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NONEXISTENCE OF NODAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATION WITH SOBOLEV-HARDY TERM

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ABSTRACT. Let B_1 be a unit ball in $\mathbf{R}^n$ $(n \geq 3)$, and $2^* = 2n/(n-2)$ be the critical Sobolev exponent for the embedding $H_0^1(B_1) \hookrightarrow L^{2^*}(B_1)$. By using a variant of Pohozǎev's identity, we prove the nonexistence of nodal solutions for the Dirichlet problem $-\Delta u - \mu \frac{u}{|x|^2} = \lambda u + |u|^{2^*-2}u$ in B_1, $u = 0$ on ∂B_1 for suitable positive numbers μ and ν.

1. INTRODUCTION

In this paper we deal with the nonexistence of nodal solutions(changing-sign solutions) for the following problem

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \lambda u + |u|^{2^*-2}u & \text{in} \quad B_1 \\ u = 0 & \text{on} \quad \partial B_1, \end{cases} \tag{1}$$

where B_1 is a unit ball in $\mathbf{R}^n$ $(n \geq 3)$, and $2^* = 2n/(n-2)$.

In recent years, much attention has been paid to the existence of nontrivial solutions to (1) where $0 \leq \mu < \bar{\mu} = (\frac{n-2}{2})^2$, $\lambda \in \mathbf{R}$. The well-known Hardy's inequality implies that the linear elliptic operator $L = -\Delta - \mu I/|x|^2$ is positive and has discrete spectrum if and only if $\mu < \bar{\mu} = (n-2)^2/4$. In particular, L has a first eigenvalue, say $\lambda_1(\mu)$, which is a solution to the problem

$$\lambda_1(\mu) = \min_{\varphi \in H_0^1(B_1)} \frac{\int_{B_1} |\nabla \varphi|^2 - \mu \int_{B_1} \varphi^2/|x|^2}{\int_{B_1} \varphi^2}.$$

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Using Pohožaev-type identity we can show that (1) has no solution since B_1 is star shaped with respect to $x = 0$ and $\lambda \le 0$. Hence we investigate the problem (1) in the confined range $0 < \lambda < \lambda_1(\mu)$ and $0 \le \mu < \bar{\mu}$.

Using the local Palais-Smale condition, Jannelli proved the following result in [8]:
(i) If $0 < \mu \le \bar{\mu} - 1$, then (1) has at least one positive solution $u \in H_0^1(\Omega)$ when $0 < \lambda < \lambda_1(\mu)$;
(ii) If $\bar{\mu} - 1 < \mu < \bar{\mu}$, then (1) has at least one positive solution $u \in H_0^1(\Omega)$ when $\lambda_*(\mu) < \lambda < \lambda_1(\mu)$, where

$$\lambda_*(\mu) = \min_{\varphi \in H_0^1(\Omega)} \Big[\int_\Omega \frac{|\nabla \varphi(x)|^2}{|x|^{2\gamma}} dx \Big/ \int_\Omega \frac{\varphi^2(x)}{|x|^{2\gamma}} dx \Big]$$

and $\gamma = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}$;
(iii) If $\bar{\mu} - 1 < \mu < \bar{\mu}$, then (1) has no positive solution for $\lambda \le \lambda_*(\mu)$.

This known result shows that any dimension n may be critical for problem (1); now it is only a matter of how μ is close to $\bar{\mu}$. This type of equations were also studied. Ruiz-Willem extended the Jannelli's result in [10]. They proved that (1) has a positive solution not just for $0 \le \mu < \bar{\mu} - 1$ but also for $\mu < 0$. The existence of nodal solutions for (1) was investigated by Cao-Feng [2] and Choi [5]. In 2003 Cao-Feng [2] obtained the following existence result by applying the min-max principles:
Let $n \ge 7$, $0 < \lambda < \lambda_1(\mu)$ and $0 < \mu < \bar{\mu} - 4 = (n+2)(n-6)/4$. Then there exists a pair of nodal solutions $u^\pm$ of (1) satisfying

$$\int_{B_1} |u|^{2^*-2} u v(u) = 0,$$

where $v(u)$ is the first eigenfunction of the weighted eigenvalue problem

$$-(\Delta u + \mu \frac{u}{|x|^2} + \lambda u) v = \gamma |u|^{2^*-2} v \ \text{ in } B_1, \quad u = 0 \ \text{ on } \partial B_1.$$

For the case without Sobolev-Hardy terms in a bounded domain $\Omega \subset \mathbf{R}^n$

$$-\Delta u = \lambda u + |u|^{2^*-2} u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{2}$$

there have been so far many works on the existence and nonexistence of nodal solutions. Especially, Ceramini-Solimini-Struwe [4] obtained existence results on nodal solutions of (2) such that if $n \ge 6$, then (2) admits a pair of nodal solutions for each $0 < \lambda < \lambda_1$ where λ_1 is the first eigenvalue of $-\Delta(\Omega)$ with zero Dirichlet boundary data and moreover, if $n \ge 7$, then under $\Omega = B_1$, (2) admits a pair of radial solutions with exactly k nodes for any integer $k \ge 0$. For the nonexistence of nodal solutions of (2), it is known that if $n \in \{3, 4, 5, 6\}$, then for small $\lambda > 0$ (2) has no nodal radial solution. Wang-Wu [11] considered the nonexistence of nodal radial solutions for (2) by using Pohožaev's identity which makes their argument simple.

In 2001 Bae-Pahk [1] considered the Dirichlet problem

$$-\Delta u = \lambda |x|^{\mu}|u|^{q-2}u + |x|^{\nu}|u|^{p-2}u \text{ in } B_1, \quad u = 0 \text{ on } \partial B_1, \tag{3}$$

where $\mu, \nu > -2$, $p = 2(n+\nu)/(n-2)$, $2 \le q < 2(n+\mu)/(n-2)$ and λ is a real parameter. They extended the previous results for the nonexistence to (3) as follows :
Assume that $\mu, \nu > -2$ and $2 \le q \le (n+2+2\nu)/(n-2)\min\{1, (2+\mu)/(2+\nu)\}$. Then there exists a constant $\tilde{\lambda} > 0$ such that for $\lambda \in (0, \tilde{\lambda})$, (3) has no nodal radial solution in $H_0^1(B_1)$.

Similarly we investigate the nonexistence of nodal solutions for (1). We prove the following nonexistence result by using a variant of Pohožaev's identity :

Theorem 1.1. *Let $n = 3, 4, 5, 6$ and $0 < \mu < \bar{\mu}$. Then there exists a constant $\lambda^* > 0$ such that for $\lambda \in (0, \lambda^*)$, (1) has no nodal radial solution.*

Remark Let $n \ge 7$, $0 < \lambda \le \lambda_*(\mu)$ and $\bar{\mu} - 1 < \mu < \bar{\mu}$. Then (1) has no nodal radial solution. We can prove the result adopting a Pohožaev-type argument, in analogy with the proof of Theorem C in [3] and Theorem 1.C. in [8].

We have the gap between the values of μ determining the existence and nonexistence of nodal solutions. We guess that the following is true: Let $n \ge 7$, and $\bar{\mu} - 4 \le \mu \le \bar{\mu} - 1$. Then (1) has no nodal radial solution for some values of λ.

2. PRELIMINARIES

In this section, we collect some known facts and present basic observations.
The imbedding of $H_0^1(B_1)$ in $L^2(B_1)$ with respect to the weight $|x|^{-2}$ is continuous.

Lemma 2.1. [7] *Suppose $0 \le \mu < \bar{\mu}$ and $\bar{\mu} = (\frac{n-2}{2})^2$. Then we have*
(i) *(Hardy's inequality)*

$$\bar{\mu} \int_{B_1} \frac{|u|^2}{|x|^2} \le \int_{B_1} |\nabla u|^2, \quad \forall \ u \in H_0^1(B_1);$$

(ii) *The constant $\bar{\mu}$ is optimal.*

Now we define the constant S_μ and investigate the properties of S_μ. Let $D^{1,2}(\mathbf{R}^n) = \{u \in L^{2^*}(\mathbf{R}^n) | \ |\nabla u| \in L^2(\mathbf{R}^n)\}$. For all $\mu \in [0, \bar{\mu})$, we define the constant

$$S_\mu := \inf_{u \in D_1^2(\mathbf{R}^n)/\{0\}} \frac{\int_{\mathbf{R}^n} |\nabla u|^2 dx - \mu \int_{\mathbf{R}^n} u^2/|x|^2 dx}{(\int_{\mathbf{R}^n} |u|^{2^*} dx)^{2/2^*}}.$$

Lemma 2.2. [6] *Suppose $0 \le \mu < \bar{\mu}$. Then we have*
(i) *S_μ is the best constant for the embedding*

$$\{u \in D^{1,2}(\mathbf{R}^n) : \int_{\mathbf{R}^n} (|\nabla u|^2 - \mu u^2/|x|^2) dx < \infty\} \hookrightarrow L^{2^*}(\mathbf{R}^n);$$

(ii) *S_μ is independent of any $\Omega \subset \mathbf{R}^n$ in the sense that if*

$$S_\mu(\Omega) = \inf_{u\in D_1^2(\Omega)/\{0\}} \frac{\int_\Omega |\nabla u|^2dx - \mu\int_\Omega u^2/|x|^2dx}{(\int_\Omega |u|^{2^*}dx)^{2/2^*}},$$

then $S_\mu(\Omega) = S_\mu(\mathbf{R}^n) = S_\mu$;
(iii) *When $\Omega = \mathbf{R}^n$, S_μ is achieved by the functions*

$$U_\epsilon(x) = \frac{C_\epsilon}{(\epsilon|x|^{\gamma'/\sqrt{\bar\mu}} + |x|^{\gamma/\sqrt{\bar\mu}})^{\sqrt{\bar\mu}}},$$

where $\gamma = \sqrt{\bar\mu} + \sqrt{\bar\mu - \mu}$, $\gamma' = \sqrt{\bar\mu} - \sqrt{\bar\mu-\mu}$, and $C_\epsilon = (\frac{4\epsilon n(\bar\mu-\mu)}{n-2})^{\sqrt{\bar\mu}/2}, \forall\, \epsilon > 0$. Moreover, the functions $U_\epsilon(x)$ are the only positive radial solutions of $-\Delta u - \frac{\mu}{|x|^2}u = |u|^{2^*-2}u$ in $\mathbf{R}^n$.

To estimate the asymptotic behavior of positive solutions of (1) near an isolated singularity at $r = 0$, we need the following Proposition and Lemmas.

Lemma 2.3. *If u is a positive radial solution of* (1) *in B_a, $0 < a \le 1$ and $u(a) = 0$, then u goes to ∞ as $r \to 0$.*

Proof. See Lemma 2.14 in [5]. □

Proposition 2.4. *Let u be a nonnegative function in $H_0^1(B_1)$ satisfying the following inequality:*

$$\int_{B_1} \nabla u\nabla\phi \le C\int_{B_1} |x|^\nu(u + u^{(n+2+2\nu)/(n-2)})\phi$$

for all $\phi \in H_0^1(B_1)$. If either $-2 < \nu \le 0$, or $-2 < \nu$ and u is radial, then u is bounded near 0.

Proof. See Proposition 2.2 in [1]. □

Lemma 2.5. *Let $\nu_i > -2$, $i = 1, 2, \cdots, k$ with $k \in \mathbf{N}$. If u is a nonnegative function in $H_r(B_a)$ for some $a > 0$ satisfying*

$$\int_{B_a} \nabla u\nabla\phi \le C\sum_{i=1}^{k}\int_{B_a} |x|^{\nu_i}(u + u^{(n+2+2\nu_i)/(n-2)})\phi \tag{4}$$

for all $\phi \in H_r(B_a)$. Then, u is bounded near 0.

Proof. See Lemma 2.3 in [1]. □

Lemma 2.6. *Let $n \ge 3$, $\lambda \in \mathbf{R}$, and $0 < \mu < \bar\mu$. If u is a positive radial solution of* (1) *in B_a, $0 < a < 1$ and $u(a) = 0$, then*

$$u(\rho) = O(\rho^{-\gamma'})$$

near 0.

Proof. Let $w(\rho) = \rho^{\gamma'} u$. Then w satisfies

$$w'' + \frac{1+2\sqrt{\bar{\mu}-\mu}}{\rho} w' + \lambda w + \rho^{-2+\frac{4}{n-2}\sqrt{\bar{\mu}-\mu}} w^{2^*-1} = 0, \ 0 < \rho < a, \quad w(a) = 0,$$

In fact, w is a positive radial solution of the equation

$$-\nabla \cdot (|x|^{\alpha} \nabla w) = \lambda |x|^{\alpha} w + |x|^{-n+\frac{2n}{n-2}\sqrt{\bar{\mu}-\mu}} |w|^{2^*-2} w \quad \text{in } B_a, \ \ w = 0 \ \text{ on } \partial B_a,$$

where $\alpha = 2 - n + 2\sqrt{\bar{\mu}-\mu}$.

Set $v(y) = ((n-2)/2\sqrt{\bar{\mu}-\mu})^{(n-2)/2} w(x)$ and $|y| = |x|^{2\sqrt{\bar{\mu}-\mu}/(n-2)}$. After a direct calculation we have

$$-\Delta v = \lambda \Big(\frac{n-2}{2\sqrt{\bar{\mu}-\mu}}\Big)^2 |y|^{\frac{(n-2-2\sqrt{\bar{\mu}-\mu})}{\sqrt{\bar{\mu}-\mu}}} v + |v|^{2^*-2} v \ \text{ in } B_a, \ \ v = 0 \ \text{ on } \partial B_a.$$

Then v satisfies (4) for some constant C. Therefore, from Lemma 2.5, v is bounded near 0. So w is bounded near 0. □

One of our methods for nonexistence of nodal solutions is to use a variant of Pohožaev-Pucci-Serrin's identity (see Proposition 1 in [9] with $\mathcal{F}(x,u,p) = \frac{1}{2}|p|^2 - F(x,u), h(x) = x, a = (n-2)/2$).

Lemma 2.7. *Let f and $\nabla_x F$ be continuous on $\bar{\Omega} \times \mathbf{R}$, where $F(x,u) = \int_0^u f(x,t)dt$. If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies $\Delta u + f(x,u) = 0$ in Ω, then*

$$\begin{aligned} &\int_{\Omega} \left[nF(x,u) - \frac{n-2}{2} u f(x,u) + x \cdot \nabla_x F(x,u) \right] \\ = \ &\int_{\partial\Omega} \left[(x \cdot \nabla u) \frac{\partial u}{\partial \mathbf{n}} - (x \cdot \mathbf{n}) \frac{|\nabla u|^2}{2} + (x \cdot \mathbf{n}) F(x,u) + \frac{n-2}{2} u \frac{\partial u}{\partial \mathbf{n}} \right], \end{aligned} \tag{5}$$

where $\partial \mathbf{n}$ denotes the exterior unit normal.

Using Lemma 2.7 we can show that (1) has no nontrivial radial solution when $\lambda \le 0$.

Lemma 2.8. *Assume that $n \ge 3$, $\lambda \le 0$ and $0 < \mu < \bar{\mu}$. If u is a nonnegative radial solution of (1) in $B_a, 0 < a \le 1$ and $u(a) = 0$, then $u \equiv 0$ in B_a.*

Proof. Since $u \in C^2(B_a/B_\delta)$ for any $0 < \delta < a$, we can apply (5) to u on B_a/B_δ. Then, as $\delta \to 0$, it follows from Lemma 2.6 that

$$\frac{1}{2}\omega_n a^n |u'(a)|^2 = \lambda \int_{B_a} |u|^2. \tag{6}$$

When $\lambda < 0$, it follows immediately from (6) that $u \equiv 0$ in B_a. When $\lambda = 0$, we deduce from (6) that $u'(a) = 0$ and then by the uniqueness theorem for initial value problems of ODE we have $u \equiv 0$ in B_a. □

3. NONEXISTENCE OF NODAL SOLUTIONS

In this chapter we prove the nonexistence result of nodal solutions to (1) using a variant of Pohožaev's identity.

For a radial solution $u \in H_r(B_1)$ of (1), equation (1) is written in the form

$$u'' + \frac{n-1}{\rho}u' + \mu\frac{u}{\rho^2} + \lambda u + |u|^{2^*-2}u = 0, \quad 0 < \rho < 1, \quad u(1) = 0. \tag{7}$$

When $u > 0$ in $(0, a)$ and $u(a) = 0$ for some $0 < a < 1$, the derivative of u at the first zero point a is estimated in terms of λ and a. We adopt some argument in [1] and [11] to obtain the following result:

Lemma 3.1. *Let* $n = 3, 4, 5, 6$, $\lambda > 0$ *and* $0 < \mu < \bar{\mu}$. *If* u *is a positive radial solution of* (1) *in* B_a, $0 < a < 1$ *and* $u(a) = 0$, *then the derivatives of* u *at* a *satisfies*

$$|u^{'}(a)| \leq C\lambda^{(n+2)/8}a^{-(n-2)/4} \tag{8}$$

for some $C > 0$.

Proof. It is easy to see that $u \in C^2(B_a/B_\delta)$ for any $0 < \delta < a$. Then the Pohožaev-Pucci-Serrin's identity (5) implies

$$\begin{aligned} \frac{a^n}{2}|u^{'}(a)|^2 &= \frac{\lambda}{\omega_n}\int_{B_a/B_\delta}|u|^2 \\ &+ \left[\frac{\delta^n}{2}(u^{'})^2 + \frac{1}{2}\mu\delta^{n-2}u^2 + \frac{1}{2}\lambda\delta^n u^2 + \frac{1}{2^*}\delta^n|u|^{2^*} + \frac{n-2}{2}\delta^{n-1}uu^{'}\right]_{r=\delta}. \end{aligned}$$

Since $u \in H_0^1(B_a)$, there exists a sequence $\{\delta_i\}$ converging to 0 such that $\delta_i^n(u^{'}(\delta_i))^2 \to 0$ as $\delta_i \to 0$. Therefore, using Lemma 2.6, we lead to

$$\frac{1}{2}\omega_n a^n|u^{'}(a)|^2 = \lambda\int_{B_a}|u|^2. \tag{9}$$

Integrating (1) on B_a/B_δ to obtain

$$\omega_n(\delta^{n-1}u^{'}(\delta) - a^{n-1}u^{'}(a)) = \int_{B_a/B_\delta}(\mu\frac{u}{\rho^2} + \lambda u + u^{2^*-1})$$

and then letting $\delta \to 0$, we observe

$$\omega_n a^{n-1}|u^{'}(a)| = \int_{B_a}(\mu\frac{u}{\rho^2} + \lambda u + u^{2^*-1}). \tag{10}$$

Combining (9) and (10) implies

$$\int_{B_a}(\mu\frac{u}{\rho^2} + \lambda u + u^{2^*-1}) = \Big(2\lambda a^{n-2}\omega_n\int_{B_a}|u|^2\Big)^{\frac{1}{2}}. \tag{11}$$

Since $2 \le 2^* - 1$ for $n = 3, 4, 5, 6$, we obtain by using Hölder's inequality,

$$\int_{B_a} |u|^2 \le \Big(\int_{B_a} 1^{(2^*-1)/(2^*-3)}\Big)^{(2^*-3)/(2^*-1)}\Big(\int_{B_a} |u|^{2^*-1}\Big)^{2/(2^*-1)}$$
$$= \Big(\frac{\omega_n}{n}a^n\Big)^{-(n-6)/(n+2)}\Big(\int_{B_a} |u|^{2^*-1}\Big)^{2(n-2)/(n+2)} \tag{12}$$

and by (11) and (12),

$$\int_{B_a} (\mu\frac{u}{\rho^2} + \lambda u + u^{2^*-1})$$
$$\le \Big[2\lambda\omega_n a^{n-2}\Big(\frac{\omega_n a^n}{n}\Big)^{-(n-6)/(n+2)}\Big(\int_{B_a} |u|^{2^*-1}\Big)^{2(n-2)/(n+2)}\Big]^{1/2}$$
$$\le \Big[2\lambda\omega_n a^{n-2}\Big(\frac{\omega_n a^n}{n}\Big)^{-(n-6)/(n+2)}\Big\{\int_{B_a} (\mu\frac{u}{\rho^2} + \lambda u + u^{2^*-1})\Big\}^{2(n-2)/(n+2)}\Big]^{1/2}$$
$$= C\lambda^{1/2}a^{(3n-2)/(n+2)}\Big[\int_{B_a} (\mu\frac{u}{\rho^2} + \lambda u + u^{2^*-1})\Big]^{(n-2)/(n+2)}.$$

Hence we have

$$\int_{B_a} (\mu\frac{u}{\rho^2} + \lambda u + u^{2^*-1}) \le C\lambda^{(n+2)/8}a^{(3n-2)/4}.$$

Then, we have the inequality from (10)

$$\omega_n a^{n-1}|u'(a)| \le C\lambda^{(n+2)/8}a^{(3n-2)/4}.$$

Thus we obtain (8) by dividing the above inequality by $\omega_n a^{n-1}$. □

For a radial solution of (7) on an annulus B_1/B_a, we obtain the lower bound of $|v'(a)|$.

Lemma 3.2. *Assume that* $0 < \lambda < \lambda_1(\mu)$ *and* $0 < \mu < \bar{\mu}$. *If* v *is a radial solution of* (7) *in* $[a, 1]$ *for some* $0 < a < 1$ *satisfying* $v(a) = v(1) = 0$ *and* $v'(a) \ne 0$, *then there holds*

$$|v'(a)| \ge \frac{C}{a} \tag{13}$$

for some $C > 0$.

Proof. Let $|v(\rho)|$ attain its maximum at $\rho = \tau$. For $\rho \in [a, \tau]$,

$$|v'(\rho)| = \rho^{1-n}\int_\rho^\tau (\mu\frac{v}{s^2} + \lambda v + v^{2^*-1})s^{n-1}ds. \tag{14}$$

Considering $\rho = a$ in (14), we have

$$|v'(\rho)| \le (\frac{a}{\rho})^{n-1}|v'(a)|$$

and

$$|v(\tau)| \le \int_a^\tau |v'(\rho)|\, d\rho \le \frac{a}{n-2}\,|v'(a)|. \tag{15}$$

Let $B_{1,a} = B_1/B_a$. Since v is a solution of (7)

$$\int_{B_{1,a}} (|\nabla v|^2 - \mu \frac{v^2}{\rho^2}) - \lambda \int_{B_{1,a}} v^2 = \int_{B_{1,a}} |v|^{2^*}.$$

By the definition of $\lambda_1(\mu)$, we have

$$\Big(1 - \frac{\lambda}{\lambda_1(\mu)}\Big) \int_{B_{1,a}} (|\nabla v|^2 - \mu \frac{v^2}{\rho^2}) \le \int_{B_{1,a}} |v|^{2^*}.$$

Also Hardy's inequality implies that

$$\int_{B_{1,a}} |\nabla v|^2 - \frac{\mu}{\bar{\mu}} \int_{B_{1,a}} |\nabla v|^2 \le \int_{B_{1,a}} |\nabla v|^2 - \mu \int_{B_{1,a}} \frac{v^2}{\rho^2}.$$

Combining the above two inequalities we obtain

$$\Big(1 - \frac{\lambda}{\lambda_1(\mu)}\Big)\Big(1 - \frac{\mu}{\bar{\mu}}\Big) \int_{B_{1,a}} |\nabla v|^2 \le \int_{B_{1,a}} |v|^{2^*}.$$

Then, by the definition of S_μ, we have

$$\Big(1 - \frac{\lambda}{\lambda_1(\mu)}\Big)\Big(1 - \frac{\mu}{\bar{\mu}}\Big) \int_{B_{1,a}} |\nabla v|^2 \le \int_{B_{1,a}} |v|^{2^*} \le S_\mu^{-2^*/2}\Big(\int_{B_{1,a}} |\nabla v|^2\Big)^{2^*/2} \tag{16}$$

Therefore we obtain

$$\int_{B_{1,a}} |\nabla v|^2 \ge C$$

for some $C > 0$. Then, we conclude from (16) that for fixed $0 < \lambda < \lambda_1(\mu)$ and $0 < \mu < \bar{\mu}$,

$$\int_{B_{1,a}} |v|^{2^*} > C$$

for some constant $C > 0$ independent of v, which implies immediately that $|v(\tau)| > C$ for some $C > 0$. Therefore, it follows from (15) that

$$|v^{'}(a)| \ge \frac{n-2}{a} |v(\tau)| \ge \frac{n-2}{a} C > 0$$

for some $C > 0$. □

Combining Lemma 3.1 and 3.2, we have the nonexistence for small $\lambda > 0$.

Proof of Theorem 1.1.

Proof. Let $w(x)$ be a radial solution of (1). Suppose that w changes sign; $w > 0$ in B_a and $w(a) = 0$ for some a with $0 < a < 1$ and $w \not\equiv 0$ in B_1/B_a. By u and v, we denote the restrictions of $w(x)$ to B_a and B_1/B_a respectively. Then $u^{'}(a) = v^{'}(a)$. Since

$$-\frac{n-2}{4} - (-1) \ge 0,$$

(8) and (13) lead to a contradiction for small $\lambda > 0$. □

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