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MULTIPLIER THEOREMS IN WEIGHTED SMIRNOV SPACES

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Abstract. The analogues of Marcinkiewicz multiplier theorem and Littlewood-Paley theorem are proved for p-Faber series in weighted Smirnov spaces defined on bounded and unbounded components of a rectifiable Jordan curve.

1. Introduction and the main results

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$, and let $G := \text{Int}\Gamma$, $G^{-} := \text{Ext}\Gamma$. Without loss of generality we assume that $0 \in G$. Let also

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} := \partial\mathbb{D}, \quad \mathbb{D}^{-} := \mathbb{C} \backslash \overline{\mathbb{D}}.$$

We denote by φ and φ_1 the conformal mappings of G^{-} and G onto $\mathbb{D}^{-}$, respectively, normalized by

$$\varphi(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0$$

and

$$\varphi_1(0) = \infty, \quad \lim_{z\to 0} z\varphi_1(z) > 0.$$

The inverse mappings of φ and φ_1 will be denoted by ψ and ψ_1, respectively.

Let $1 \leq p < \infty$. A function f is said to belongs to the *Smirnov space* $E_p(G)$ if it is analytic in G and satisfies

$$\sup_{0\leq r<1} \int_{\Gamma_r} |f(z)|^p |dz| < \infty,$$

where Γ_r is the image of the circle $\{z \in \mathbb{C} : |z| = r\}$ under a conformal mapping of $\mathbb{D}$ onto G. The functions belong to $E_p(G)$ have nontangential limits almost

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everywhere (a.e.) on Γ, and these limit functions belong to the Lebesgue space $L_p(\Gamma)$. The Smirnov space $E_p(G)$ is a Banach space with respect to the norm

$$\|f\|_{E_p(G)} := \|f\|_{L_p(\Gamma)} = \left(\int_{\Gamma} |f(z)|^p |dz| \right)^{1/p}.$$

The Smirnov spaces $E_p(G^-), 1 \leq p < \infty$ are defined similarly. It is known that $\varphi' \in E_1(G^-)$, $\varphi_1' \in E_1(G)$ and ψ', $\psi_1' \in E_1(\mathbb{D}^-)$. The general information about Smirnov spaces can be found in [3, pp. 168–185] and [4, pp. 438–453].

Let ω be a weight function (nonnegative, integrable function) on Γ and let $L_p(\Gamma, \omega)$ be the ωweighted Lebesgue space on Γ, i.e., the space of measurable functions on Γ for which

$$\|f\|_{L_p(\Gamma,\omega)} := \left(\int_{\Gamma} |f(z)|^p \omega(z) |dz| \right)^{1/p} < \infty.$$

The *ω-weighted Smirnov spaces* $E_p(G, \omega)$ and $E_p(G^-, \omega)$ are defined as

$$E_p(G, \omega) := \{ f \in E_1(G) : f \in L_p(\Gamma, \omega) \}$$

and

$$E_p\left(G^-, \omega\right) := \left\{ f \in E_1\left(G^-\right) : f \in L_p(\Gamma, \omega) \right\}.$$

We also define the following subspace of $E_p(G^-, \omega)$:

$$\widetilde{E}_p\left(G^-, \omega\right) := \left\{ f \in E_p\left(G^-, \omega\right) : f(\infty) = 0 \right\}.$$

Let $1 < p < \infty$. For $k = 0, 1, 2, \ldots$, the functions $\varphi^k (\varphi')^{1/p}$ and $\varphi_1^{k-2/p} (\varphi_1')^{1/p}$ have poles of order k at the points ∞ and 0, respectively. Hence, there exist polynomials $F_{k,p}$ and $\widetilde{F}_{k,p}$ of degree k, and functions $E_{k,p}$ and $\widetilde{E}_{k,p}$ analytic in G^- and G, respectively, such that the following relations holds:

$$\begin{aligned} [\varphi(z)]^k (\varphi'(z))^{1/p} &= F_{k,p}(z) + E_{k,p}(z), \quad z \in G^- \\ [\varphi_1(z)]^{k-2/p} (\varphi_1'(z))^{1/p} &= \widetilde{F}_{k,p}(1/z) + \widetilde{E}_{k,p}(z), \quad z \in G \setminus \{0\}. \end{aligned}$$

The polynomials $F_{k,p}$ and $\widetilde{F}_{k,p}$ $(k = 0, 1, 2, \ldots)$ are called the p-Faber polynomials for G and G^-, respectively. It is clear that $\widetilde{F}_{0,p}(1/z) = 0$.

It is known that the integral representations

$$F_{k,p}(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{w^k (\psi'(w))^{1-1/p}}{\psi(w) - z} dw, \quad z \in G, \ R \geq 1$$

$$\widetilde{F}_{k,p}(1/z) = -\frac{1}{2\pi i} \int_{|w|=R} \frac{w^k w^{-2/p} (\psi_1'(w))^{1-1/p}}{\psi_1(w) - z} dw, \quad z \in G^-, \ R \geq 1$$

and the expansions

$$\frac{(\psi'(w))^{1-1/p}}{\psi(w)-z}=\sum_{k=0}^{\infty}\frac{F_{k,p}(z)}{w^{k+1}},\quad z\in G,\ w\in\mathbb{D}^-, \tag{1}$$

$$\frac{w^{-2/p}(\psi_1'(w))^{1-1/p}}{\psi_1(w)-z}=\sum_{k=1}^{\infty}-\frac{\widetilde{F}_{k,p}(1/z)}{w^{k+1}},\quad z\in G^-,\ w\in\mathbb{D}^-, \tag{2}$$

holds (see [6]).

Let $f\in E_p(G,\omega)$. Since $f\in E_1(G)$, by Cauchy's integral formula, we have

$$f(z)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f(\varsigma)}{\varsigma-z}d\varsigma=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f(\psi(w))(\psi'(w))^{1/p}(\psi'(w))^{1-1/p}}{\psi(w)-z}dw,\quad z\in G.$$

Hence, by taking into account (1) we can associate with f the series

$$f(z)\sim\sum_{k=0}^{\infty}a_k(f)F_{k,p}(z),\quad z\in G, \tag{3}$$

where

$$a_k(f):=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f(\psi(w))(\psi'(w))^{1/p}}{w^{k+1}}dw,\quad k=0,1,2,\ldots.$$

By the Cauchy formula and (2) we can also associate with $f\in\widetilde{E}_p(G^-,\omega)$ the series

$$f(z)\sim\sum_{k=1}^{\infty}\widetilde{a}_k(f)\widetilde{F}_{k,p}(1/z),\quad z\in G^-, \tag{4}$$

where

$$\widetilde{a}_k(f):=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f(\psi_1(w))(\psi_1'(w))^{1/p}w^{2/p}}{w^{k+1}}dw,\quad k=1,2,\ldots.$$

The series (3) and (4) are called the p-Faber series, and the coefficients $a_k(f)$ and $\widetilde{a}_k(f)$ are called the p-Faber coefficients of the corresponding functions.

Definition 1. A rectifiable Jordan curve Γ is called a *Carleson curve* if the condition

$$\sup_{z\in\Gamma}\sup_{\varepsilon>0}\frac{1}{\varepsilon}\,|\Gamma(z,\varepsilon)|<\infty$$

holds, where $\Gamma(z,\varepsilon)$ is the portion of Γ in the open disk of radius ε centered at z, and $|\Gamma(z,\varepsilon)|$ its length.

Definition 2. Let $1 < p < \infty$. A weight function ω belongs to the *Muckenhoupt class* $A_p(\Gamma)$ if the condition

$$\sup_{z\in\Gamma}\sup_{\varepsilon>0}\left(\frac{1}{\varepsilon}\int\limits_{\Gamma(z,\varepsilon)}\omega(\tau)\,|d\tau|\right)\left(\frac{1}{\varepsilon}\int\limits_{\Gamma(z,\varepsilon)}[\omega(\tau)]^{-1/(p-1)}\,|d\tau|\right)^{p-1}<\infty$$

holds.

The Carleson curves and Muckenhoupt classes $A_p(\Gamma)$ were studied in details in [1].

We consider the sequences $\{\lambda_k\}_0^\infty$ of complex numbers which satisfies the following conditions for all natural numbers k and m:

$$|\lambda_k|\leq c,\qquad \sum_{k=2^{m-1}}^{2^m-1}|\lambda_k-\lambda_{k+1}|\leq c. \tag{5}$$

For a given weight function ω on Γ we define two weights on $\mathbb{T}$ by setting $\omega_0 := \omega\circ\psi$ and $\omega_1 := \omega\circ\psi_1$.

We shall denote by $c_1, c_2, \ldots$ the constants (in general, different in different relations) depending only on numbers that are not important for the questions of interest.

Our main results are the following:

Theorem 1. *Let Γ be a Carleson curve, $1 < p < \infty$, $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$. If $f \in E_p(G,\omega)$ with the p-Faber series* (3) *and $\{\lambda_k\}_0^\infty$ is a sequence of complex numbers which satisfies the condition* (5)*, then there exists a function $F \in E_p(G,\omega)$ which has the p-Faber series*

$$F(z)\sim\sum_{k=0}^{\infty}\lambda_k a_k(f)\,F_{k,p}(z),\quad z\in G,$$

and $\|F\|_{L_p(\Gamma,\omega)} \leq c_1\|f\|_{L_p(\Gamma,\omega)}$.

Similar theorem holds for $f \in \widetilde{E}_p(G^-,\omega)$:

Theorem 2. *Let Γ be a Carleson curve, $1 < p < \infty$, $\omega \in A_p(\Gamma)$ and $\omega_1 \in A_p(\mathbb{T})$. If $f \in \widetilde{E}_p(G^-,\omega)$ with the p-Faber series* (4) *and $\{\lambda_k\}_0^\infty$ is a sequence of complex numbers which satisfies the condition* (5)*, then there exists a function $F \in \widetilde{E}_p(G^-,\omega)$ which has the p-Faber series*

$$F(z)\sim\sum_{k=1}^{\infty}\lambda_k\widetilde{a}_k(f)\,\widetilde{F}_{k,p}(1/z),\quad z\in G^-$$

and $\|F\|_{L_p(\Gamma,\omega)} \leq c_2\|f\|_{L_p(\Gamma,\omega)}$.

For Fourier series in Lebesgue spaces on the interval $[0, 2\pi]$ the multiplier theorem was proved by Marcinkiewicz in [11] (see also, [16, Vol. II, p. 232]). For weighted Lebesgue spaces with Muckenhoupt weights the similar theorem can be deduced from Theorem 2 of [9]. The analogue of Theorem 1 in nonweighted Smirnov spaces was cited by V. Kokilashvili without proof in [8].

We introduce the notations

$$\Delta_{k,p}(f)(z) := \sum_{j=2^{k-1}}^{2^k-1} a_j(f) F_{j,p}(z)$$

and

$$\widetilde{\Delta}_{k,p}(f)(z) := \sum_{j=2^{k-1}}^{2^k-1} \widetilde{a}_j(f) \widetilde{F}_{j,p}(1/z)$$

for $f \in E_p(G, \omega)$ and $f \in \widetilde{E}_p(G^-, \omega)$, respectively. By virtue of Theorems 1 and 2 we prove the following Littlewood-Paley type theorems:

Theorem 3. *Let Γ be a Carleson curve, $1 < p < \infty$, $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$. If $f \in E_p(G, \omega)$, then the two-sided estimate*

$$c_3 \|f\|_{L_p(\Gamma,\omega)} \leq \left\| \left(\sum_{k=0}^{\infty} |\Delta_{k,p}(f)|^2 \right)^{1/2} \right\|_{L_p(\Gamma,\omega)} \leq c_4 \|f\|_{L_p(\Gamma,\omega)} \tag{6}$$

holds.

Theorem 4. *Let Γ be a Carleson curve, $1 < p < \infty$, $\omega \in A_p(\Gamma)$ and $\omega_1 \in A_p(\mathbb{T})$. If $f \in \widetilde{E}_p(G^-, \omega)$, then the two-sided estimate*

$$c_5 \|f\|_{L_p(\Gamma,\omega)} \leq \left\| \left(\sum_{k=0}^{\infty} \left|\widetilde{\Delta}_{k,p}(f)\right|^2 \right)^{1/2} \right\|_{L_p(\Gamma,\omega)} \leq c_6 \|f\|_{L_p(\Gamma,\omega)} \tag{7}$$

holds.

Such theorems were firstly proved by J. E. Littlewood and R. Paley in [10] for the spaces $L_p(\mathbb{T})$, $1 < p < \infty$ (see also, [16, Vol II, pp. 222–241]) and play an important role in the various problems of approximation theory. For example, in [14], M. F. Timan obtained an improvement of the inverse approximation theorems by trigonometric polynomials in Lebesgue spaces $L_p(\mathbb{T})$, $1 < p < \infty$ by aim of the Littlewood-Paley theorems. Timan also improved the direct approximation theorem by using the same results [15]. By considering the analogue of Littlewood-Paley theorems in Smirnov spaces $E_p(G)$, V. Kokilashvili obtained very good results on polynomial approximation in these spaces [8]. For the spaces $L_p(\mathbb{T}, \omega)$, where $\omega \in A_p(\mathbb{T})$, the Littlewood-Paley type theorem can be obtained from Theorem 1 of [9].

In Theorems 1-4, it is assumed that Γ to be a Carleson curve and the weight functions to be Muckenhoupt weights. Because, proofs of Theorems 1-4 depend on the boundedness of the Cauchy singular operator, and the Cauchy singular operator is bounded on the space $L_p(\Gamma, \omega)$ if and only if Γ is a Carleson curve and $\omega \in A_p(\Gamma)$ (see Theorem 5).

2. Auxiliary results

Let Γ be rectifiable Jordan curve and $f \in L_1(\Gamma)$. The functions f^+ and f^- defined by

$$f^+(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z}d\varsigma, \quad z \in G, \tag{8}$$

and

$$f^-(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z}d\varsigma, \quad z \in G^-, \tag{9}$$

are analytic in G and G^-, respectively, and $f^-(\infty) = 0$.

It is known that [5, Lemma 3] if Γ is a Carleson curve and $\omega \in A_p(\Gamma)$, then $f^+ \in E_p(G, \omega)$ and $f^- \in E_p(G^-, \omega)$ for $f \in L_p(\Gamma, \omega)$, $1 < p < \infty$.

Since $f \in L_1(\Gamma)$, the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to 0}\frac{1}{2\pi i}\int_{\Gamma\setminus\Gamma(z,\varepsilon)} \frac{f(\varsigma)}{\varsigma - z}\,d\varsigma$$

exists and is finite for almost all $z \in \Gamma$ (see [1, pp. 117–144]). $S_\Gamma(f)(z)$ is called the *Cauchy singular integral* of f at $z \in \Gamma$.

The functions f^+ and f^- have nontangential limits a.e. on Γ and the formulas

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z), \quad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z) \tag{10}$$

holds for almost every $z \in \Gamma$ [4, p. 431]. Hence we have

$$f = f^+ - f^- \tag{11}$$

a.e. on Γ.

For $f \in L_1(\Gamma)$, we associate the function $S_\Gamma(f)$ taking the value $S_\Gamma(f)(z)$ a.e. on Γ. The linear operator S_Γ defined in such way is called the *Cauchy singular operator.* The following theorem, which is analogously deduced from David's theorem (see [2]), states the necessary and sufficient condition for boundedness of S_Γ in $L_p(\Gamma, \omega)$ (see also [1, pp. 117–144]).

Theorem 5. *Let Γ be a rectifiable Jordan curve, $1 < p < \infty$, and let ω be a weight function on Γ. The inequality*

$$\|S_\Gamma(f)\|_{L_p(\Gamma,\omega)} \le c_7 \|f\|_{L_p(\Gamma,\omega)}$$

holds for every $f \in L_p(\Gamma, \omega)$ *if and only if* Γ *is a Carleson curve and* $\omega \in A_p(\Gamma)$.

Let $\mathcal{P}$ be the set of all algebraic polynomials (with no restrictions on the degree), and let $\mathcal{P}(\mathbb{D})$ be the set of traces of members of $\mathcal{P}$ on $\mathbb{D}$. If we define the operators $T_p : \mathcal{P}(\mathbb{D}) \to E_p(G, \omega)$ and $\widetilde{T}_p : \mathcal{P}(\mathbb{D}) \to \widetilde{E}_p(G^-, \omega)$ as

$$T_p(P)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P(w)(\psi'(w))^{1-1/p}}{\psi(w) - z} dw, \quad z \in G$$

and

$$\widetilde{T}_p(P)(z) := -\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P(w) w^{-2/p} (\psi_1'(w))^{1-1/p}}{\psi_1(w) - z} dw, \quad z \in G^-,$$

then it is clear that

$$T_p\left(\sum_{k=0}^{n} \alpha_k w^k\right) = \sum_{k=0}^{n} \alpha_k F_{k,p}(z), \quad \widetilde{T}_p\left(\sum_{k=0}^{n} \alpha_k w^k\right) = \sum_{k=1}^{n} \alpha_k \widetilde{F}_{k,p}(1/z).$$

Taking into account (8), we get

$$T_p(P)(z') = \left[(P \circ \varphi)(\varphi')^{1/p}\right]^+(z')$$

for $z' \in G$. Taking the limit $z' \to z \in \Gamma$ over all nontangential paths inside Γ, we obtain by (10)

$$T_p(P)(z) = \frac{1}{2}\left[(P \circ \varphi)(\varphi')^{1/p}\right](z) + S_\Gamma\left[(P \circ \varphi)(\varphi')^{1/p}\right](z)$$

for almost all $z \in \Gamma$. Similarly, by considering (9) and taking the limit along all nontangential paths outside Γ, by (10) we get

$$\widetilde{T}_p(P)(z) = \frac{1}{2}\left[(P \circ \varphi_1)\varphi_1^{-2/p}(\varphi_1')^{1/p}\right](z) - S_\Gamma\left[(P \circ \varphi_1)\varphi_1^{-2/p}(\varphi_1')^{1/p}\right](z)$$

a.e. on Γ.

Therefore we can state the following theorem as a corollary of Theorem 5:

Theorem 6. *Let* Γ *be a Carleson curve,* $1 < p < \infty$, *and let* ω *be a weight function on* Γ. *The following assertions hold*:

(a) *If* $\omega \in A_p(\Gamma)$ *and* $\omega_0 \in A_p(\mathbb{T})$, *then the linear operator*

$$T_p : \mathcal{P}(\mathbb{D}) \subset E_p(\mathbb{D}, \omega_0) \to E_p(G, \omega)$$

is bounded.

(b) *If* $\omega \in A_p(\Gamma)$ *and* $\omega_1 \in A_p(\mathbb{T})$, *then the linear operator*

$$\widetilde{T}_p : \mathcal{P}(\mathbb{D}) \subset E_p(\mathbb{D}, \omega_1) \to \widetilde{E}_p\left(G^-, \omega\right)$$

is bounded.

Hence, the operators T_p and $\widetilde{T}_p$ can be extended as bounded linear operators to $E_p(\mathbb{D},\omega_0)$ and $E_p(\mathbb{D},\omega_1)$, respectively, and we have the representations

$$T_p(g)(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g(w)(\psi'(w))^{1-1/p}}{\psi(w)-z}dw, \quad g \in E_p(\mathbb{D},\omega_0),$$

and

$$\widetilde{T}_p(g)(z) := -\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g(w)\,w^{-2/p}(\psi_1'(w))^{1-1/p}}{\psi_1(w)-z}dw, \quad g \in E_p(\mathbb{D},\omega_1).$$

Lemma 1. *Let Γ be a Carleson curve, $1<p<\infty$, and $\omega \in A_p(\Gamma)$. Further let g be an analytic function in $\mathbb{D}$, which has the Taylor expansion $g(w) = \sum_{k=0}^{\infty}\alpha_k(g)\,w^k$.*

(a) *If $g \in E_p(\mathbb{D},\omega_0)$ and $\omega_0 \in A_p(\mathbb{T})$, then $T_p(g)$ has the p-Faber coefficients $\alpha_k(g)$, $k=0,1,2,\ldots$.*
(b) *If $g \in E_p(\mathbb{D},\omega_1)$ and $\omega_0 \in A_p(\mathbb{T})$, then $\widetilde{T}_p(g)$ has the p-Faber coefficients $\alpha_k(g)$, $k=0,1,2,\ldots$.*

Proof. Let's prove the statement (b). The statement (a) can be proved similarly.

If we set

$$g_r(w) := g(rw), \quad 0<r<1,$$

and take into account that every function in $E_1(\mathbb{D})$ coincides with the Poisson integral of its boundary function, we have by [12, Theorem 10]

$$\|g_r - g\|_{L_p(\mathbb{T},\omega_1)} \to 0, \quad r \to 1^-,$$

and then the boundedness of the operator $\widetilde{T}_p$ yields

$$\left\|\widetilde{T}_p(g_r) - \widetilde{T}_p(g)\right\|_{L_p(\Gamma,\omega)} \to 0, \quad r \to 1^-. \tag{12}$$

The series $\sum_{k=0}^{\infty}\alpha_k(g)\,r^k w^k$ converges uniformly on $\mathbb{T}$, hence,

$$\begin{aligned}
\widetilde{T}_p(g_r)(z) &= -\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g_r(w)\,w^{-2/p}(\psi_1'(w))^{1-1/p}}{\psi_1(w)-z}dw \\
&= \sum_{k=0}^{\infty}\alpha_k(g)\,r^k\left\{-\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{w^k w^{-2/p}(\psi_1'(w))^{1-1/p}}{\psi_1(w)-z}dw\right\} \\
&= \sum_{k=0}^{\infty}\alpha_k(g)\,r^k\widetilde{F}_{k,p}(1/z)
\end{aligned}$$

for $z \in G^{-}$. By a simple calculation one can see that

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\widetilde{F}_{m,p}\left(\frac{1}{\psi_1(w)}\right) w^{2/p} \left(\psi_1'(w)\right)^{1/p}}{w^{k+1}} dw = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$$

and as a corollary of this

$$\widetilde{a}_k\left(\widetilde{T}_p(g_r)\right) = \alpha_k(g)\, r^k, \quad k = 0, 1, 2, \ldots .$$

Therefore,

$$\widetilde{a}_k\left(\widetilde{T}_p(g_r)\right) \to \alpha_k(g), \quad r \to 1^{-}. \tag{13}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned}
&\left|\widetilde{a}_k\left(\widetilde{T}_p(g_r)\right) - \widetilde{a}_k\left(\widetilde{T}_p(g)\right)\right| \\
= &\left|\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\left[\widetilde{T}_p(g_r) - \widetilde{T}_p(g)\right](\psi_1(w))\, w^{2/p}\left(\psi_1'(w)\right)^{1/p}}{w^{k+1}} dw\right| \\
\leq &\ \frac{1}{2\pi}\int_{\mathbb{T}} \left|\left(\widetilde{T}_p(g_r) - \widetilde{T}_p(g)\right)(\psi_1(w))\right| \left|\left(\psi_1'(w)\right)^{1/p}\right| |dw| \\
\leq &\ \frac{1}{2\pi}\left(\int_{\mathbb{T}} \left|\left(\widetilde{T}_p(g_r) - \widetilde{T}_p(g)\right)(\psi_1(w))\right|^p \omega(\psi_1(w))\, |\psi_1'(w)|\, |dw|\right)^{1/p} \\
&\times \left(\int_{\mathbb{T}} \left[\omega(\psi_1(w))\right]^{-1/p-1} |dw|\right)^{1-1/p} \\
= &\ \frac{1}{2\pi}\left\|\widetilde{T}_p(g_r) - \widetilde{T}_p(g)\right\|_{L_p(\Gamma,\omega)} \left(\int_{\mathbb{T}} \left[\omega_1(w)\right]^{-1/p-1} |dw|\right)^{1-1/p},
\end{aligned}$$

and by (12)

$$\widetilde{a}_k\left(\widetilde{T}_p(g_r)\right) \to \widetilde{a}_k\left(\widetilde{T}_p(g)\right)$$

as $r \to 1^{-}$. This and (13) yield that

$$\widetilde{a}_k\left(\widetilde{T}_p(g)\right) = \alpha_k(g), \quad k = 0, 1, 2, \ldots$$

which proves the part (b) of Lemma 1. □

3. Proofs of the main results

We need the following lemma to prove Theorem 1 and Theorem 2.

Lemma 2. *Let $\omega \in A_p(\mathbb{T})$, $1 < p < \infty$, and let $\{\lambda_k\}_0^\infty$ be a sequence which satisfies the condition* (5)*. If the function $g \in E_p(\mathbb{D}, \omega)$ has the Taylor series*

$$g(w) = \sum_{k=0}^{\infty} \alpha_k(g)\, w^k, \quad w \in \mathbb{D},$$

then there exists a function $g^* \in E_p(\mathbb{D}, \omega)$ which has the Taylor series

$$g^*(w) = \sum_{k=0}^{\infty} \lambda_k \alpha_k(g)\, w^k, \quad w \in \mathbb{D},$$

and satisfies $\|g^*\|_{L_p(\mathbb{T},\omega)} \leq c_8 \|g\|_{L_p(\mathbb{T},\omega)}$.

Proof. Let $c_k(g)$ $(k = \ldots, -1, 0, 1, \ldots)$ denote the Fourier coefficients of the boundary function of g. By Theorem 3.4 in [3, p. 38] we have

$$c_k(g) = \begin{cases} \alpha_k(g), & k \geq 0 \\ 0, & k < 0. \end{cases}$$

By Theorem 2 of [9], there is a function $h \in L_p(\mathbb{T}, \omega)$ with Fourier coefficients $c_k(h) = \lambda_k c_k(g)$ and $\|h\|_{L_p(\mathbb{T},\omega)} \leq c_9 \|g\|_{L_p(\mathbb{T},\omega)}$. If we take $g^* := h^+$, then $g^* \in E_p(\mathbb{D}, \omega)$. For Taylor coefficients of g^*, we have by (11)

$$\begin{aligned} \alpha_k(g^*) &= \alpha_k\left(h^+\right) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{h^+(w)}{w^{k+1}}dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{h(w)}{w^{k+1}}dw + \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{h^-(w)}{w^{k+1}}dw \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{h(w)}{w^{k+1}}dw = c_k(h) = \lambda_k c_k(g) = \lambda_k \alpha_k(g) \end{aligned}$$

for $k = 0, 1, 2, \ldots$. On the other hand,

$$\|g^*\|_{L_p(\mathbb{T},\omega)} = \left\|h^+\right\|_{L_p(\mathbb{T},\omega)} \leq c_{10} \|h\|_{L_p(\mathbb{T},\omega)} \leq c_{11} \|g\|_{L_p(\mathbb{T},\omega)},$$

and the lemma is proved. □

We set for $f \in E_p(G, \omega)$

$$f_0(w) := f(\psi(w))\,(\psi'(w))^{1/p}, \quad w \in \mathbb{T},$$

and for $f \in \widetilde{E}_p(G^-, \omega)$

$$f_1(w) := f(\psi_1(w))\,(\psi_1'(w))^{1/p}\, w^{2/p}, \quad w \in \mathbb{T}.$$

It is clear that $f_0 \in L_p(\mathbb{T}, \omega_0)$ and $f_1 \in L_p(\mathbb{T}, \omega_1)$. Hence, if $\omega_0, \omega_1 \in A_p(\mathbb{T})$, then $f_0^+ \in E_p(\mathbb{D}, \omega_0)$, $f_0^- \in E_p(\mathbb{D}^-, \omega_0)$, $f_1^+ \in E_p(\mathbb{D}, \omega_1)$, $f_1^- \in E_p(\mathbb{D}^-, \omega_1)$.

Proof of Theorem 1. Let $f \in E_p(G,\omega)$. By the definitions of the coefficients $a_k(f)$ and f_0 from (11), we get

$$\begin{aligned} a_k(f) &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}}dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}}dw - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^-(w)}{w^{k+1}}dw \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}}dw = \alpha_k\left(f_0^+\right) \end{aligned}$$

for $k = 0, 1, 2, \ldots$. This means that the p-Faber coefficients of f are the Taylor coefficients of f_0^+ at the origin, that is,

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k(f)\, w^k, \quad w \in \mathbb{D}.$$

By Lemma 2, there is a function $F_0 \in E_p(\mathbb{D},\omega_0)$ which has the Taylor coefficients $\alpha_k(F_0) = \lambda_k a_k(f)$ for $k = 0, 1, 2, \ldots$, and

$$\|F_0\|_{L_p(\mathbb{T},\omega_0)} \le c_{12}\left\|f_0^+\right\|_{L_p(\mathbb{T},\omega_0)}.$$

Hence, $T_p(F_0) \in E_p(G,\omega)$ and by Lemma 1 the p-Faber coefficients of $T_p(F_0)$ are $\alpha_k(F_0) = \lambda_k a_k(f)$, that is,

$$T_p(F_0)(z) \sim \sum_{k=0}^{\infty} \lambda_k a_k(f)\, F_{k,p}(z), \quad z \in G.$$

On the other hand, boundedness of T_p, (10) and the boundedness of the Cauchy singular operator in $L_p(\mathbb{T},\omega_0)$ yield

$$\begin{aligned} \|T_p(F_0)\|_{L_p(\Gamma,\omega)} &\le \|T_p\|\,\|F_0\|_{L_p(\mathbb{T},\omega_0)} \le c_{13}\left\|f_0^+\right\|_{L_p(\mathbb{T},\omega_0)} \\ &\le c_{14}\|f_0\|_{L_p(\mathbb{T},\omega_0)} = c_{14}\|f\|_{L_p(\Gamma,\omega)}. \end{aligned}$$

Hence taking $F := T_p(F_0)$ finishes the proof of Theorem 1. □

Proof of Theorem 2. By considering the formula of the p-Faber coefficients of $f \in \widetilde{E}_p(G^-,\omega)$,

$$\begin{aligned} \widetilde{a}_k(f) &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_1(w)}{w^{k+1}}dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_1^+(w)}{w^{k+1}}dw - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_1^-(w)}{w^{k+1}}dw \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_1^+(w)}{w^{k+1}}dw = \alpha_k\left(f_1^+\right), \end{aligned}$$

i.e., the p-Faber coefficients of f are the Taylor coefficients of f_1^+. By Lemma 2, there exists a function $F_1 \in E_p(\mathbb{D},\omega_1)$ such that

$$F_1(w) = \sum_{k=0}^{\infty} \lambda_k \widetilde{a}_k(f)\, w^k, \quad w \in \mathbb{D},$$

and

$$\|F_1\|_{L_p(\mathbb{T},\omega_1)} \leq c_{15} \ \left\|f_1^+\right\|_{L_p(\mathbb{T},\omega_1)}.$$

Setting $F := \widetilde{T}_p(F_1)$, we obtain by Lemma 1

$$F(z) \sim \sum_{k=1}^{\infty} \lambda_k \widetilde{a}_k(f)\, \widetilde{F}_{k,p}(1/z), \quad z \in G^-,$$

and by boundedness of $\widetilde{T}_p$ and (10) we obtain

$$\begin{aligned} \|F\|_{L_p(\Gamma,\omega)} &= \left\|\widetilde{T}_p(F_1)\right\|_{L_p(\Gamma,\omega)} \leq \left\|\widetilde{T}_p\right\| \|F_1\|_{L_p(\mathbb{T},\omega_1)} \\ &\leq c_{15} \ \left\|f_1^+\right\|_{L_p(\mathbb{T},\omega_1)} \leq c_{16} \ \|f_1\|_{L_p(\mathbb{T},\omega_1)} = c_{16} \ \|f\|_{L_p(\Gamma,\omega)}, \end{aligned}$$

since the singular operator is bounded in $L_p(\mathbb{T},\omega_1)$. □

Proof of Theorem 3. Let $\{r_k\}_0^{\infty}$ be the sequence of Rademacher functions and let $t \in [0,1]$ be not dyadic rational number. If we set $\lambda_0 := r_0(t)$ and

$$\lambda_j := r_k(t), \quad 2^{k-1} \leq j < 2^k,$$

then the sequence $\{\lambda_j\}_0^{\infty}$ satisfies the condition (5). By Theorem 1 there exists a function $F \in E_p(G,\omega)$ such that

$$F(z) \sim \sum_{j=0}^{\infty} \lambda_j a_j(f)\, F_{j,p}(z) = \sum_{k=0}^{\infty} r_k(t)\, \Delta_{k,p}(f)(z)$$

and

$$\|F\|_{L_p(\Gamma,\omega)} \leq c_{17} \|f\|_{L_p(\Gamma,\omega)}.$$

On the other hand, since

$$F(z) \sim \sum_{k=0}^{\infty} r_k(t)\, \Delta_{k,p}(f)(z)$$

and $\{\lambda_j\}_0^{\infty}$ satisfies (5), there is $F^* \in E_p(G,\omega)$ for which

$$F^*(z) \sim \sum_{k=0}^{\infty} \lambda_k r_k(t)\, \Delta_{k,p}(f)(z) = \sum_{k=0}^{\infty} a_k(f)\, F_{k,p}(z)$$

and

$$\|F^*\|_{L_p(\Gamma,\omega)} \leq c_{18} \|F\|_{L_p(\Gamma,\omega)}$$

holds. Since there is no two different functions in $E_p(G,\omega)$ have the same p-Faber series we have $F^* = f$ and hence

$$c_{19} \|f\|_{L_p(\Gamma,\omega)} \leq \|F\|_{L_p(\Gamma,\omega)} \leq c_{17} \|f\|_{L_p(\Gamma,\omega)}.$$

From this we obtain

$$c_{20} \|f\|_{L_p(\Gamma,\omega)}^p \leq \int_{\Gamma} \left|\sum_{k=0}^{\infty} r_k(t)\, \Delta_{k,p}(f)(z)\right|^p \omega(z)\, |dz| \leq c_{21} \|f\|_{L_p(\Gamma,\omega)}^p. \tag{14}$$

By Theorem 8.4 in [16, Vol I, p. 213] we get

$$\begin{aligned} c_{22}\left(\sum_{k=0}^{\infty}\left|\Delta_{k,p}(f)(z)\right|^{2}\right)^{1/2} &\leq \left(\int_{0}^{1}\left|\sum_{k=0}^{\infty} r_{k}(t)\,\Delta_{k,p}(f)(z)\right|^{p}dt\right)^{1/p} \\ &\leq c_{23}\left(\sum_{k=0}^{\infty}\left|\Delta_{k,p}(f)(z)\right|^{2}\right)^{1/2}. \end{aligned} \tag{15}$$

If we integrate all sides of (14) over $[0,1]$, change the order of integration and use (15) we obtain (6). □

Proof of Theorem 4 is similar to that of Theorem 3.

Let Γ be a Carleson curve, $1<p<\infty$ and $\omega \in A_p(\Gamma)$. For $f \in L_p(\Gamma,\omega)$ we have $f^{+} \in E_p(G,\omega)$ and $f^{-} \in \widetilde{E}_p(G^{-},\omega)$. Hence we can associate the series

$$f^{+}(z) \sim \sum_{k=0}^{\infty} a_k\left(f^{+}\right)F_{k,p}(z), \quad z \in G$$

and

$$f^{-}(z) \sim \sum_{k=1}^{\infty} \widetilde{a}_k\left(f^{-}\right)\widetilde{F}_{k,p}(1/z), \quad z \in G^{-}.$$

Since $f=f^{+}-f^{-}$ almost everywhere on Γ, we can associate with f the formal series

$$f(z) \sim \sum_{k=0}^{\infty} a_k\left(f^{+}\right)F_{k,p}(z)-\sum_{k=1}^{\infty} \widetilde{a}_k\left(f^{-}\right)\widetilde{F}_{k,p}(1/z) \tag{16}$$

almost everywhere on Γ. This series is called the *p-Faber-Laurent* series of the function $f \in L_p(\Gamma,\omega)$ (see [6]).

We can state the following corollary of Theorem 1 and Theorem 2.

Corollary. *Let Γ be a Carleson curve, $1<p<\infty$, $\omega \in A_p(\Gamma)$ and $\omega_0,\omega_1 \in A_p(\mathbb{T})$. If $f \in L_p(\Gamma,\omega)$ has the p-Faber-Laurent series* (16) *and $\{\lambda_k\}_0^{\infty}$ is a sequence of complex numbers which satisfies the condition* (5)*, then there exists a function $F \in L_p(\Gamma,\omega)$ which has the p-Faber-Laurent series*

$$F(z) \sim \sum_{k=0}^{\infty} \lambda_k a_k\left(f^{+}\right)F_{k,p}(z)-\sum_{k=1}^{\infty} \lambda_k\widetilde{a}_k\left(f^{-}\right)\widetilde{F}_{k,p}(1/z)$$

and satisfies $\|F\|_{L_p(\Gamma,\omega)} \leq c_{24}\|f\|_{L_p(\Gamma,\omega)}$.

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