\
 & \langle \overline{6k+3}, \overline{3k-3}, 6k+2, 3k-2, \overline{6k}, \overline{3k} \rangle.
\end{array}$$

$$\begin{array}{ll}
(F_3) & \langle \overline{0}, 3k-2, \overline{6k+3}, 3k-1, 6k+1, \overline{3k+1} \rangle, \\
 & \langle \overline{1}, 3k-1, \overline{0}, 3k, 6k+2, \overline{3k+2} \rangle, \\
 & \vdots \\
 & \langle \overline{3k}, 6k-2, \overline{3k-1}, 6k-1, 3k-3, \overline{6k+1} \rangle, \\
 & \langle \overline{3k+1}, 6k-1, \overline{3k}, 6k, 3k-2, \overline{6k+2} \rangle. \\
(F_4) & \langle \overline{3k+2}, 6k, \overline{3k+1}, \overline{6k+1}, 3k-1, 6k+3 \rangle, \\
 & \langle \overline{3k+3}, 6k+1, \overline{3k+2}, \overline{6k+2}, 3k, 0 \rangle, \\
 & \vdots \\
 & \langle \overline{6k+2}, 3k-4, \overline{6k+1}, \overline{3k-3}, 6k-1, 3k-1 \rangle, \\
 & \langle \overline{6k+3}, 3k-3, \overline{6k+2}, \overline{3k-2}, 6k, 3k \rangle.
\end{array}$$

Now, take the sequence

$$\mu = (3k-2, 3k-1, -(3k+1), -(3k-2), 3k+1, -(3k-1)),$$

and we have $\sigma_\mu = (0, 3k-2, 6k-3, 3k-4, 6k+2, 3k-1)$. As before, it can be easily checked that μ is an f-sequence. We choose the flag $\psi_5^* = (\phi^+, \phi^-, \phi^-, \phi^+, \phi^+, \phi^+)$. Then, Using the γ_6-cycle $\phi_5^*(\sigma_\mu)$, we generate an n-class $F_5 = \{\psi_5^*(\tau_*^j(\sigma_\mu)) \mid j \in \mathbb{Z}\}$ as below:

$$
\begin{array}{ll}
(F_5) & \langle\ 0,\ \overline{3k-2}, \overline{6k-3}, 3k-4, 6k+2, 3k-1\rangle, \\
 & \langle\ 1,\ \overline{3k-1}, \overline{6k-2}, 3k-3, 6k+3,\ 3k\ \rangle, \\
 & \qquad \vdots \\
 & \langle 6k+2, \overline{3k-4}, \overline{6k-5}, 3k-6,\ 6k,\ 3k-3\rangle, \\
 & \langle 6k+3, \overline{3k-3}, \overline{6k-4}, 3k-5, 6k+1, 3k-2\rangle.
\end{array}
$$

For p, q in $\mathbb{Z}_n$ with $q-p = 3k+1$, we have the following observations.

(i) Each edge $p\,q$ appears in the forms $q\,p$ as the fifth edge of a cycle in F_5.

(ii) Each edge $\overline{p}\,\overline{q}$ appears in the form $\overline{q}\,\overline{p}$ as the last edge of a cycle in F_2 or F_3.

(iii) Each edge $\overline{p}\,q$ appears in the form $q\,\overline{p}$ as the last edge of a cycle in F_1 or F_4.

(iv) Each edge $p\,\overline{q}$ appears in the form $\overline{q}\,p$ as the third edge of a cycle in F_5.

Since no other edges of the above cycles have distance $3k+1$, each of the edges $p\,q$, $\overline{p}\,\overline{q}$, $\overline{p}\,q$ and $p\,\overline{q}$ with $p-q = 3k+1$ appears exactly once in the above cycles. Similarly, we can check that the same is true for all p, q with $3k-2 \leq p-q \leq 3k$ as well.

We now handle the edges of distance $3k+2 = \frac{n}{2}$. These edges are diagonal edges and appear in every cycle in F_1, F_2, F_3 and F_4. In fact, this is the reason we produce $\frac{n}{2}$-classes instead of n-classes when the relevant f-sequence contains the distance $\frac{n}{2}$. For p, q in $\mathbb{Z}_n$ with $q-p = 3k+2$, we have the following observations.

(i) Each edge $p\,q$ appears in the form $q\,p$ or $q\,p$ as the fourth edge of a cycle in F_3.

(ii) Each edge $\overline{p}\,\overline{q}$ appears in the form $\overline{p}\,\overline{q}$ or $\overline{q}\,\overline{p}$ as the fourth edge of a cycle in F_1.

(iii) Each edge $\overline{p}\,q$ appears in the form $\overline{p}\,q$ or $q\,\overline{p}$ as the fourth edge of a cycle in F_2 or F_4. If $\overline{p}\,q$ appears in F_2 then $p\,\overline{q}$ appears in F_4, and vice versa.

Since no other edges are diagonal edges, each diagonal edge appears exactly once in γ_6-cycles in F_1, F_2, F_3 and F_4, and no diagonal edges appear in F_5.

Consequently, the 6-cycles in

$$
\left(\bigcup_{i=0}^{k-2}(C_i \cup D_i)\right) \bigcup \left(\bigcup_{i=1}^{5} F_i\right)
$$

constitute a γ_6-decomposition of $K_{n(2)}$.

Example 3.2. Let $n = 6 \cdot 1 + 4 = 10$. Then $\mathcal{D}_n = \{\pm 1, \pm 2, \pm 3, \pm 4, 5\}$. By the procedure in Case (2), we have $\eta = (1, -2, 3, 5, -3, -4)$ and $\mu = (1, 2, -4, -1, 4, -2)$, and so $\sigma_\eta = (0, 1, 9, 2, 7, 4)$ and $\sigma_\mu = (0, 1, 3, 9, 8, 2)$. The four 5-classes and one 10-class are as below:

(F_1)
$$\langle \overline{0},\overline{1},9,\overline{2},\overline{7},4\rangle,\ \langle \overline{1},\overline{2},0,\overline{3},\overline{8},5\rangle,\ \langle \overline{2},\overline{3},1,\overline{4},\overline{9},6\rangle,\ \langle \overline{3},\overline{4},2,\overline{5},\overline{0},7\rangle,\ \langle \overline{4},\overline{5},3,\overline{6},\overline{1},8\rangle.$$

(F_2)
$$\langle \overline{5},\overline{6},4,7,\overline{2},\overline{9}\rangle,\ \langle \overline{6},\overline{7},5,8,\overline{3},\overline{0}\rangle,\ \langle \overline{7},\overline{8},6,9,\overline{4},\overline{1}\rangle,\ \langle \overline{8},\overline{9},7,0,\overline{5},\overline{2}\rangle,\ \langle \overline{9},\overline{0},8,1,\overline{6},\overline{3}\rangle.$$

(F_3)
$$\langle \overline{0},1,\overline{9},2,7,\overline{4}\rangle,\ \langle \overline{1},2,\overline{0},3,8,\overline{5}\rangle,\ \langle \overline{2},3,\overline{1},4,9,\overline{6}\rangle,\ \langle \overline{3},4,\overline{2},5,0,\overline{7}\rangle,\ \langle \overline{4},5,\overline{3},6,1,\overline{8}\rangle.$$

(F_4)
$$\langle \overline{5},6,\overline{4},\overline{7},2,9\rangle,\ \langle \overline{6},7,\overline{5},\overline{8},3,0\rangle,\ \langle \overline{7},8,\overline{6},\overline{9},4,1\rangle,\ \langle \overline{8},9,\overline{7},\overline{0},5,2\rangle,\ \langle \overline{9},0,\overline{8},\overline{1},6,3\rangle.$$

(F_5)
$$\langle 0,\overline{1},\overline{3},9,8,2\rangle,\ \langle 1,\overline{2},\overline{4},0,9,3\rangle,\ \langle 2,\overline{3},\overline{5},1,0,4\rangle,\ \langle 3,\overline{4},\overline{6},2,1,5\rangle,\ \langle 4,\overline{5},\overline{7},3,2,6\rangle,$$
$$\langle 5,\overline{6},\overline{8},4,3,7\rangle,\ \langle 6,\overline{7},\overline{9},5,4,8\rangle,\ \langle 7,\overline{8},\overline{0},6,5,9\rangle,\ \langle 8,\overline{9},\overline{1},7,6,0\rangle,\ \langle 9,\overline{0},\overline{2},8,7,1\rangle.$$

The γ_6-cycles at the first rows of the classes F_1, F_2, F_3 and F_4 are drawn in Figure 2. We see that all edges between A_2 and A_7 appear. By rotating the these cycles up to 4 clicks, we obtain all diagonal edges. Looking at edges between A_2 and A_9, and edges between A_4 and A_7, we find the same is true for all the edges of distance 3.

Let H be the bipartite graph $K_{6,6}$ such that each partite set is partitioned into three 2-element sets. Let the two partite sets be $B_1 \cup B_2 \cup B_3$ and $B_4 \cup B_5 \cup B_6$, respectively, where $B_i = \{b_i, \overline{b_i}\}$ for $i = 1, 2, \ldots, 6$. We denote this graph by $H(B_1, B_2, B_3; B_4, B_5, B_6)$.

A 6-cycle which consists of exactly one vertex from each set B_i for $i = 1, 2, \ldots, 6$ will be called a γ_6-*cycle* for H.

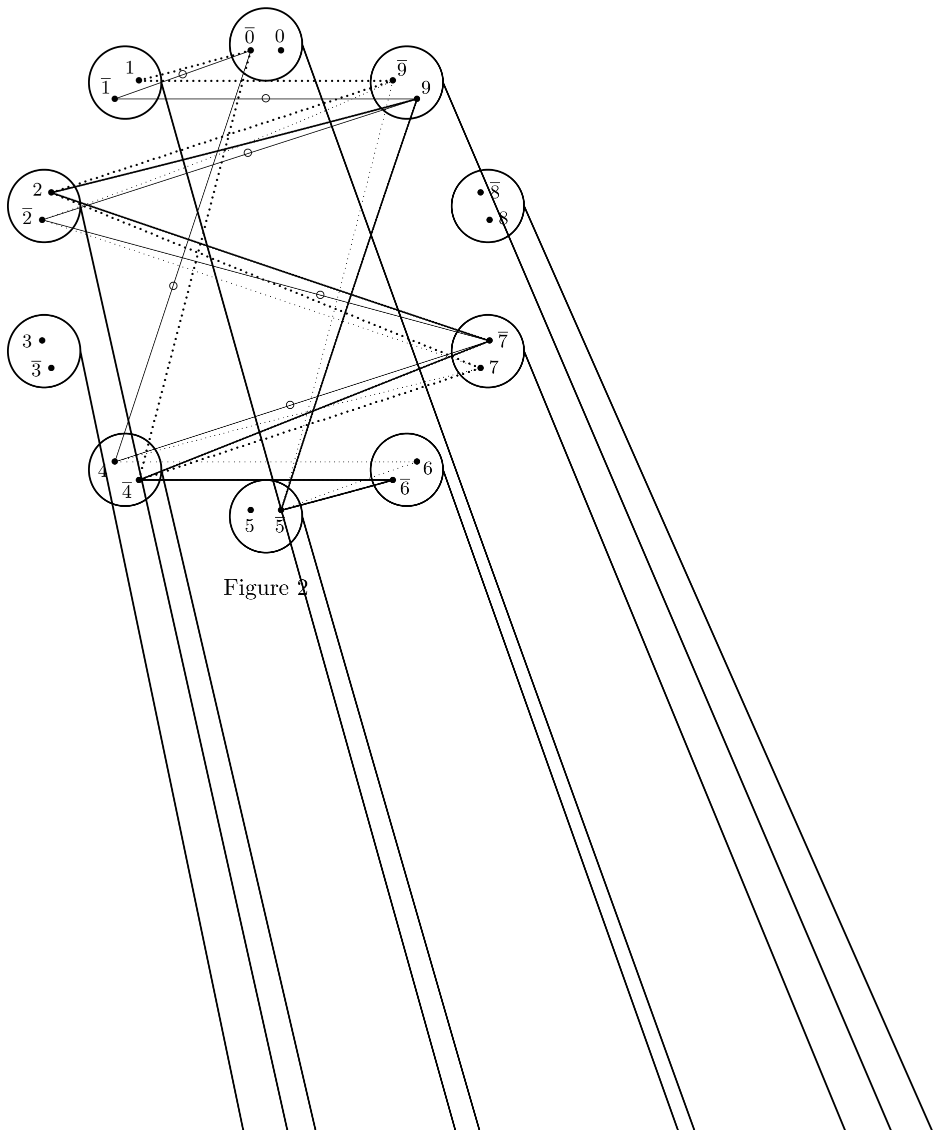

Figure 2

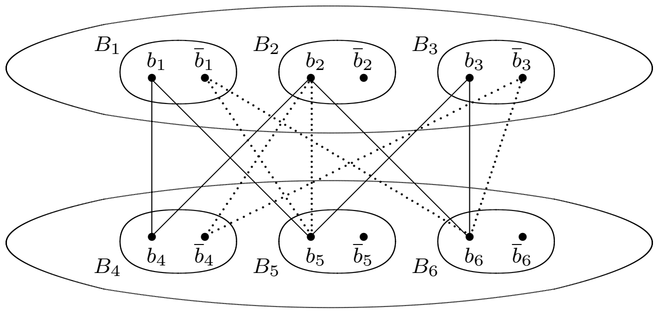

Figure 3

Lemma 3.1. *The graph H has a γ_6-decomposition.*

Proof. We have six disjoint γ_6-cycles for H, which constitute a required decomposition as below. The two cycles in the first column are shown in Figure 3.

$$\langle b_1, b_4, b_2, b_6, b_3, b_5\rangle, \quad \langle b_1, b_6, \overline{b_2}, b_4, \overline{b_3}, \overline{b_5}\rangle, \quad \langle \overline{b_1}, \overline{b_4}, b_3, \overline{b_5}, b_2, \overline{b_6}\rangle,$$
$$\langle \overline{b_1}, b_6, \overline{b_3}, \overline{b_4}, b_2, b_5\rangle, \quad \langle b_1, \overline{b_4}, \overline{b_2}, b_5, \overline{b_3}, \overline{b_6}\rangle, \quad \langle \overline{b_1}, b_4, b_3, \overline{b_6}, \overline{b_2}, \overline{b_5}\rangle.$$

□

Case (3). Suppose $n \equiv 0 \pmod 6$ and put $n = 6k$ with $k \geq 1$. When $n = 6$, we have the following γ_6-decomposition for $K_{6(2)}$.

$$\langle 0,1,3,\overline{4},\overline{2},\overline{5}\rangle, \ \langle 1,2,4,\overline{0},\overline{3},\overline{5}\rangle, \ \langle 2,3,0,\overline{1},\overline{4},\overline{5}\rangle, \ \langle 3,4,1,\overline{2},\overline{0},\overline{5}\rangle, \ \langle 4,0,2,\overline{3},\overline{1},\overline{5}\rangle,$$
$$\langle \overline{0},1,\overline{3},\overline{2},4,5\rangle, \ \langle \overline{1},2,\overline{4},\overline{3},0,5\rangle, \ \langle \overline{2},3,\overline{0},\overline{4},1,5\rangle, \ \langle \overline{3},4,\overline{1},\overline{0},2,5\rangle, \ \langle \overline{4},0,\overline{2},\overline{1},3,5\rangle.$$

To use an induction, assume that $K_{6k(2)}$ has a γ_6-decomposition, and consider $K_{(6(k+1))(2)}$. We partition $K_{(6(k+1))(2)}$ into two graphs $K_{6(2)}$ and $K_{6k(2)}$ and the edges between two vertices, one from $K_{6(2)}$ and another from $K_{6k(2)}$. Now, $K_{6(2)}$ and $K_{6k(2)}$ are γ_6-decomposable by the above table and the induction hypothesis, respectively. Let the partite sets in $K_{6(2)}$ be A_i for $i = 1, 2, \ldots, 6$, and let the partite sets in $K_{6k(2)}$ be B_j for $j = 1, 2, \ldots, 6k$. Let $H_{pq} = H(A_{3p+1}, A_{3p+2}, A_{3p+3}; B_{3q+1}, B_{3q+2}, B_{3q+3})$ for $p = 0, 1$ and $q = 0, 1, \ldots, 2k-1$. Each of them is γ_6-decomposable by Lemma 3.1. Since edges between vertices in $K_{6(2)}$ and vertices in $K_{6k(2)}$ can be partitioned into edges of H_{pq}, $K_{(6(k+1))(2)}$ has a γ_6-decomposition. Consequently, $K_{n(2)}$ is γ_6-decomposable for all $n = 6k$ with $k \geq 1$.

Case (4). Suppose $n \equiv 3 \pmod 6$ and put $n = 6k + 3$ with $k \geq 1$. We prove this case by an induction on k. If $k = 1$, then $n = 9$. Since 6 divides $\binom{9}{2} = 36$, $K_{9(2)}$ has a γ_6-decomposition by Theorem 1.3. Now, assume that $K_{(6k+3)(2)}$ has a γ_6-decomposition, and consider $K_{(6(k+1)+3)(2)}$. We partition $K_{(6(k+1)+3)(2)}$ into two graphs $K_{6(2)}$ and $K_{(6k+3)(2)}$ and the edges between two vertices, one from $K_{6(2)}$ and another from $K_{(6k+3)(2)}$. Now, $K_{6(2)}$ and

$K_{(6k+3)(2)}$ are γ_6-decomposable by the table in Case (3) and the induction hypothesis, respectively. As in Case (3), the edges between vertices in $K_{6(2)}$ and vertices in $K_{(6k+3)(2)}$ can be partitioned into $2(2k+1)$ classes, each classes inducing a copy of H, which is γ_6-decomposable by Lemma 3.1. Thus, $K_{(6k+3)(2)}$ is γ_6-decomposable. Consequently, $K_{n(2)}$ is γ_6-decomposable for all $n = 6k$ with $k \geq 1$.

4. A remark

For Case (1) of the preceding section, we can proceed by induction on the number n of partite sets as follow. First, we produce a γ_6-decomposition of $K_{7(2)}$ in any method such as computer-aided search. Then, we suppose $k \geq 2$ and $K_{(6(k-1)+1)(2)}$ is γ_6-decomposable, and show that $K_{(6k+1)(2)}$ is also γ_6-decomposable. Partition $K_{(6k+1)(2)}$ into two graphs $K_{(6(k-1)+1)(2)}$ and $K_{6(2)}$ and the edges between vertices, one from each graph. By induction hypothesis, $K_{(6(k-1)+1)(2)}$ is γ_6-decomposable. Let $A_0 = \{a, \overline{a}\}$ be a partite set of $K_{(6(k-1)+1)(2)}$. Then the vertex set $A_0 \cup K_{6(2)}$ induces a subgraph of $K_{(6k+1)(2)}$ which is isomorphic to $K_{7(2)}$, and this graph was shown to be γ_6-decomposable. Now, partition the vertices in $K_{(6(k-1)+1)(2)} \setminus A_0$ into $2(k-1)$ classes $\{A_{i1}, A_{i2}, A_{i3}\}$ of 3 partite sets for $i = 1, 2, \ldots, 2(k-1)$. Also partition the vertices in $K_{6(2)}$ into 2 classes $\{B_{j1}, B_{j2}, B_{j3}\}$ of 3 partite sets for $j = 1, 2$. Then, $H(A_{i1}, A_{i2}, A_{i3}; B_{j1}, B_{j2}, B_{j3})$ is an induced subgraph of $K_{(6k+1)(2)}$ which is γ_6-decomposable by Lemma 3.1, for $i = 1, 2, \ldots, 2(k-1)$ and $j = 1, 2$. Consequently, $K_{(6k+1)(2)}$ is γ_6-decomposable for all $k \geq 1$.

For Case (2), a similar induction can be applied. First, we produce a γ_6-decomposition of $K_{10(2)}$. Then, we partition $K_{(6k+4)(2)}$ into two graphs $K_{(6(k-1)+4)(2)}$ and $K_{6(2)}$ and the edges between two vertices, one from each graph. Take vertices in $K_{6(2)}$ and four partite sets of $K_{(6(k-1)+4)(2)}$. Then, these vertices induce a subgraph of $K_{(6k+4)(2)}$ which is isomorphic to $K_{10(2)}$. The remaining edges of $K_{(6k+4)(2)}$ are partitioned into edge-disjoint subgraphs each of which is isomorphic to H. Consequently, $K_{(6k+4)(2)}$ is γ_6-decomposable for all $k \geq 1$.

However, the above γ_6-decompositions do not have good symmetry as the γ_6-decompositions in Section 3. In Section 3, we constructed a circular γ_6-decomposition when $n \equiv 1 \pmod 6$, and when $n \equiv 4 \pmod 6$ the γ_6-decomposition could be partitioned into full classes and half classes. There, the full classes are circular and the half classes are not circular but are *almost* circular in the sense that the orderings of partite sets for the γ_6-cycles in the classes are circular.

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