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ANALYTIC FUNCTIONS SHARING THREE VALUES DM IN ONE ANGULAR DOMAIN

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Abstract. We investigate the uniqueness of transcendental analytic functions that share three values DM in one angular domain instead of the whole complex plane.

1. Introduction and main results

In this paper, a transcendental meromorphic (analytic) function is meromorphic (analytic) in the whole complex plane $\mathbb{C}$ and not rational. We assume that the reader is familiar with the Nevanlinna's theory of meromorphic functions and the standard notations such as $m(r, f)$, $T(r, f)$. For references, see [2]. We say that two meromorphic functions f and g share the value a ($a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) in $X \subseteq \mathbb{C}$ provided that in X, we have $f(z) = a$ if and only if $g(z) = a$. We will state whether a shared value is by DM (differential multiplicities), or by IM (ignoring multiplicities). R. Nevanlinna (see [4]) proved that if two meromorphic functions f and g have five distinct IM shared values in $X = \mathbb{C}$, then $f(z) \equiv g(z)$. After his very work, the uniqueness of meromorphic functions with shared values in the whole complex plane attracted many investigations (for references, see [7]). E. Mues consider DM shared values and proved the following theorem.

Theorem A ([3]). *There are no two distinct nonconstant analytic functions f and g that share three distinct values DM in $X = \mathbb{C}$.*

In [8], Zheng took into account of the uniqueness dealing with five shared values in some angular domains of $\mathbb{C}$. It is an interesting topic to investigate the uniqueness with shared values in the remaining part of the complex plane removing an unbounded closed set. In [9], Zheng continued to investigate this subject and obtain some results on uniqueness of meromorphic functions with five or four shared values in one angular domain.

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We may ask: *What can be said to an analogous result as Theorem A in one angular domain?*

Nevanlinna's theory on angular domain (see [1]) will play a key role in this paper. Let f be a meromorphic function on the angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\le\arg z\le\beta\}$, where $0<\beta-\alpha\le 2\pi$. Following Nevanlinna define

$$(1)\ A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\right\}\frac{dt}{t},$$

$$(2)\ B_{\alpha,\beta}(r,f) = \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,$$

$$(3)\ C_{\alpha,\beta}(r,f) = 2\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^\omega}-\frac{|b_n|^\omega}{r^{2\omega}}\right)\sin\omega(\theta_n-\alpha),$$

$$(4)\ D_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f),$$

where $\omega=\frac{\pi}{\beta-\alpha}$, $1\le r<\infty$ and $b_n=|b_n|e^{i\theta_n}$ are the poles of f on $\overline{\Omega}(\alpha,\beta)$ appearing according to their multiplicities. If we only consider the distinct poles of f, we denote the corresponding angular counting function by $\overline{C}_{\alpha,\beta}(r,f)$. Nevanlinna's angular characteristic is defined as follows:

$$(5)\qquad S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

Throughout, we denote respectively by $R(r,*)$ and $R_{\alpha,\beta}(r,*)$ quantities satisfying

$$R(r,*)=O\left(\log\left(rT(r,*)\right)\right), r\notin E,$$

and

$$R_{\alpha,\beta}(r,*)=O\left(\log\left(rS_{\alpha,\beta}(r,*)\right)\right), r\notin E,$$

where E denotes a set of positive real numbers with finite linear measure. The notation E is not necessarily the same for its every time occurrence in the context.

Now we show our main result which can answer the above question.

Theorem 1. *There are no two distinct transcendental analytic functions f and g that share three distinct values a_1, a_2, a_3 DM in one angular domain $X=\{z:\alpha<\arg z<\beta\}$ with $0\le\alpha<\beta\le 2\pi$, provided that*

$$\lim_{r\to\infty}\frac{S_{\alpha,\beta}(r,f)}{\log\left(rT(r,f)\right)}=\infty,\quad (r\notin E).$$

2. Lemmas

Lemma A ([5], [6], [10]). *Suppose that $g(z)$ is a non-constant meromorphic function in the plane and that $\Omega(\alpha,\beta)$ is an angular domain, where $0<\beta-\alpha\le 2\pi$. Then*

(i) ([1], *Chap. 1*) *for any complex number $a\ne\infty$,*

$$S_{\alpha,\beta}\left(r,\frac{1}{g-a}\right)=S_{\alpha,\beta}(r,g)+\varepsilon(r,a),$$

where $\varepsilon(r,a) = O(1)$ $(r \to \infty)$;

(ii) ([1], *p. 138*) *for any* $1 \le r < R$,

$$A_{\alpha,\beta}\left(r, \frac{g'}{g}\right) \le K\left\{\left(\frac{R}{r}\right)^{\omega}\int_1^R \frac{\log^+ T(t,g)}{t^{1+\omega}}dt + \log^+ \frac{r}{R-r} + \log\frac{R}{r} + 1\right\}$$

and

$$B_{\alpha,\beta}\left(r, \frac{g'}{g}\right) \le \frac{4\omega}{r^{\omega}} m\left(r, \frac{g'}{g}\right),$$

where $\omega = \frac{\pi}{\beta-\alpha}$ *and* K *is a positive constant not depending on* r *and* R.

Remark. It follows from Lemma A(ii) that

$$D_{\alpha,\beta}(r, \frac{g'}{g}) = A_{\alpha,\beta}(r, \frac{g'}{g}) + B_{\alpha,\beta}(r, \frac{g'}{g}) = R_{\alpha,\beta}(r,g) = R(r,g).$$

Lemma B ([9]). *Suppose that* $f(z)$ *is a non-constant meromorphic function in the plane and that* $\Omega(\alpha,\beta)$ *is an angular domain, where* $0 < \beta - \alpha \le 2\pi$. *Then for arbitrary* q *distinct* $a_j \in \overline{\mathbb{C}}$ $(1 \le j \le q)$, *we have*

$$\begin{aligned}(q-2)S_{\alpha,\beta}(r,f) &\le \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + R_{\alpha,\beta}(r,f)\\ &= \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + R(r,f),\end{aligned}$$

where the term $\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right)$ *will be replaced by* $\overline{C}_{\alpha,\beta}(r,f)$ *when some* $a_j = \infty$.

Lemma 1. *Suppose that* $f(z)$ *is a non-constant meromorphic function in the plane and that* $\Omega(\alpha,\beta)$ *is an angular domain, where* $0 < \beta-\alpha \le 2\pi$. *Let* $P(f) = a_0 f^m + a_1 f^{m-1} + \cdots + a_m$ $(a_0 \ne 0)$ *be a polynomial in* f *with degree* m, *where the coefficients* $a_j (j = 0, 1, \ldots, m)$ *are constants, and let* $b_j (j = 1, 2, \ldots, q)$ $(q > m)$ *be* q *distinct finite complex numbers. Then*

$$D_{\alpha,\beta}\left(r, \frac{P(f)\cdot f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) = R(r,f).$$

Proof. One can deduce that

$$\frac{P(f)}{(f-b_1)(f-b_2)\cdots(f-b_q)} = \sum_{j=1}^{q}\frac{A_j}{f-b_j}$$

holds, where A_j are nonzero constants. Hence we deduce by Lemma A(ii) and the lemma of logarithmic derivative of meromorphic function in the complex plane,

$$\begin{aligned} & D_{\alpha,\beta}\left(r, \frac{P(f)\cdot f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) \\ = & D_{\alpha,\beta}\left(r, \sum_{j=1}^{q}\frac{A_j\cdot f'}{f-b_j}\right) \\ \leq & \sum_{j=1}^{q} D_{\alpha,\beta}\left(r, \frac{f'}{f-b_j}\right) + \sum_{j=1}^{q} D_{\alpha,\beta}\left(r, A_j\right) + O(1) \\ = & R(r,f). \end{aligned}$$

□

Lemma 2. *Let f and g be two distinct transcendental meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 IM in $X = \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$. Then*

(i) $S_{\alpha,\beta}(r,f) = S_{\alpha,\beta}(r,g) + R(r,f)$, $S_{\alpha,\beta}(r,g) = S_{\alpha,\beta}(r,f) + R(r,g)$;

(ii) $\sum_{j=1}^{4} \overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j}) = 2S_{\alpha,\beta}(r,f) + R(r,f)$;

(iii) $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-b}) = S_{\alpha,\beta}(r,f) + R(r,f)$, $\overline{C}_{\alpha,\beta}(r, \frac{1}{g-b}) = S_{\alpha,\beta}(r,g) + R(r,g)$, *where* $b \neq a_j$ $(j = 1,2,3,4)$;

(iv) $C^*_{\alpha,\beta}(r, \frac{1}{f'}) = R(r,f)$, $C^*_{\alpha,\beta}(r, \frac{1}{g'}) = R(r,g)$, *where* $C^*_{\alpha,\beta}$ $(r, \frac{1}{f'})$ *and* $C^*_{\alpha,\beta}(r, \frac{1}{g'})$ *are respectively the counting functions of the zeros of f' that are not zeros of $f - a_j$ $(j = 1,2,3,4)$, and the zeros of g' that are not zeros of $g - a_j$ $(j = 1,2,3,4)$;*

(v) $\sum_{j=1}^{4} C^{**}_{\alpha,\beta}(r, f(z) = a_j = g(z)) = R(r,f)$, *where* $C^{**}_{\alpha,\beta}(r, f(z) = a_j = g(z))$ *is the counting function for common multiple zeros of $f - a_j$ and $g - a_j$ $(j = 1,2,3,4)$, counting the smaller one of the two multiplicities at each of the points.*

Proof. From Lemma B we have

$$\begin{aligned} 2S_{\alpha,\beta}(r,f) &\leq \sum_{j=1}^{4} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + R(r,f) \\ &\leq C_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) + R(r,f) \\ &\leq S_{\alpha,\beta}(r,f) + S_{\alpha,\beta}(r,g) + R(r,f), \end{aligned}$$

and by interchanging f and g we obtain (i) and (ii).

Again by Lemma B and (ii) we have

$$\begin{aligned} 3S_{\alpha,\beta}(r,f) &\leq \sum_{j=1}^{4}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-b}\right)+R(r,f)\\ &= 2S_{\alpha,\beta}(r,f)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-b}\right)+R(r,f), \end{aligned}$$

i.e.,

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-b}\right)=S_{\alpha,\beta}(r,f)+R(r,f).$$

By interchanging f and g, we get

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{g-b}\right)=S_{\alpha,\beta}(r,g)+R(r,g).$$

Thus we obtain (iii).

Without loss of generality we will assume that $a_4=\infty$. This is allowed because if all the shared values are finite, then we can consider

$$F=(f-a_4)^{-1} \quad \text{and} \quad G=(g-a_4)^{-1}.$$

Set

$$\Psi=\frac{f'g'(f-g)^2}{(f-a_1)(f-a_2)(f-a_3)(g-a_1)(g-a_2)(g-a_3)}. \tag{6}$$

It is easy to see from Lemma 1 and (6) that

$$D_{\alpha,\beta}(r,\Psi)=R(r,f)+R(r,g).$$

If $z_0\in X$ is a point such that $f(z_0)=g(z_0)=a_j$ for some $j=1,2,3,4$, then from (6) we see that Ψ will be analytic at z_0. Thus we can deduce that

$$\begin{aligned} &C^*_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+C^*_{\alpha,\beta}\left(r,\frac{1}{g'}\right)+\sum_{j=1}^{4}C^{**}_{\alpha,\beta}(r,f(z)=a_j=g(z))\\ \leq\ &C_{\alpha,\beta}\left(r,\frac{1}{\Psi}\right)\\ \leq\ &S_{\alpha,\beta}\left(r,\frac{1}{\Psi}\right)\\ =\ &S_{\alpha,\beta}(r,\Psi)+O(1)\\ =\ &D_{\alpha,\beta}(r,\Psi)+C_{\alpha,\beta}(r,\Psi)+O(1)\\ =\ &R(r,f)+R(r,g). \end{aligned}$$

From (i) we see that $R(r,f)=R(r,g)$. Therefore we obtain (v) and (vi). □

Lemma 3. *Let f and g be two distinct transcendental meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 IM in $X = \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$. Then*

$$D_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) = R(r, f).$$

Proof. We assume that $a_j (j = 1, 2, 3, 4)$ are finite, then it follows from Lemma A and Lemma B that

$$\begin{aligned} 2S_{\alpha,\beta}(r, f) &\leq \sum_{j=1}^{4} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + R(r, f) \\ &\leq C_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) + R(r, f) \\ &\leq S_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) + R(r, f) \\ &\leq S_{\alpha,\beta}(r, f) + S_{\alpha,\beta}(r, g) + R(r, f), \end{aligned}$$

namely,

$$S_{\alpha,\beta}(r, f) \leq S_{\alpha,\beta}(r, g) + R(r, f).$$

Similarly, we have

$$S_{\alpha,\beta}(r, g) \leq S_{\alpha,\beta}(r, f) + R(r, g).$$

It implies from above discussion that

$$S_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) = C_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) + R(r, f).$$

Hence we have

$$D_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) = R(r, f).$$

We now assume $a_4 = \infty$. Let $b \neq a_j$ $(j = 1, 2, 3, 4)$, $F(z) = \frac{1}{f(z)-b}$, and $G(z) = \frac{1}{g(z)-b}$. Then $b_j = \frac{1}{a_j-b}(j = 1, 2, 3)$ and $b_4 = 0$ are IM shared values of $F(z)$ and $G(z)$ in X. From the discussion above, we have

$$D_{\alpha,\beta}\left(r, \frac{1}{F-G}\right) = R(r, f). \tag{7}$$

From Lemma 2(iii) we have

$$C_{\alpha,\beta}\left(r, \frac{1}{f-b}\right) = S_{\alpha,\beta}(r, f) + R(r, f).$$

This implies from Lemma A that

$$C_{\alpha,\beta}(r, F) = S_{\alpha,\beta}(r, F) + R(r, F).$$

Hence

$$D_{\alpha,\beta}(r, F) = R(r, F). \tag{8}$$

Similarly, we have

$$D_{\alpha,\beta}(r, G) = R(r, G). \tag{9}$$

From the equalities (7), (8), and (9), we get

$$\begin{aligned} D_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) &= D_{\alpha,\beta}\left(r, \frac{FG}{F-G}\right) \\ &\leq D_{\alpha,\beta}\left(r, \frac{1}{F-G}\right) + D_{\alpha,\beta}(r, F) + D_{\alpha,\beta}(r, G) \\ &= R(r, f). \end{aligned}$$

This completes the proof of the lemma. □

Lemma 4. *Let f and g be two distinct transcendental meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 IM in $X = \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$. Then*

$$D_{\alpha,\beta}\left(r, \frac{f}{f-g}\right) + D_{\alpha,\beta}\left(r, \frac{g}{f-g}\right) + D_{\alpha,\beta}\left(r, \frac{f'}{f-g}\right) + D_{\alpha,\beta}\left(r, \frac{g'}{f-g}\right) + D_{\alpha,\beta}\left(r, \frac{f'g'}{f-g}\right) = R(r, f).$$

Proof. Set

$$F = \frac{1}{f}, \quad G = \frac{1}{g}.$$

Then $b_j = \frac{1}{a_j}$ $(j = 1, 2, 3, 4)$ are IM shared values of F and G in X. From Lemma 3 we have

$$D_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) = R(r, f)$$

and

$$D_{\alpha,\beta}\left(r, \frac{fg}{f-g}\right) = D_{\alpha,\beta}\left(r, \frac{1}{F-G}\right) = R(r, F) = R(r, f).$$

Since

$$\left(\frac{f}{f-g}\right)^2 = \frac{f}{f-g} + \frac{fg}{(f-g)^2},$$

we have

$$\begin{aligned} 2D_{\alpha,\beta}\left(r, \frac{f}{f-g}\right) &\leq D_{\alpha,\beta}\left(r, \frac{f}{f-g}\right) \\ &\quad + D_{\alpha,\beta}\left(r, \frac{fg}{f-g}\right) + D_{\alpha,\beta}\left(r, \frac{1}{f-g}\right) + O(1) \\ &\leq D_{\alpha,\beta}\left(r, \frac{f}{f-g}\right) + R(r, f). \end{aligned}$$

Hence

$$D_{\alpha,\beta}\left(r, \frac{f}{f-g}\right) = R(r, f).$$

Similarly, we have

$$D_{\alpha,\beta}\left(r,\frac{g}{f-g}\right)=R(r,f).$$

Furthermore, we have

$$D_{\alpha,\beta}\left(r,\frac{f'}{f-g}\right)\le D_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+D_{\alpha,\beta}\left(r,\frac{f}{f-g}\right)=R(r,f),$$
$$D_{\alpha,\beta}\left(r,\frac{g'}{f-g}\right)\le D_{\alpha,\beta}\left(r,\frac{g'}{g}\right)+D_{\alpha,\beta}\left(r,\frac{g}{f-g}\right)=R(r,f),$$
$$D_{\alpha,\beta}\left(r,\frac{f'g'}{f-g}\right)\le D_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+D_{\alpha,\beta}\left(r,\frac{g'}{g}\right)+D_{\alpha,\beta}\left(r,\frac{fg}{f-g}\right)=R(r,f).$$

□

3. Proof of Theorem 1

We assume that the conclusion of Theorem 1 is not true, namely, there exist two distinct transcendental analytic functions $f(z)$ and $g(z)$ that share three distinct values a_1, a_2, a_3 DM in X, provided that

$$\lim_{r\to\infty}\frac{S_{\alpha,\beta}(r,f)}{\log(rT(r,f))}=\infty\quad(r\notin E).$$

Without loss of generality, we assume $a_1=0$, $a_2=1$, $a_3=c$. Set

$$\Phi=\frac{(f')^2(g')^2(f-g)}{f(f-1)(f-c)g(g-1)(g-c)}.$$

If $z_0\in X$ is a zero of both f and g, with multiplicities p and q, respectively. Since 0 is DM shared values of f and g in X, we have $p\neq q$. Then by computation, we have

$$\Phi(z)=O\left((z-z_0)^t\right),$$

where $t=p+q-4+\min\{p,q\}\ge 0$. Hence zeros of f in X are not poles of Φ in X. Similarly, zeros of $f-1$ or $f-c$ in X are not poles of Φ in X. Therefore we get that Φ is analytic in X. Obviously,

$$\Phi=\frac{f'g'}{f-g}\cdot\Psi,$$

where Ψ is the function defined in (6). From Lemma 4 we have

$$\begin{aligned}S_{\alpha,\beta}(r,\Phi)&=D_{\alpha,\beta}\left(r,\frac{f'g'}{f-g}\right)+D_{\alpha,\beta}(r,\Psi)\\&=R(r,f).\end{aligned}$$

We denote by $\overline{C}^*_{\alpha,\beta}(r)$ the counting function of zeros of $f, f-1, f-c$ (or $g, g-1, g-c$) with multiplicities more than 2, each point counts only once. Then

$$\overline{C}^*_{\alpha,\beta}(r)\le C_{\alpha,\beta}\left(r,\frac{1}{\Phi}\right)\le S_{\alpha,\beta}(r,\Phi)+O(1)=R(r,f).\tag{10}$$

Since $0, 1, c$ are DM shared values of f and g in X, from Lemma 2(iv) we have

$$\begin{aligned}&\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-c}\right)\\ &=C_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+C_{\alpha,\beta}\left(r,\frac{1}{g'}\right)+R(r,f).\end{aligned}$$

Since f and g are analytic in X, from the above equality, Lemma A, Lemma B, Lemma 2(i) and Lemma 2(ii) we have

$$\begin{aligned}2S_{\alpha,\beta}(r,f)&=\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-c}\right)+R(r,f)\\ &=C_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+C_{\alpha,\beta}\left(r,\frac{1}{g'}\right)+R(r,f)\\ &=S_{\alpha,\beta}\left(r,\frac{1}{f'}\right)-D_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+S_{\alpha,\beta}\left(r,\frac{1}{g'}\right)-D_{\alpha,\beta}\left(r,\frac{1}{g'}\right)+R(r,f)\\ &=S_{\alpha,\beta}(r,f')-D_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+S_{\alpha,\beta}(r,g')-D_{\alpha,\beta}\left(r,\frac{1}{g'}\right)+R(r,f)\\ &=D_{\alpha,\beta}(r,f')-D_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+D_{\alpha,\beta}(r,g')-D_{\alpha,\beta}\left(r,\frac{1}{g'}\right)+R(r,f)\\ &\le D_{\alpha,\beta}(r,f)-D_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+D_{\alpha,\beta}(r,g)-D_{\alpha,\beta}\left(r,\frac{1}{g'}\right)+R(r,f)\\ &=2S_{\alpha,\beta}(r,f)-D_{\alpha,\beta}\left(r,\frac{1}{f'}\right)-D_{\alpha,\beta}\left(r,\frac{1}{g'}\right)+R(r,f),\end{aligned}$$

namely,

$$D_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+D_{\alpha,\beta}\left(r,\frac{1}{g'}\right)=R(r,f).$$

Hence we get from Lemma A that

$$\begin{aligned}&D_{\alpha,\beta}\left(r,\frac{1}{f}\right)+D_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)+D_{\alpha,\beta}\left(r,\frac{1}{f-c}\right)\\ &\le D_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+R(r,f)=R(r,f).\end{aligned}$$

If

$$C_{\alpha,\beta}\left(r,\frac{1}{f}\right)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)=R(r,f),$$

then we have

$$\begin{aligned}S_{\alpha,\beta}(r,f)&=C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+D_{\alpha,\beta}\left(r,\frac{1}{f}\right)+O(1)\\ &=C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+R(r,f)\end{aligned}$$

$$\begin{aligned} &= \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)+R(r,f)\\ &\le \frac{1}{2}C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+R(r,f)\\ &= \frac{1}{2}S_{\alpha,\beta}\left(r,\frac{1}{f}\right)+R(r,f), \end{aligned}$$

namely,

$$S_{\alpha,\beta}(r,f)=R(r,f).$$

This contradicts the condition of Theorem 1. Hence we have

$$C_{\alpha,\beta}\left(r,\frac{1}{f}\right)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right)\ne R(r,f). \tag{11}$$

Similarly, we have

$$C_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-1}\right)\ne R(r,f),$$

$$C_{\alpha,\beta}\left(r,\frac{1}{f-c}\right)-\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-c}\right)\ne R(r,f).$$

Set

$$\begin{aligned} \Omega &= 2\frac{f''}{f'}-3\left(\frac{f'}{f}+\frac{f'}{f-1}+\frac{f'}{f-c}\right)-2\frac{g''}{g'}\\ &= 2\left(\frac{g'}{g}+\frac{g'}{g-1}+\frac{g'}{g-c}\right)+\frac{f'-2g'}{f-g}. \end{aligned}$$

Then from Lemma A, Lemma 4 we have

$$D_{\alpha,\beta}(r,\Omega)\le R(r,f). \tag{12}$$

If $z_1\in X$ is a zero of both f and g, (or both $f-1$ and $g-1$; or both $f-c$ and $g-c$), with multiplicities 2 and 1, respectively; and if $z_2\in X$ is a zero of both f and g, (or both $f-1$ and $g-1$; or both $f-c$ and $g-c$), with multiplicities 1 and 2, respectively. Then by simple computation, we get that both z_1 and z_2 are not poles of $\Omega(z)$ in X. Hence we can deduce by Lemma 2(iv) and (10) that

$$C_{\alpha,\beta}(r,\Omega)\le R(r,f). \tag{13}$$

Hence combining (12) and (13), we have

$$S_{\alpha,\beta}(r,\Omega)\le R(r,f).$$

If $z_1\in X$ is a zero of both f and g, with multiplicities 2 and 1, respectively. Then by computation we have

$$\Psi(z_1)=\frac{2}{c^2}(g'(z_1))^2,$$

$$\Omega(z_1)=-\left(1+\frac{1}{c}\right)g'(z_1).$$

Hence we get

$$\frac{\Omega^2(z_1)}{2\Psi(z_1)} = (c+1)^2.$$

If

$$\frac{\Omega^2(z)}{2\Psi(z)} - (c+1)^2 \not\equiv 0,$$

then

$$\begin{aligned} & C_{\alpha,\beta}\left(r, \frac{1}{f}\right) - \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) - R(r,f) \\ \leq\ & C_{\alpha,\beta}\left(r, \frac{1}{\frac{\Omega^2}{2\Psi} - (c+1)^2}\right) \\ \leq\ & 2S_{\alpha,\beta}(r,\Omega) + S_{\alpha,\beta}(r,\Psi) + O(1) \\ =\ & R(r,f). \end{aligned}$$

This contradicts to (11). Hence

$$\frac{\Omega^2(z)}{2\Psi(z)} - (c+1)^2 \equiv 0.$$

If $z_3 \in X$ is a zero of both $f-1$ and $g-1$, with multiplicities 2 and 1, respectively. Then by computation we have

$$\frac{\Omega^2(z_3)}{2\Psi(z_3)} = (2-c)^2.$$

Hence

$$(c+1)^2 = (2-c)^2,$$

namely, $c = \frac{1}{2}$.

If $z_4 \in X$ is a zero of both $f-c$ and $g-c$, with multiplicities 2 and 1, respectively. Then by computation we have

$$\frac{\Omega^2(z_4)}{2\Psi(z_4)} = (2c-1)^2.$$

Hence

$$(c+1)^2 = (2c-1)^2.$$

This implies $c = 2$. We obtain a contradiction. Therefore, we complete the proof of Theorem 1.

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