J. Korean Math. Soc. **45** (2008), No. 6, pp. 1601–1611

A SUPPLEMENT TO PRECISE ASYMPTOTICS IN THE LAW OF THE ITERATED LOGARITHM FOR SELF-NORMALIZED SUMS

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Abstract. Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with zero means, variance one, and set $S_n = \sum_{i=1}^n X_i$, $n \geq 1$. Gut and Spătaru [3] established the precise asymptotics in the law of the iterated logarithm and Li, Nguyen and Rosalsky [7] generalized their result under minimal conditions. If $\mathbf{P}(|S_n| \geq \varepsilon\sqrt{2n \log\log n})$ is replaced by $\mathbf{E}\{|S_n|/\sqrt{n} - \varepsilon\sqrt{2\log\log n}\}_+$ in their results, the new one is called the moment version of precise asymptotics in the law of the iterated logarithm. We establish such a result for self-normalized sums, when X belongs to the domain of attraction of the normal law.

1. Introduction and main result

Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbf{E}X = 0$ and $S_n = \sum_{i=1}^n X_i$. Also let $\log x = \ln(x \vee e)$ and $\log\log x = \log(\log x)$. Hsu and Robbins [5] established the well-known complete convergence, if $\mathbf{E}X^2 < \infty$, then

$$\sum_{n=1}^{\infty} \mathbf{P}(|S_n| \geq \varepsilon n) < \infty, \qquad \forall\varepsilon > 0.$$

Katz [6] extended this result as follows: If $\alpha > 1/2, \alpha p \geq 1$ and $\mathbf{E}|X|^p < \infty$, then

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbf{P}(|S_n| \geq \varepsilon n^{\alpha}) < \infty, \qquad \forall\varepsilon > 0. \tag{1.1}$$

Many authors considered various extensions of the results of Hsu-Robbins and Katz. Some of them study the precise asymptotics of the infinite sums as $\varepsilon \to 0$ (c.f. Heyde [4] and Spătaru [9]). But, this kind of results do not hold for $\alpha = 1/2$. However, by replacing n^{α} by $\sqrt{n \log\log n}$, Gut and Spătaru [3]

Received January 23, 2007; Revised January 4, 2008.

2000 *Mathematics Subject Classification.* 60F15, 62E20.

Key words and phrases. precise asymptotics, law of iterated logarithm, self-normalized sums.

This work was supported by the Korea Research Foundation Grant funded by Korea Government (MOEHRD) (KRF-2006-353-C00004).

established the following result called the precise asymptotics of the law of the iterated logarithm.

Theorem A. *Suppose that* $\mathbf{E}X^2 = 1$ *and* $\mathbf{E}X^2(\log\log|X|)^{1+\delta} < \infty$ *for some* $\delta > 0$, *and that* $a_n = O(\sqrt{n}/(\log\log n)^h)$ *for some* $h > 1/2$. *Then*

$$\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\sum_{n=1}^{\infty}\frac{1}{n}\mathbf{P}(|S_n| \geq \varepsilon\sqrt{2n\log\log n}+a_n) = 1. \tag{1.2}$$

Li, Nguyen, and Rosalsky [7] generalized this result under minimal conditions as follows:

Theorem B. *Suppose that* $-1/2 < b \leq 1$, *that* $\mathbf{E}X^2 = 1$ *and* $\mathbf{E}X^2 I\{|X| \geq t\} = o(\frac{1}{\log\log t})$ *as* $t \to \infty$, *and that* $\{a_n; n \geq 1\}$ *is a sequence of real numbers such that*

$$\lim_{n\to\infty}(\frac{\log\log n}{n})^{1/2}a_n = \gamma \in [-\infty, \infty].$$

Then

$$\begin{aligned}&\lim_{\varepsilon\searrow 1}(\varepsilon^2-1)^{b+1/2}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n}\mathbf{P}(|S_n| \geq \varepsilon\sqrt{2n\log\log n}+a_n)\\ &= e^{-\sqrt{2}\gamma}2^b\sqrt{2/\pi}\Gamma(b+1/2),\end{aligned} \tag{1.3}$$

where $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$, $s > 0$ *is the gamma function.*

On the other hand, compared to complete convergence, Chow [2] established a moment version of (1.1) as follows: If $p \geq 1$, $\alpha > 1/2$, $\alpha p \geq 1$ and $\mathbf{E}\{|X|^p + |X|\log|X|\} < \infty$, then

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\mathbf{E}\{|S_n| - \varepsilon n^\alpha\}_+ < \infty, \quad \forall\varepsilon > 0, \tag{1.4}$$

where $a_+ = \max(a, 0)$. Pang et al. [8] obtained the precise rates in the law of the logarithm for the moment convergence of i.i.d. random variables.

In this paper, we prove the following theorem, which is a moment version of (1.3) for self-normalized sums.

Throughout this paper let $\{X, X_n; n \geq 1\}$ be a sequence of nondegenerate i.i.d. symmetric random variables, set $S_n = \sum_{i=1}^n X_i$, $V_n^2 = \sum_{i=1}^n X_i^2$ and $l(x) = \mathbf{E}X^2 I\{|X| \leq x\}$. We also let $[x]$ denote the largest integer $\leq x$, $\sum_{i=1}^0 \eta_i = 0$, and A denote a positive constant, whose values can differ in different places. $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$.

Theorem 1.1. *Suppose that* $a > -1$, *that* $l(x)$ *is a slowly varying function at* ∞, *satisfying* $l(x) \leq c_1\exp(c_2(\log x)^\beta)$ *for some* $c_1 > 0$, $c_2 > 0$ *and* $0 \leq \beta < 1$, *and that* $\alpha_n(\varepsilon)$ *is a nonnegative function of* ε *such that*

$$\alpha_n(\varepsilon)\log\log n \to \tau < \infty \quad \textit{as} \quad n \to \infty. \tag{1.5}$$

Then

(1.6)

$$\lim_{\varepsilon \searrow \sqrt{1+a}} \frac{1}{-\log(\varepsilon^2 - 1 - a)} \sum_{n=1}^{\infty} \frac{(\log n)^a}{n} \mathbf{E}\{|S_n|/V_n - \sqrt{2\log\log n}(\varepsilon + \alpha_n(\varepsilon))\}_+$$
$$= \frac{\exp(-2\tau\sqrt{1+a})}{\sqrt{2\pi}(1+a)}.$$

If $b > 0$*, then*

(1.7)

$$\lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^b \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathbf{E}\{|S_n|/V_n - \sqrt{2\log\log n}(\varepsilon + \alpha_n(\varepsilon))\}_+$$
$$= \frac{\Gamma(b)}{\sqrt{2\pi}(1+a)} \exp\{-2\tau\sqrt{1+a}\}.$$

Remark 1.1. Note that X belonging to the domain of attraction of the normal law is well known to be equivalent to $l(x)$ being a slowly varying function at ∞. We note also that $l(x) \le c_1 \exp(c_2(\log x)^\beta)$ is a weak enough assumption, which is satisfied by a large class of slowly varying functions such as $(\log\log x)^p$ and $(\log x)^p$ for some $0 < p < \infty$.

2. The proof of Theorem 1.1

For convenience, we need some notation. Put $c = \inf\{x \ge 1 : l(x) > 0\}$ and

$$\eta_n = \inf\{s : s \ge c+1, \frac{l(s)}{s^2} \le \frac{(\log\log n)^2}{n}\}.$$

Furthermore, for each n and $1 \le i \le n$, we let

$$\begin{aligned} &\bar{X}_{ni} = X_i I\{|X_i| \le \eta_n\}, \quad && \bar{S}_n = \sum_{i=1}^n \bar{X}_{ni}, \quad \bar{V}_n^2 = \sum_{i=1}^n \bar{X}_{ni}^2, \\ &S_n^i = S_n - X_i, && V_n^i = (V_n^2 - X_i^2)^{1/2}, \\ &\bar{S}_n^i = \bar{S}_n - \bar{X}_{ni}, && \bar{V}_n^i = (\bar{V}_n^2 - \bar{X}_{ni}^2)^{1/2}. \end{aligned}$$

Note that $nl(\eta_n) \sim \eta_n^2(\log\log n)^2$. Thus we have

$$l(\eta_n) \le c_1 \exp(c_2(\log \eta_n)^\beta) \le c_1 \exp(c_2(\log n)^\beta)$$

for n large enough, and we also have that $l(\eta_n)$ and $c_1 \exp(c_2(\log n)^\beta)/l(\eta_n)$ are slowly varying functions at ∞ (see [1, Chapter 1]). Using these facts, it follows easily that

$$c_1 \exp(c_2(\log j)^\beta)/l(\eta_j) \ge \frac{1}{2} c_1 \exp(c_2(\log k)^\beta)/l(\eta_k)$$

for all $j \geq k$ and k large enough, and hence

$$\begin{aligned}
&\frac{A}{l(\eta_k)(\log k)^\beta(\log\log k)^2}\\
&\leq \frac{\exp(c_2(\log k)^\beta)}{2l(\eta_k)}\sum_{j=k}^\infty \frac{1}{j\exp(c_2(\log j)^\beta)(\log j)(\log\log j)^2} \qquad (2.1)\\
&\leq \sum_{j=k}^\infty \frac{1}{jl(\eta_j)(\log j)(\log\log j)^2}.
\end{aligned}$$

We first will prove Theorem 1.1 in the case that $X, X_1, X_2, \ldots$ are normal random variables. Let N be a standard normal variable, we have the following proposition.

Proposition 2.1. *Let $a > -1$ and $\alpha_n(\varepsilon)$ be a nonnegative function of ε satisfying* (1.5). *Then*

(2.2)
$$\lim_{\varepsilon\searrow\sqrt{1+a}} \frac{1}{-\log(\varepsilon^2-1-a)}\sum_{n=1}^\infty \frac{(\log n)^a}{n}\mathbf{E}\{|N|-(\varepsilon+\alpha_n(\varepsilon))\sqrt{2\log\log n}\}_+ = \frac{\exp\{-2\tau\sqrt{1+a}\}}{\sqrt{2\pi}(1+a)}.$$

If $b > 0$, then
(2.3)
$$\lim_{\varepsilon\searrow\sqrt{1+a}} (\varepsilon^2-1-a)^b\sum_{n=1}^\infty \frac{(\log n)^a(\log\log n)^b}{n}\mathbf{E}\{|N|-(\varepsilon+\alpha_n(\varepsilon))\sqrt{2\log\log n}\}_+ = \frac{\Gamma(b)}{\sqrt{2\pi}(1+a)}\exp\{-2\tau\sqrt{1+a}\}.$$

Proof. Let $\psi_n(\varepsilon) = (\varepsilon+\alpha_n(\varepsilon))\sqrt{2\log\log n}$. Note that the limit in (2.2) and (2.3) does not depend on any finite terms of the infinite series, and we have

$$\mathbf{P}(|N| \geq x) \sim \frac{2}{\sqrt{2\pi}x}e^{-x^2/2} \quad \text{as} \quad x \to \infty.$$

Consider (2.2), for any $x > 0$, by (1.5)

$$\begin{aligned}
&\mathbf{P}(|N| \geq x+\psi_n(\varepsilon))\\
&\sim \frac{2}{\sqrt{2\pi}(x+\psi_n(\varepsilon))}\exp\{-\frac{1}{2}(x+\psi_n(\varepsilon))^2\}\\
&\sim \frac{2}{\sqrt{2\pi}}\frac{1}{x+\varepsilon\sqrt{2\log\log n}}\exp\{-\frac{1}{2}(x+\varepsilon\sqrt{2\log\log n})^2\}\\
&\quad\times\exp\{-2\varepsilon\alpha_n(\varepsilon)\log\log n - x\alpha_n(\varepsilon)\sqrt{2\log\log n}\} \quad \text{as} \quad n\to\infty,
\end{aligned}$$

uniformly in $\varepsilon \in (\sqrt{1+a}, \sqrt{1+a}+\delta)$ for some $\delta > 0$. So, for any $x > 0$ and $0 < \theta < 1$, there exist $\delta > 0$ and n_0 such that for all $n \geq n_0$ and $\varepsilon \in (\sqrt{1+a}, \sqrt{1+a}+\delta)$,

(2.4)
$$\begin{aligned}&\frac{2}{\sqrt{2\pi}}\frac{1}{x+\varepsilon\sqrt{2\log\log n}}\exp\{-\frac{1}{2}(x+\varepsilon\sqrt{2\log\log n})^2\}\exp\{-2\tau\sqrt{1+a}-\theta\}\\ &\leq \mathbf{P}(|N| \geq x+\psi_n(\varepsilon))\\ &\leq \frac{2}{\sqrt{2\pi}}\frac{1}{x+\varepsilon\sqrt{2\log\log n}}\exp\{-\frac{1}{2}(x+\varepsilon\sqrt{2\log\log n})^2\}\exp\{-2\tau\sqrt{1+a}+\theta\}.\end{aligned}$$

Write

$$\begin{aligned}&\lim_{\varepsilon\searrow\sqrt{1+a}}\frac{1}{-\log(\varepsilon^2-1-a)}\sum_{n=1}^{\infty}\frac{(\log n)^a}{n}\mathbf{E}\{|N|-\psi_n(\varepsilon)\}_+\\ &= \lim_{\varepsilon\searrow\sqrt{1+a}}\frac{1}{-\log(\varepsilon^2-1-a)}\int_{e^e}^{\infty}\frac{(\log y)^a}{y}\int_0^{\infty}\mathbf{P}(|N| \geq x+\psi_y(\varepsilon))dxdy\end{aligned}\tag{2.5}$$

and

(2.6)
$$\begin{aligned}&\lim_{\varepsilon\searrow\sqrt{1+a}}\frac{1}{-\log(\varepsilon^2-1-a)}\int_{e^e}^{\infty}\frac{(\log y)^a}{y}\int_0^{\infty}\frac{\exp\{-\frac{1}{2}(x+\varepsilon\sqrt{2\log\log y})^2\}}{x+\varepsilon\sqrt{2\log\log y}}dxdy\\ &= \lim_{\varepsilon\searrow\sqrt{1+a}}\frac{1}{-\log(\varepsilon^2-1-a)}\int_{e^e}^{\infty}\frac{(\log y)^a}{y}\int_{\varepsilon\sqrt{2\log\log y}}^{\infty}\frac{1}{x}\exp\{-\frac{1}{2}x^2\}dxdy\\ &= \lim_{\varepsilon\searrow\sqrt{1+a}}\frac{1}{-\log(\varepsilon^2-1-a)}\int_{\sqrt{2}\varepsilon}^{\infty}\frac{t}{\varepsilon^2}\exp\{\frac{1+a}{2\varepsilon^2}t^2\}\int_t^{\infty}\frac{1}{x}\exp\{-\frac{1}{2}x^2\}dxdt\\ &= \frac{1}{1+a}\lim_{\varepsilon\searrow\sqrt{1+a}}\frac{1}{-\log(\varepsilon^2-1-a)}\int_{\sqrt{2}\varepsilon}^{\infty}\frac{1}{x}\exp\{-\frac{1}{2}x^2\}[\exp\{\frac{1+a}{2\varepsilon^2}x^2\}-e^{1+a}]dx\\ &= \frac{1}{2(1+a)}\lim_{\varepsilon\searrow\sqrt{1+a}}\frac{1}{-\log(\varepsilon^2-1-a)}\int_{\varepsilon^2-1-a}^{\infty}\frac{1}{s}e^{-s}ds\\ &= \frac{1}{2(1+a)}\lim_{\varepsilon\searrow\sqrt{1+a}}\frac{1}{-\log(\varepsilon^2-1-a)}\left[\int_{\varepsilon^2-1-a}^{1}\frac{1}{s}e^{-s}ds+\int_1^{\infty}\frac{1}{s}e^{-s}ds\right]\\ &= \frac{1}{2(1+a)}.\end{aligned}$$

Hence, from (2.4)$\sim$ (2.6), (2.2) is obtained by letting $\theta \to 0$.

We now proceed to show (2.3). This follows by the same method as in (2.5) and (2.6)

(2.7)
$$\begin{aligned}&\lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\mathbf{E}\{|N|-\psi_n(\varepsilon)\}_+\\ &= \lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\int_{e^e}^{\infty}\frac{(\log y)^a(\log\log y)^b}{y}\int_0^{\infty}\mathbf{P}(|N| \geq x+\psi_y(\varepsilon))dxdy\end{aligned}$$

and

(2.8)

$$\lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\int_{e^e}^{\infty}\frac{(\log y)^a(\log\log y)^b}{y}\int_0^{\infty}\frac{\exp\{-\frac{1}{2}(x+\varepsilon\sqrt{2\log\log y})^2\}}{x+\varepsilon\sqrt{2\log\log y}}dxdy$$
$$=\lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\int_{\sqrt{2}\varepsilon}^{\infty}\frac{1}{x}\exp\{-\frac{1}{2}x^2\}\left[\int_{\sqrt{2}\varepsilon}^{x}\frac{t}{\varepsilon^2}\left(\frac{t^2}{2\varepsilon^2}\right)^b\exp\{\frac{1+a}{2\varepsilon^2}t^2\}dt\right]dx.$$

If $0<b<1$, then using integration by parts

(2.9)

$$\lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\int_{\sqrt{2}\varepsilon}^{\infty}\frac{1}{x}\exp\{-\frac{1}{2}x^2\}\left[\int_{\sqrt{2}\varepsilon}^{x}\frac{t}{\varepsilon^2}\left(\frac{t^2}{2\varepsilon^2}\right)^b\exp\{\frac{1+a}{2\varepsilon^2}t^2\}dt\right]dx$$
$$=\frac{1}{(1+a)^{1+b}}\lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\int_{\sqrt{2}\varepsilon}^{\infty}\frac{1}{x}\exp\{-\frac{1}{2}x^2\}\int_{1+a}^{\frac{1+a}{2\varepsilon^2}x^2}t^be^tdtdx$$
$$=\frac{1}{(1+a)^{1+b}}\lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\int_{\sqrt{2}\varepsilon}^{\infty}\frac{1}{x}\exp\{-\frac{1}{2}x^2\}\left(\frac{1+a}{2\varepsilon^2}x^2\right)^b\exp\{\frac{1+a}{2\varepsilon^2}x^2\}dx$$
$$-\frac{b}{(1+a)^{1+b}}\lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\int_{\sqrt{2}\varepsilon}^{\infty}\frac{1}{x}\exp\{-\frac{1}{2}x^2\}\int_{1+a}^{\frac{1+a}{2\varepsilon^2}x^2}t^{b-1}e^tdtdx$$
$$=:I_1-I_2.$$

We have

(2.10)

$$I_1=\frac{1}{1+a}\lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\int_{\sqrt{2}\varepsilon}^{\infty}(2\varepsilon^2)^{-b}x^{2b-1}\exp\{-(\frac{1}{2}-\frac{1+a}{2\varepsilon^2})x^2\}dx$$
$$=\frac{1}{2(1+a)}\lim_{\varepsilon\searrow\sqrt{1+a}}\int_{\varepsilon^2-1-a}^{\infty}s^{b-1}e^{-s}ds$$
$$=\frac{1}{2(1+a)}\Gamma(b)$$

and by (2.6)

(2.11)

$$0\le I_2\le\frac{b(1+a)^{b-1}}{(1+a)^{1+b}}\lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\int_{\sqrt{2}\varepsilon}^{\infty}\frac{1}{x}\exp\{-(\frac{1}{2}-\frac{1+a}{2\varepsilon^2})x^2\}dx$$
$$=\frac{b}{2(1+a)^2}\lim_{\varepsilon\searrow\sqrt{1+a}}(\varepsilon^2-1-a)^b\int_{\varepsilon^2-1-a}^{\infty}s^{-1}e^{-s}ds$$
$$=\frac{b}{2(1+a)^2}\lim_{\varepsilon\searrow\sqrt{1+a}}[(\varepsilon^2-1-a)^b\cdot(-\log(\varepsilon^2-1-a))]\frac{\int_{\varepsilon^2-1-a}^{\infty}s^{-1}e^{-s}ds}{-\log(\varepsilon^2-1-a)}$$
$$=0.$$

Thus, letting $\theta\to 0$, (2.3) follows from (2.4) and (2.7)∼(2.11).

If $b \geq 1$, then we write via integration by parts again

(2.12)
$$\begin{aligned}
&\int_{\sqrt{2}\varepsilon}^{x} \frac{t}{\varepsilon^2} \left(\frac{t^2}{2\varepsilon^2}\right)^b \exp\{\frac{1+a}{2\varepsilon^2}t^2\}dt = \frac{1}{(1+a)^{b+1}} \int_{1+a}^{\frac{1+a}{2\varepsilon^2}x^2} t^b e^t dt \\
&= \frac{1}{(1+a)^{b+1}} \left[(\frac{1+a}{2\varepsilon^2}x^2)^b \exp\{\frac{1+a}{2\varepsilon^2}x^2\} - (1+a)^b e^{1+a} \right] \\
&\quad + \frac{1}{(1+a)^{b+1}} \sum_{i=1}^{[b]-1} (-1)^i \left(\prod_{j=b-i+1}^{b} j \right) \left[\left(\frac{1+a}{2\varepsilon^2}x^2 \right)^{b-i} \exp\{\frac{1+a}{2\varepsilon^2}x^2\} - (1+a)^{b-i} e^{1+a} \right] \\
&\quad + \frac{(-1)^{[b]} \prod_{j=0}^{[b]-1}(b-j)}{(1+a)^{b+1}} \int_{1+a}^{\frac{1+a}{2\varepsilon^2}x^2} t^{b-[b]} e^t dt \\
&= J_{x1} + J_{x2} + J_{x3}.
\end{aligned}$$

We have by (2.10),

$$\lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - 1 - a)^b \int_{\sqrt{2}\varepsilon}^{\infty} \frac{1}{x} \exp\{-\frac{1}{2}x^2\} J_{x1} dx = \frac{1}{2(1+a)} \Gamma(b), \tag{2.13}$$

(2.14)
$$\begin{aligned}
&\lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - 1 - a)^b \int_{\sqrt{2}\varepsilon}^{\infty} \frac{1}{x} \exp\{-\frac{1}{2}x^2\} J_{x2} dx \\
&\quad = \sum_{i=1}^{[b]-1} \frac{(-1)^i (\prod_{j=b-i+1}^{b} j)}{(1+a)^{1+i}} \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - 1 - a)^b \\
&\qquad\qquad \times \int_{\sqrt{2}\varepsilon}^{\infty} (\frac{1}{2\varepsilon^2})^{b-i} x^{2b-2i-1} \exp\{-(\frac{1}{2} - \frac{1+a}{2\varepsilon^2})x^2\} dx \\
&\quad = \sum_{i=1}^{[b]-1} \frac{(-1)^i (\prod_{j=b-i+1}^{b} j)}{2(1+a)^{1+i}} \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - 1 - a)^i \int_{\varepsilon^2-1-a}^{\infty} s^{b-i-1} e^{-s} ds \\
&\quad = 0
\end{aligned}$$

and

(2.15)
$$\begin{aligned}
&\lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - 1 - a)^b \left| \int_{\sqrt{2}\varepsilon}^{\infty} \frac{1}{x} \exp\{-\frac{1}{2}x^2\} J_{x3} dx \right| \\
&\leq \frac{\prod_{j=0}^{[b]-1}(b-j)}{(1+a)^{b+1}} \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - 1 - a)^b \int_{\sqrt{2}\varepsilon}^{\infty} (\frac{1+a}{2\varepsilon^2})^{b-[b]} x^{2b-2[b]-1} \exp\{-(\frac{1}{2} - \frac{1+a}{2\varepsilon^2})x^2\} dx \\
&= \frac{\prod_{j=0}^{[b]-1}(b-j)}{2(1+a)^{[b]+1}} \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - 1 - a)^{[b]} \int_{\varepsilon^2-1-a}^{\infty} s^{b-[b]-1} e^{-s} ds \\
&= 0.
\end{aligned}$$

Therefore, letting $\theta \to 0$, (2.3) follows from (2.4), (2.7), (2.8) and (2.12)$\sim$(2.15), which completes the proof. □

The following is the proof of Theorem 1.1 in general case via the non-uniform Berry-Esseen bound for self-normalized sums.

Proof of Theorem 1.1. We will only verify (1.7) since the proof of (1.6) is similar. Let $\psi_n(\varepsilon) = \sqrt{2\log\log n}(\varepsilon + \alpha_n(\varepsilon))$. By Proposition 2.1 (2.3), it suffices to prove that

$$\lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^b \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} |\mathbf{E}\{|S_n|/V_n - \psi_n(\varepsilon)\}_+ - \mathbf{E}\{|N| - \psi_n(\varepsilon)\}_+| = 0. \tag{2.16}$$

Write

$$\begin{aligned}
&|\mathbf{E}\{|S_n|/V_n - \psi_n(\varepsilon)\}_+ - \mathbf{E}\{|N| - \psi_n(\varepsilon)\}_+| \\
&\le \int_0^\infty |\mathbf{P}(\frac{|S_n|}{V_n} \ge x + \psi_n(\varepsilon)) - \mathbf{P}(|N| \ge x + \psi_n(\varepsilon))| dx
\end{aligned}$$

and

$$\begin{aligned}
&|\mathbf{P}(\frac{|S_n|}{V_n} \ge x + \psi_n(\varepsilon)) - \mathbf{P}(|N| \ge x + \psi_n(\varepsilon))| \\
&\le |\mathbf{P}(\frac{|S_n|}{V_n} \ge x + \psi_n(\varepsilon)) - \mathbf{P}(\frac{|\bar{S}_n|}{\bar{V}_n} \ge x + \psi_n(\varepsilon))| \\
&\quad + |\mathbf{P}(\frac{|\bar{S}_n|}{\bar{V}_n} \ge x + \psi_n(\varepsilon)) - \mathbf{P}(|N| \ge x + \psi_n(\varepsilon))| \\
&\le |\mathbf{P}(\frac{S_n}{V_n} \ge x + \psi_n(\varepsilon)) - \mathbf{P}(\frac{\bar{S}_n}{\bar{V}_n} \ge x + \psi_n(\varepsilon))| \\
&\quad + |\mathbf{P}(-\frac{S_n}{V_n} \ge x + \psi_n(\varepsilon)) - \mathbf{P}(-\frac{\bar{S}_n}{\bar{V}_n} \ge x + \psi_n(\varepsilon))| \\
&\quad + |\mathbf{P}(\frac{|\bar{S}_n|}{\bar{V}_n} \ge x + \psi_n(\varepsilon)) - \mathbf{P}(|N| \ge x + \psi_n(\varepsilon))| \\
&=: K_{x1} + K_{x2} + K_{x3}.
\end{aligned}$$

To prove (2.16), it suffices to show that

$$\lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^b \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \int_0^\infty K_{xi} dx = 0, \qquad i = 1, 2, 3. \tag{2.17}$$

Note that for any $s, t \in R$, $c \ge 0$ and $x \ge 1$,

$$\begin{aligned}
x\sqrt{c + t^2} &= \sqrt{(x^2-1)c + t^2 + c + (x^2-1)t^2} \\
&\ge \sqrt{(x^2-1)c + t^2 + 2t\sqrt{(x^2-1)c}} \\
&= t + \sqrt{(x^2-1)c}.
\end{aligned}$$

Thus we have

$$\{s + t \ge x\sqrt{c + t^2}\} \subset \{s \ge (x^2-1)^{1/2}\sqrt{c}\}.$$

Hence, for any $x > 0$

$$
\begin{aligned}
K_{x1} &\le \mathbf{P}(S_n \ge V_n(x+\psi_n(\varepsilon)), \max_{1\le i\le n} |X_i| > \eta_n) \\
&\quad + \mathbf{P}(\bar{S}_n \ge \bar{V}_n(x+\psi_n(\varepsilon)), \max_{1\le i\le n} |X_i| > \eta_n) \\
&\le \sum_{i=1}^{n} \mathbf{P}(S_n^{(i)} \ge ((x+\psi_n(\varepsilon))^2-1)^{1/2} V_n^{(i)}, |X_i| > \eta_n) \\
&\quad + \sum_{i=1}^{n} \mathbf{P}(\bar{S}_n^{(i)} \ge ((x+\psi_n(\varepsilon))^2-1)^{1/2} \bar{V}_n^{(i)}, |X_i| > \eta_n) \\
&\le \sum_{i=1}^{n} \mathbf{P}(S_n^{(i)} \ge ((x+\psi_n(\varepsilon))^2-1)^{1/2} V_n^{(i)}) \mathbf{P}(|X_i| > \eta_n) \\
&\quad + \sum_{i=1}^{n} \mathbf{P}(\bar{S}_n^{(i)} \ge ((x+\psi_n(\varepsilon))^2-1)^{1/2} \bar{V}_n^{(i)}) \mathbf{P}(|X_i| > \eta_n).
\end{aligned}
\tag{2.18}
$$

Notice that $\eta_n^2 \sim nl(\eta_n)/(\log\log n)^2$ and $l(\eta_n)$ is a slowly varying function at ∞. From (1.5), (2.1), (2.18), Lemma 4.3 in [10] we have for some $0 \le \beta < 1$

$$
\begin{aligned}
&\lim_{\varepsilon\searrow\sqrt{1+a}} (\varepsilon^2-a-1)^b \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \int_0^{\infty} K_{x1} dx \\
&= \lim_{\varepsilon\searrow\sqrt{1+a}} (\varepsilon^2-a-1)^b \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \left[\int_0^{\psi_n(\varepsilon)} + \int_{\psi_n(\varepsilon)}^{\infty}\right] K_{x1} dx \\
&\le A \lim_{\varepsilon\searrow\sqrt{1+a}} (\varepsilon^2-a-1)^b \sum_{n=1}^{\infty} (\log n)^a (\log\log n)^{b-1/2} (\log n)^{-(\varepsilon+\alpha_n(\varepsilon))^2} \mathbf{P}(|X| > \eta_n) \\
&\quad + A \lim_{\varepsilon\searrow\sqrt{1+a}} (\varepsilon^2-a-1)^b \sum_{n=1}^{\infty} (\log n)^a (\log\log n)^b \int_{\psi_n(\varepsilon)}^{\infty} \exp\{-\frac{(x+\psi_n(\varepsilon))^2}{2}\} dx \mathbf{P}(|X| > \eta_n) \\
&\le A \lim_{\varepsilon\searrow\sqrt{1+a}} (\varepsilon^2-a-1)^b \sum_{n=1}^{\infty} (\log n)^{-1} (\log\log n)^{b-1/2} \mathbf{P}(|X| > \eta_n) \\
&\le A \lim_{\varepsilon\searrow\sqrt{1+a}} (\varepsilon^2-a-1)^b \sum_{k=1}^{\infty} \eta_k^{-2} \mathbf{E} X^2 I\{\eta_k < |X| \le \eta_{k+1}\} \sum_{n=1}^{k} (\log n)^{-1} (\log\log n)^{b-1/2} \\
&\le A \lim_{\varepsilon\searrow\sqrt{1+a}} (\varepsilon^2-a-1)^b \sum_{k=1}^{\infty} \frac{1}{l(\eta_k)(\log k)^{\beta}(\log\log k)^2} \mathbf{E} X^2 I\{\eta_k < |X| \le \eta_{k+1}\} \\
&\le A \lim_{\varepsilon\searrow\sqrt{1+a}} (\varepsilon^2-a-1)^b \sum_{j=1}^{\infty} \frac{\mathbf{E} X^2 I\{|X| \le \eta_{j+1}\}}{jl(\eta_j)(\log j)(\log\log j)^2} = 0.
\end{aligned}
$$

Similarly,

$$
\lim_{\varepsilon\searrow\sqrt{1+a}} (\varepsilon^2-a-1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \int_0^{\infty} K_{x2} dx = 0.
$$

From Corollary 2.1 in [10] and (2.1), for some $0 \leq \beta < 1$

$$\begin{aligned}
&\lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^b \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \int_0^{\infty} K_{x3} dx \\
&= \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^b \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \left[\int_0^{\psi_n(\varepsilon)} + \int_{\psi_n(\varepsilon)}^{\infty} \right] K_{x3} dx \\
&\leq A \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^b \sum_{n=1}^{\infty} (\log n)^{a-(\varepsilon+\alpha_n(\varepsilon))^2} (\log\log n)^{b+2} \frac{\mathbf{E}|X|^3 I\{|X| \leq \eta_n\}}{n^{3/2}(l(\eta_n))^{3/2}} \\
&\leq A \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^b \sum_{k=1}^{\infty} \eta_k \mathbf{E} X^2 I\{\eta_{k-1} < |X| \leq \eta_k\} \sum_{n=k}^{\infty} \frac{(\log\log n)^{b+2}}{n^{3/2}(\log n)(l(\eta_n))^{3/2}} \\
&\leq A \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^b \sum_{k=1}^{\infty} \frac{1}{l(\eta_k)(\log k)^{\beta} (\log\log k)^2} \mathbf{E} X^2 I\{\eta_{k-1} < |X| \leq \eta_k\} \\
&\leq A \lim_{\varepsilon \searrow \sqrt{1+a}} (\varepsilon^2 - a - 1)^b \sum_{j=1}^{\infty} \frac{\mathbf{E} X^2 I\{|X| \leq \eta_j\}}{j l(\eta_j)(\log j)(\log\log j)^2} = 0.
\end{aligned}$$

The proof is complete. □

Acknowledgments. The author would like to express his gratitude to the referee for his/her insightful comments and invaluable suggestions, and the author gratefully acknowledges the many helpful suggestions of Dr. Wang during the preparation of the paper.

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, Vol. 27. Cambridge University Press, Cambridge, 1987.

[2] Y. S. Chow, *On the rate of moment convergence of sample sums and extremes*, Bull. Inst. Math. Acad. Sin. (N.S.) **16** (1988), no. 3, 177–201.

[3] A. Gut and A. Spătaru, *Precise asymptotics in the law of the iterated logarithm*, Ann. Probab. **28** (2000), no. 4, 1870–1883.

[4] C. C. Heyde, *A supplement to the strong law of large numbers*, J. Appl. Probab. **12** (1975), no. 1, 173–175.

[5] P. L. Hsu and H. Robbins, *Complete convergence and the law of large numbers*, Proc. Natl. Acad. Sci. USA **33** (1947), no. 2, 25–31.

[6] M. L. Katz, *The probability in the tail of a distribution*, Ann. Math. Statist. **34** (1963), no. 1, 312–318.

[7] D. Li, B.-E. Nguyen, and A. Rosalsky, *A supplement to precise asymptotics in the law of the iterated logarithm*, J. Math. Anal. Appl. **302** (2005), no. 1, 84–96.

[8] T.-X. Pang, Z.-Y. Lin, Y. Jiang, and K.-S. Hwang, *Precise rates in the law of the logarithm for the moment convergence of i.i.d. random variables*, J. Koran Math. Soc. **45** (2008), no. 4, 993–1005.

[9] A. Spătaru, *Precise asymptotics in Spitzer's law of large numbers*, J. Theoret. Probab. **12** (1999), no. 3, 811–819.

[10] Q. Wang and B. Y. Jing, *An exponential nonuniform Berry-Esseen bound for self-normalized sums*, Ann. Probab. **27** (1999), no. 4, 2068–2088.

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