H_∞ Multi-Step Prediction for Linear Discrete-Time Systems: A Distributed Algorithm

Hao-qian Wang, Huan-shui Zhang, and Hong Hu

Abstract: A new approach to H_∞ multi-step prediction is developed by applying the innovation analysis theory. Although the predictor is derived by resorting to state augmentation, nevertheless, it is completely different from the previous works with state augmentation. The augmented state here is considered just as a theoretical mathematic tool for deriving the estimator. A distributed algorithm for the Riccati equation of the augmented system is presented. By using the reorganized innovation analysis, calculation of the estimator does not require any augmentation. A numerical example demonstrates the effect in reducing computing burden.

Keywords: Distributed algorithm, H_∞ estimation, innovation, Krein space.

1. INTRODUCTION

The estimation problem of a stochastic process, given observations of a related random process, is encountered in many areas of science and engineering [1]. There are two well-known approaches for such problems. One is based on the L_2 or H_2 criterion, where the estimators are designed to minimize the mean squared error, for example, the Wiener filtering and Kalman formulation. Another is based on the H_∞ performance. An H_∞ estimator is to minimize the maximum energy gain from the disturbances to the estimation errors; see [2,3] for the continuous-time case and [4] for the discrete-time one, and is applicable to situations where no statistical information of input noises is available. Recently, much of the attention has been paid on the H_∞ estimate problem and some improvements have been made (see, e.g., [5-9] and the references therein).

Of the two different approaches, the H_2 estimate theory, including filtering, smoothing and prediction, has been the most researched in past decades. The estimation under H_∞ performance, inspired by Zames [10], has been extensively studied since 1980 and has been approached by numerous authors using various techniques. It should be pointed out that previous works are mainly concentrated on the filtering and the one-step prediction, whereas less attention has been paid to the more complicated multi-step prediction problem. The H_∞ prediction problem, where an estimator of the current state is sought based on measurements in a past infinite interval, was addressed through system augmentation. However, the multi-step prediction process would be computational too expensive. The H_∞ prediction problem has been studied in [11] without resorting to system augmentation for the first time, and the estimator is obtained by minimizing quadratic functions. Unfortunately, the derivation is complex, and the existence condition of the l-step predictor is not easy to be verified.

In this paper, we aim to present a new approach to the H_∞ multi-step ahead prediction problem by introducing a reorganized innovation sequence and the projection formula in an indefinite space, i.e., the so-called Krein space. The key behind this simplicity is the presentation of a distributed algorithm for the Riccati equation of the augmented system.

Notations: Whenever the Krein space elements and the Euclidean space elements satisfy the same set of constraints, we shall denote them by the same letters with the former identified by bold faces and the latter by normal faces.

2. PROBLEM STATEMENT

We consider the following linear system for the H_∞ estimation problem

$$x(t+1) = \Phi_t x(t) + \Gamma_t u(t), \tag{1}$$

Manuscript received November 2, 2005; revised November 26, 2006 and August 1, 2007; accepted October 25, 2007. Recommended by Editorial Board member Young Il Lee under the direction of Editor Tae-Woong Yoon. This work was supported by the projects of NSFC (No. 60574016) and NSFC (No. 60772046).

Hao-qian Wang is with the Department of Automation, Tsinghua University, Beijing 100084, P. R. China (e-mail: wanghaoqian@tsinghua.edu.cn).

Huan-shui Zhang is with the School of Control Science and Engineering, Shandong University, Jinan 250061, P. R. China (e-mail: hszhang@sdu.edu.cn).

Hong Hu is with the Shenzhen Graduate School of Harbin Institute of Technology, Shenzhen 518055, P. R. China (e-mail: honghu@hit.edu.cn).

$$y(t) = H_t x(t) + v(t), \tag{2}$$

$$z(t) = L_t x(t), \tag{3}$$

where $x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^m, u(t) \in \mathbb{R}^r, v(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^p$ represent the state, measurement output, input noise, measurement noise and the signal to be estimated, respectively. It is assumed that the input and measurement noises are deterministic signals and are from $L_2[0, N]$, where N is a time-horizon of the estimation problem under investigation.

The H_∞ l-step prediction problem can be stated as follows.

Given scalar $\gamma > 0$, positive integer l and observations $\{y(j)\}_{j=0}^{t-l}$, find an estimator $\breve{z}(t \mid t-l)$ of $z(t)$, if exist, such that the following inequality is satisfied

$$\sup_{(x_0,u,v)\neq 0} \frac{\sum_{t=0}^{N} [\breve{z}(t \mid t-l) - z(t)]^T [\breve{z}(t \mid t-l) - z(t)]}{x_0^T \Pi_0^{-1} x_0 + \sum_{t=0}^{N} u^T(t)u(t) + \sum_{t=0}^{N-l} v^T(t)v(t)} < \gamma^2, \tag{4}$$

where $x_0 = x(0)$, and Π_0 is a given positive definite matrix which reflects the relative uncertainty of the initial state to the input and measurement noises.

3. H_∞ MULTI-STEP PREDICTION

For the sake of comparison, we first review the existing results for multiple-step prediction with augmentation, and then present the new results with distributed algorithm.

3.1. Multi-step prediction by augmentation

In view of (4), the cost function for multi-step prediction game is

$$J(t) = x_0^T \Pi_0^{-1} x_0 + \sum_{t=0}^{N} u^T(t)u(t) + \sum_{t=0}^{N} v^T(t-l)v(t-l) - \gamma^{-2} \sum_{t=0}^{N} v_z^T(t) v_z(t), \tag{5}$$

where $v(t-l) = 0, t < l \ (l > 0)$, and $v_z(t) = \breve{z}(t \mid t-l) - z(t)$.

Firstly, we recall some related results obtained in [12]. The predictor is derived with the help of the following augmented state space

$$x_a(t+1) = \Phi_a(t) x_a(t) + \Gamma_a(t) u(t), \tag{6}$$

$$y(t-l) = H_a(t) x_a(t) + v(t-l), \tag{7}$$

$$L_t x(t) = L_a(t) x_a(t), \tag{8}$$

where

$$x_a^T(t) = \begin{bmatrix} x^T(t) & x^T(t-1) & \cdots & x^T(t-l) \end{bmatrix}^T, \tag{9}$$

$$\Phi_a(t) = \begin{bmatrix} \Phi_t & 0 & \cdots & 0 & 0 \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}, \Gamma_a(t) = \begin{bmatrix} \Gamma_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{10}$$

$$H_a(t) = \begin{cases} 0 & 0 \le t < l, \\ \begin{bmatrix} 0 & 0 & \cdots & H_{t-l} \end{bmatrix}, & t \ge l, \end{cases} \tag{11}$$

$$L_a(t) = \begin{bmatrix} L_t & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{12}$$

According to the forms of (9)-(12), the cost function of the game then becomes

$$J(t) = x_a^T(0) \mathrm{diag}(\Pi_0^{-1}, 0_{nl \times nl}) x_a(0) + \sum_{t=0}^{N-1} \|u(t)\|^2 + \sum_{t=0}^{N} \|y(t-l) - H_a(t) x_a(t)\|^2 - \gamma^{-2} \sum_{t=0}^{N} \|\breve{z}(t \mid t-l) - L_a(t) x_a(t)\|^2. \tag{13}$$

Note that the H_∞ multi-step prediction is equivalent to the H_∞ filtering problem for the augmented system (6)-(8). The results in [12] are summarized as followed.

Lemma 1: Considering the system (1)-(3) and the associated performance criterion (4), and a given scalar $\gamma > 0$, then we have

1) An H_∞ estimator $\breve{z}(t \mid t-l)$ that achieves (4) exists if and only if the following Riccati equations

$$P_a(t+1) = \Gamma_a(t)\Gamma_a^T(t) + \Phi_a(t)\Sigma_a(t)\Phi_a^T(t), \tag{14}$$

$$\Sigma_a(t) = P_a(t)\left\{I + [H_a^T(t)H_a(t) - \gamma^{-2} L_a^T(t) L_a(t)] P_a(t)\right\}^{-1}, \tag{15}$$

$$\Sigma_a(0) = \mathrm{diag}(\Pi_0^{-1}, 0_{nl \times nl}), \tag{16}$$

or equivalently,

$$\Pi_a(t) = -P_a(l+1, t) H_{t-l}^T [I_m + H_{t-l} P_{l+1,l+1}(t) H_{t-l}^T]^{-1} \times H_{t-l} P_a^T(l+1, t) + P_a(t), \tag{17}$$

$$\Sigma_a(t) = \gamma^{-2} \Pi_a(1, t) L_t^T [I_p - \gamma^{-2} L_t \Pi_{11}(t) L_t^T]^{-1} \times L_t \Pi_a^T(1, t) + \Pi_a(t), \tag{18}$$

$$P_a(t+1) = \Gamma_a(t)\Gamma_a^T(t) + \Phi_a(t)\Sigma_a(t)\Phi_a^T(t) \tag{19}$$

have a solution so that $\Sigma_a(t) \ge 0, 0 < t \le N$, where $P_a(t)$ is partitioned as

$$P_a(t) = \{P_{ij}(t), 1 \le i, j \le l+1\}, \tag{20}$$

with the dimension of $P_{ij}(t)$ is $n \times n$, $P_a(i,t)$ represents the i-th column blocks of $P_a(t)$, and $\Pi_a(t)$ is partitioned similarly to $P_a(t)$.

2) In this case, the H_∞ multi-step prediction is obtained as

$$\breve{z}(t \mid t-l) = L_a(t)\breve{x}_a(t), \tag{21}$$

where $\breve{x}_a(t)$ is computed by the following equation:

$$\begin{aligned}\breve{x}_a(t+1) = {} & \Phi_a(t)P_a(t)H_a^T(t)[I + H_a(t)P_a(t)H_a^T(t)]^{-1} \\ & \times[y(t+1-l) - H_a(t+1)\Phi_a(t)x_a(t)] \\ & +\Phi_a(t)x_a(t).\end{aligned} \tag{22}$$

Remark 1: As shown in Lemma above, $P_a(t) = \{P_{ij}(t), 1 \le i, j \le l+1\}$ is the key to design the multi-step predictor. The calculation is very complicated because of the high dimension of the augmented system. For the sake of simplicity, we shall propose a distributed algorithm.

3.2. A new method for multi-step prediction

In view of (1)-(3), we introduce a system in Krein space [11,13]

$$\mathbf{x}_a(t+1) = \Phi_a(t)\mathbf{x}_a(t) + \Gamma_a(t)\mathbf{u}(t), \tag{23}$$

$$\begin{bmatrix} \mathbf{y}(t-l) \\ \breve{\mathbf{z}}(t \mid t-l) \end{bmatrix} = \begin{bmatrix} H_a(t) \\ L_a(t) \end{bmatrix}\mathbf{x}_a(t) + \begin{bmatrix} \mathbf{v}(t-l) \\ \mathbf{v}_z(t) \end{bmatrix}, t \ge l, \tag{24}$$

$$\breve{\mathbf{z}}(t \mid t-l) = L_a(t)\mathbf{x}_a(t) + \mathbf{v}_z(t), 0 \le t < l, \tag{25}$$

where $\Phi_a(t)$, $H_a(t)$, $L_a(t)$, $\Gamma_a(t)$ are defined as (10)-(12), $\mathbf{u}(i)$, $\mathbf{v}(i)$ and $\mathbf{v}_z(i)$ are assumed to be uncorrelated white noises

$$\begin{aligned}&\langle \mathbf{u}(i), \mathbf{u}(j)\rangle = Q_u(i)\delta_{ij},\ \langle \mathbf{v}(i), \mathbf{v}(j)\rangle = Q_v(i)\delta_{ij}, \\ &\langle \mathbf{v}_z(i), \mathbf{v}_z(j)\rangle = Q_u(i)\delta_{ij}, \\ &Q_u(i) = I_r, Q_v(i) = I_m, Q_{v_z}(i) = -I_p\gamma^2,\end{aligned} \tag{26}$$

here the bilinear form $\langle , \rangle$ is an operator to obtain the covariance matrix, i.e., $\langle u, u\rangle = E(uu^*)$, where $E()$ denotes "expectation".

Remark 2: Note that $Q_{v_z}(i) < 0$, then system (23)-(25) is no longer a system in Euclidean space but an indefinite space (Krein space).

The linear H_2 estimation theory in Krein space has been well studied in [11]. For the convenience of discussion, we recall the projection in Krein space.

Given elements $\mathbf{s}(i)$ and $\{\mathbf{y}_0, \mathbf{y}_1, \cdots, \mathbf{y}_j\}$ in Krein space, we define $\hat{\mathbf{s}}(i \mid j)$ to be the projection of $\mathbf{s}(i)$ onto $L\{\mathbf{y}_0, \mathbf{y}_1, \cdots, \mathbf{y}_j\}$, then

$$\mathbf{s}(i) = \hat{\mathbf{s}}(i \mid j) + \tilde{\mathbf{s}}(i \mid j), \tag{27}$$

where $\hat{\mathbf{s}}(i \mid j) \in L\{\mathbf{y}_0, \mathbf{y}_1, \cdots, \mathbf{y}_j\}$ and $\tilde{\mathbf{s}}(i \mid j)$ satisfy orthogonality condition

$$\tilde{\mathbf{s}}(i \mid j) \perp L\{\mathbf{y}_0, \mathbf{y}_1, \cdots, \mathbf{y}_j\}, \tag{28}$$

or equivalently, $\langle \tilde{\mathbf{s}}(i \mid j), \mathbf{y}_k\rangle = 0$ for any $k = 0, 1, \cdots, j$. As is well known, the projection in Euclidean space always exists and unique. However, this is not always the case in Krein space. Actually, the Krein space projection exists and is unique if and only if a certain Gramian matrix is nonsingular.

Let $\mathbf{y}_z(t)$ be the observation of system (24)-(25) at time t, then we have,

$$\mathbf{y}_z(t) = \begin{cases} \breve{\mathbf{z}}(t \mid t-l), & 0 \le t < l \\ \begin{bmatrix} \mathbf{y}(t-l) \\ \breve{\mathbf{z}}(t \mid t-l) \end{bmatrix}, & t \ge l. \end{cases} \tag{29}$$

Theorem 1: Consider the system (23)-(25). The matrix $P_a(t)$ is as

$$\begin{aligned}&P_a(t) \triangleq \langle \tilde{\mathbf{x}}_a(t), \tilde{\mathbf{x}}_a(t)\rangle, \\ &P_a(0) = diag(\Pi_0^{-1}, 0_{lp \times lp}),\end{aligned} \tag{30}$$

where

$$\tilde{\mathbf{x}}_a(t) = \mathbf{x}_a(t) - \hat{\mathbf{x}}_a(t \mid t-1), \tag{31}$$

and $\hat{\mathbf{x}}_a(t \mid t-1)$ is the projection of $\mathbf{x}_a(t)$ on the linear space

$$L\{\mathbf{y}_z(0), \mathbf{y}_z(1), \cdots, \mathbf{y}_z(t-1)\}. \tag{32}$$

Then the matrix $P_a(t)$ is the solution to Riccati equations (14)-(16) or (17)-(19).

Proof: For $t > l$, using the Kalman filtering formulation in Krein space [11,13], it follows

$$\begin{aligned}\hat{\mathbf{x}}_a(t+1 \mid t) = {} & \Phi_a(t)\hat{\mathbf{x}}(t \mid t-1) + \Phi_a(t)P_a(t)\begin{bmatrix} H_a(t) \\ L_a(t) \end{bmatrix}^T \\ & \times Q_{w_z}^{-1}(t)\mathbf{w}_z(t),\end{aligned} \tag{33}$$

where $\mathbf{w}_z(t)$ is the innovation, i.e., one step prediction error of observation $\mathbf{y}_z(t), t \ge l$,

$$\mathbf{w}_z(t) = \begin{bmatrix} H_a(t) \\ L_a(t) \end{bmatrix}\tilde{\mathbf{x}}_a(t) + \begin{bmatrix} \mathbf{v}(t-l) \\ \mathbf{v}_z(t) \end{bmatrix}, \tag{34}$$

and $Q_{w_z}(t)$ is calculated by

$$Q_{w_z}(t) = \langle \mathbf{w}_z(t), \mathbf{w}_z(t)\rangle. \tag{35}$$

Since $\tilde{\mathbf{x}}_a(t+1)$ is uncorrelated with $\mathbf{w}_z(t)$, it follows from the above equation that

$$P_a(t+1)+\Phi_a(t)P_a(t)\begin{bmatrix}H_a(t)\\L_a(t)\end{bmatrix}^T Q_{w_z}^{-1}(t)\begin{bmatrix}H_a(t)\\L_a(t)\end{bmatrix} \times P_a(t)\Phi_a^T(t)=\Phi_a(t)P_a(t)\Phi_a^T(t)+\Gamma_a(t)\Gamma_a^T(t). \quad (36)$$

Obviously, (36) can be rewritten as (17)-(19).

For $t\le l$, similar results can be easily rewritten as (17)-(19) with the initial value $P_a(0)$. Now we complete the proof. □

In view of (9) and (30), the matrix $P_a(t)$ has the form of

$$P_a(t)=\{P_{ij}(t), i,j=1,2,\cdots,l+1\}, \quad (37)$$

with

$$P_{ij}(t)=\langle\tilde{\mathbf{x}}(t-i+1),\tilde{\mathbf{x}}(t-j+1)\rangle, \quad (38)$$

$$\tilde{\mathbf{x}}(t-s+1)=\mathbf{x}(t-s+1)-\hat{\mathbf{x}}(t-s+1\,|\,t-1), \quad s=i,j, \quad (39)$$

where $\hat{\mathbf{x}}(t-s+1\,|\,t-1),(s=i,j)$ is the projection of $\mathbf{x}(t-s+1)$ onto the linear space of (32). Thus the computation of $P_a(t)$ is converted into the problem of calculating $P_{ij}(t)\,(i,j=1,2,\cdots,l+1)$. To this end, we introduce a reorganized innovation sequence and the associated Riccati equation.

From (29) and let $t_{l+1}\triangleq t-l-1$, the linear space (32) is equivalent to

$$L\{\mathbf{y}(0),\cdots,\mathbf{y}(t_{l+1});\mathbf{y}_s(0),\cdots,\mathbf{y}(t-1)\} \quad (40)$$

or

$$L\{\mathbf{y}_f(0),\cdots,\mathbf{y}_f(t_{l+1});\mathbf{y}_s(t-l),\cdots,\mathbf{y}_s(t-1)\}, \quad (41)$$

where for $i=0,1,\cdots,t_{l+1}$,

$$\mathbf{y}_f(i)\triangleq\begin{bmatrix}\mathbf{y}(i)\\ \breve{\mathbf{z}}(i\,|\,i-l)\end{bmatrix}=\begin{bmatrix}H_i\\L_i\end{bmatrix}\mathbf{x}(i)+\begin{bmatrix}\mathbf{v}(i)\\ \mathbf{v}_z(i)\end{bmatrix}, \quad (42)$$

and

$$\mathbf{y}_s(i)\triangleq\breve{\mathbf{z}}(i\,|\,i-l),\ i=1,2,\cdots,t-1. \quad (43)$$

Definition 1: The estimation $\hat{\xi}(i\,|\,k,j)$ is defined as the projection (if exists) of $\xi(i)$ onto the linear space of

$$L\{\mathbf{y}(0),\cdots,\mathbf{y}(j);\mathbf{y}_s(0),\cdots,\mathbf{y}_s(k)\}.$$

Based on Definition 1, the reorganized innovation sequence associated with (40) (or equivalently (41)) is

$$\{\mathbf{w}(\tau,\tau),0\le\tau\le t_{l+1};\mathbf{w}(\tau,t_{l+1}),t-l\le\tau\le t-1\}, \quad (44)$$

where

$$\mathbf{w}(\tau,\tau)\triangleq\mathbf{y}_f(\tau)-\hat{\mathbf{y}}_f(\tau\,|\,\tau-1,\tau-1), \quad (45)$$

$$\mathbf{w}(\tau,t_{l+1})\triangleq\mathbf{y}_s(\tau)-\hat{\mathbf{y}}_s(\tau\,|\,\tau-1,t_{l+1}),\ \tau>t_{l+1}, \quad (46)$$

while $\hat{\mathbf{y}}_f(\tau\,|\,\tau-1,\tau-1)$ and $\hat{\mathbf{y}}_s(\tau\,|\,\tau-1,t_{l+1})$ are defined as in Definition 1, i.e., the projection of $\mathbf{y}_f(\tau)$ and $\mathbf{y}_s(\tau)$ onto linear space

$$L\{\mathbf{y}_f(0),\cdots,\mathbf{y}_f(\tau-1)\}$$

and

$$L\{\mathbf{y}_f(0),\cdots,\mathbf{y}_f(t_{l+1});\mathbf{y}_s(t-l),\cdots,\mathbf{y}_s(\tau-1)\},$$

respectively, and $\mathbf{y}_s(\tau)=\breve{\mathbf{z}}(t\,|\,t-l)$.

Similar to the discussion in [14], it is not difficult to show that (44) is the innovation sequence and spans the same linear space as (41), where (44) is termed as the reorganized innovation process.

Definition 2: The matrix

$$P_i^j(t)\triangleq\langle\mathbf{x}(t+j),\mathbf{e}(t+i,t)\rangle;\ j\ge 0, i>0 \quad (47)$$

is termed as the cross covariance matrix of the state $\mathbf{x}(t+j)$ with state estimation error $\mathbf{e}(t+i,t)$,

$$\mathbf{e}(t+i,t)=\mathbf{x}(t+i)-\hat{\mathbf{x}}(t+i\,|\,t+i-1,t), i>0, \quad (48)$$

and $\hat{\mathbf{x}}(t+i\,|\,t+i-1,t)$ is as in Definition 1.

As to be shown, the matrix $P_i^j(t)$ plays an important role for the distributed algorithm of $P_a(t)$.

Theorem 2: The cross-covariance matrix $P_i^j(t)$ can.be calculated by the following process.

1) The case of $i=j$.

For $i=1$, $P_1^1(t)$ is a solution to the following equation

$$P_1^1(t)=\Phi_tP_1^1(t-1)\Phi_t^T-\Phi_tP_1^1(t-1)\begin{bmatrix}H_t\\L_t\end{bmatrix}^T \times Q_w^{-1}(t,t)\begin{bmatrix}H_t\\L_t\end{bmatrix}P_1^1(t-1)\Phi_t^T+\Gamma_t\Gamma_t^T, \quad (49)$$

where $P_1^1(-1)=P_0$ and

$$Q_w(t,t)=\begin{bmatrix}H_t\\L_t\end{bmatrix}P_1^1(t-1)\begin{bmatrix}H_t\\L_t\end{bmatrix}^T+\begin{bmatrix}I_m & 0\\0 & -\gamma^2I_p\end{bmatrix}. \quad (50)$$

For $i>1$, $P_i^i(t)$ is a solution to the following Riccati equation

$$P_i^i(t)=\Phi_{t+i-1}P_{i-1}^{i-1}(t)\Phi_{t+i-1}^T-\Phi_{t+i-1}P_{i-1}^{i-1}(t)H_{t+i-1}^T$$

$$\times Q_w^{-1}(t+i-1,t)H_{t+i-1}P_{i-1}^{i-1}(t)\Phi_{t+i-1}^T + \Gamma_{t+i-1}\Gamma_{t+i-1}^T,\ P_1^1(t), \quad (51)$$

where

$$Q_w(t+i-1,t) = L_{t+i-1}P_{i-1}^{i-1}(t)L_{t+i-1}^T - \gamma^2 I_p. \quad (52)$$

2) The case of $i \neq j$.
For $j > i$,

$$P_i^j(t) = \Phi_{t+j-1}P_i^{j-1}(t),\ P_i^i(t), \quad (53)$$

else for $j < i$,

$$P_i^j(t) = P_{i-1}^j(t)A^T(t+i-1,t),\ P_j^j(t), \quad (54)$$

where

$$A(t+i-1,t) = \Phi_{t+i-1} - \Phi_{t+i-1}P_{i-1}^{i-1}(t)H_{t+i-1}^T \times Q_w^{-1}(t+i-1,t)H_{t+i-1}. \quad (55)$$

Proof: By applying the reorganized innovation analysis, the proof is completed similarly as [14]. □

Now, we present the distributed algorithm for computing $P_a(t)$ based on the reorganized innovation sequence and the associated Riccati equations aforementioned.

Theorem 3: The solution to the Riccati equation (17)-(19) associated with the augmented system (6)-(8) is given by $P_a(t) = \{P_{ij}(t), 1 \le i,\ j \le l+1\}$, where $P_{ij}(t)$ compute by

$$P_{ij}(t) = P_{l-j+2}^{l-i+2}(t_{l+1}) - \sum_{s=1}^{j-1} P_{l-s+1}^{l-i+2}(t_{l+1})L_{t-s}^T \times Q_w^{-1}(t-s,t_{l+1})L_{t-s}[P_{l-s+1}^{l-j+2}(t_{l+1})]^T, \quad (56)$$

where $t_{l+1} = t-l-1$,

$$Q_w(t-s,t_{l+1}) = L_{t-s}P_{l-s+1}^{l-s+1}(t_{l+1})L_{t-s}^T - \gamma^2 I_p, \quad (57)$$

and $P^{\cdot}(t_{l+1})$ are calculated by Theorem 2.

Proof: Note that

$$\tilde{\mathbf{x}}(t-j+1) = \mathbf{x}(t-j+1) - \hat{\mathbf{x}}(t-j+1 \mid t-1, t_{l+1}),$$

where $\hat{\mathbf{x}}(t-j+1 \mid t-1, t_{l+1})$ is the projection of $\mathbf{x}(t-j+1)$ onto the linear space of (40). Since $\mathbf{w}(\cdot,\cdot)$ is white noise, using projection formula yields

$$\tilde{\mathbf{x}}(t-j+1) = \mathbf{e}(t-j+1,t_{l+1}) - \sum_{s=1}^{j-1} P_{l-s+1}^{l-j+2}(t_{l+1})L_{t-s}^T \times Q_w^{-1}(t-s,t_{l+1})\mathbf{w}(t-s,t_{l+1}),$$

where

$$\mathbf{w}(t-s,t_{l+1}) = L_{t-s}\mathbf{e}(t-s,t_{l+1}) + \mathbf{v}_z(t-s). \quad (58)$$

Thus we have $P_{ij}(t)$ as shown in (56). □

By Theorem 2, a simple method for multi-step prediction can be obtained easily as in the following theorem.

Theorem 4: Consider the system (1)-(3). For a given scalar $\gamma > 0$,

1) An H_∞ estimator $\breve{z}(t \mid t-l)$ that achieves (4) exists if and only if

$$\begin{bmatrix} H_{t-l}P_{l+1,l+1}(t)H_{t-l}^T + I_m & H_{t-l}P_{l+1,1}(t)L_t^T \\ L_tP_{1,l+1}(t)H_{t-l}^T & L_tP_{11}(t)L_t^T - \gamma^2 I_p \end{bmatrix}$$

and

$$\begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_p \end{bmatrix}$$

have the same inertias.

2) If $\breve{z}(t \mid t-l), t \ge l$ exists, then the H_∞ multi- step predictor $\breve{z}(t \mid t-l)$ is given by

$$\breve{z}(t \mid t-l) = L_a(t)\breve{x}_a(t),$$

where $\breve{x}_a(t)$ is computed by

$$\breve{x}_a(t+1) = \Phi_a(t)\breve{x}_a(t) + K_a(t+1)y(t-l+1) - K_a(t+1)H_a(t+1)\Phi_a(t)\breve{x}_a(t)], \quad (59)$$

$$K_a(t) = P_a(l+1,t)H_{t-l}^T[I + H_{t-l}P_{l+1,l+1}(t)H_{t-l}^T]^{-1}, \quad (60)$$

with

$$P_a(l+1,t) = \begin{bmatrix} P_{1,l+1}(t) \\ \vdots \\ P_{l+1,l+1}(t) \end{bmatrix}.$$

The matrices $P_{11}(t)$ and $P_{i,l+1}(t), i = 1,2,\cdots,l+1$ can be computed as

$$P_{11}(t) = P_{l+1}^{l+1}(t_{l+1}), P_{l+1,1}(t) = P_{l+1}^1(t_{l+1}), \quad (61)$$

$$P_{i,l+1}(t) = P_1^{l-i+2}(t_{l+1}) - \sum_{s=1}^{l} P_{l-s+1}^{l-i+2}(t_{l+1}) \times L_{t-s}^T Q_w^{-1}(t-s,t_{l+1})L_{t-s}[P_{l-s+1}^{l-i+2}(t_{l+1})]^T. \quad (62)$$

Proof: Based on Lemma 1, for a given $\gamma > 0$, an H_∞ estimator $\breve{z}(t \mid t-l)$ that achieves (4) exists if and only if $\Sigma_a(t) > 0, (0 < t \le N)$, where $\Sigma_a(t)$ is as in (15) and $P_a(t)$ can be computed by Theorem 3. Recall [11,13], $\Sigma_a(t) > 0, (0 < t \le N)$ is equivalent to that

$$\begin{bmatrix} H_a(t) \\ L_a(t) \end{bmatrix} P_a(t) \begin{bmatrix} H_a(t) \\ L_a(t) \end{bmatrix}^T + \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_p \end{bmatrix}$$

and

$$\begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_p \end{bmatrix}$$

have the same inertia. Furthermore, we note that

$$\begin{bmatrix} H_a(t) \\ L_a(t) \end{bmatrix} P_a(t) \begin{bmatrix} H_a(t) \\ L_a(t) \end{bmatrix}^T + \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_p \end{bmatrix}$$
$$= \begin{bmatrix} H_{t-l} P_{l+1,l+1}(t) H_{t-l}^T + I_m & H_{t-l} P_{l+1,1}(t) L_t^T \\ L_t P_{1,l+1}(t) H_{t-l}^T & L_t P_{11}(t) L_t^T - \gamma^2 I_p \end{bmatrix}.$$

A multi-step predictor $\check{z}(t|t-l)$ is given by (20)-(22), which can be easily rewritten as (59)-(62).

Remark 3: Note that only the block matrices $P_{i,l+1}(t), i = 1, 2, \cdots, l+1$ and $P_{11}(t)$ are required for the multi-step predictor in Theorem 4. We need not to compute all the blocks of $P_a(t) = \{P_{ij}(t), i, j = 1, 2, \cdots, l+1\}$ as in (17)-(19).

3.3. Operation count for computing $P_a(t)$

In the above subsection, we have revisited the previous work and presented a new approach for multi-step prediction. For the purpose of comparison, we shall give the operation numbers of calculating $P_a(t)$ by the two approaches.

Traditionally, as additions are much faster than multiplications and divisions, counted together, which is used as the operation count. For a convenient notation, denote $MD(\cdot)$ the number of multiplications and divisions (the multiplication is from right to left), the *MD1* and *MD2* are defined as the numbers of calculating $P_a(t)$ with $P_a(t-1)$ by previous work and the new approach in this paper. It is clear that $MD1 > MD2$ when l is appropriately large. Moreover, when l is bigger, the difference is bigger (suppose that p is fixed). For example, consider the system (1)-(3) with $n = 4$, $m = 1$, $p = 3$ and $r = 1$. We investigate the relationship between the *MD* number and l shown as Table 1.

Table 1. Comparisons of computational cost.

l	1	2	3	5	8
MD1	1444	2620	4148	8260	17068
MD2	1459	2274	3217	5487	9852

4. CONCLUSION

In this paper, we have approached the problem of H_∞ multi-step prediction by using the innovation reorganization analysis and the projection formula in an indefinite linear space. The main contribution is that we have presented a distributed algorithm to the Riccati equations. A numerical example has clearly indicated the low computational cost of our algorithm.

REFERENCES

[1] T. Kailath, A. Sayed, and B. Hassibi, *Linear Estimation*, Prentice-Hall, Englewood, Cliffs, NJ, 1999.

[2] K. M. Nagpal and P. P. Khargonekar, "Filtering and smoothing in an H_∞ setting," *IEEE Trans. on Automatic Control*, vol. 36, pp. 152-166, 1991.

[3] E. Fridman, U. Shaked, and L. H. Xie, "Robust H_∞ filtering of linear systems with time-varying delay," *IEEE Trans. on Automatic Control*, vol. 48, no. 1, pp. 159-165, 2003.

[4] U. Shaked, "H_∞ optimal estimation-old and new result," *Proc. of the 21st Brazilian Automatic Control Conference*, Uberlandia, MG, Brasil, September 1998.

[5] P. Colaneri and A. Ferrante, "A *J*-spectral factorization approach for H_∞ estimation problem in discrete time," *IEEE Trans. on Automatic Control*, vol. 47, no. 12, pp. 2108-2113, 2002.

[6] P. Colaneri and A. Ferrante, "Algebraic riccati equation and *J*-spectral factorization for H_∞ estimation," *System & Control Letters*, vol. 51, no. 5, pp. 383-393, 2004.

[7] P. Colaneri and A. Ferrante, "Algebraic riccati equation and *J*-spectral factorization for H_∞ filtering and deconvolution," *SIAM J. Contr. and Opt.*, vol. 45, no. 1, pp. 123-145, 2006.

[8] H. Gao and C. Wang, "A delay-dependent approach to robust H_∞ filtering for uncertain discretetime state-delayed systems," *IEEE Trans. on Signal Processing*, vol. 52, no. 6, pp. 1631-1640, 2004.

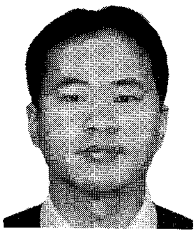
[9] L. Mirkin and G. Tadmor, "Yet another H_∞ disctetization," *IEEE Trans. on Automatic Control*, vol. 38, pp. 891-894, 2003.

[10] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate," *IEEE Trans. on Automatic Control*, vol. 26, pp. 301-320, 1981.

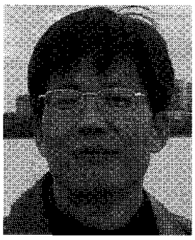
[11] B. Hassibi, A. H. Sayed, and T. Kailath, *Indefinite Quadratic Estimation and Control: A Unified Approach to H2 and H_∞ Theories*, SIAM Studies in Applied Mathematics Series, 1998.

[12] Y. Theodor and U. Shaked, "Game theory approach to H_∞ optimal discrete-time fixed-point and fixed-lag smoothing," *IEEE Trans. on

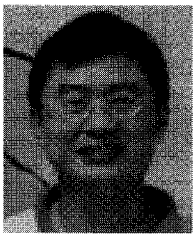
Automatic Control, vol. 39, pp. 1944-1948, 1994.
[13] H. Zhang, L. Xie, Y. C. Soh, and D. Zhang, "H_∞ fixed-lag smoothing for linear time-varying discrete time systems," *Automatica*, vol. 41, no. 5, pp. 839-846, 2005.
[14] H. Zhang, L. Xie, D. Zhang, and Y. C. Soh, "A reorganized innovation approach to linear estimation," *IEEE Trans. on Automatic Control*, vol. 49, no. 10, pp. 1810-1814, 2004.



Hao-qian Wang received the Ph.D. degree in Harbin Institute of Technology in 2005. His research interests include signal processing, descriptor systems, and video coding.



Huan-shui Zhang received the Ph.D. degree in Northeast University in 1997. His research interests include robust filtering, optimal control, and signal processing.



Hong Hu received the Ph.D. degree in Toronto University in 2003. His research interests include micro/nano systems, motion control and micro-electronics manufacturing.